THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 1, JANUARY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII

1936

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

THE FIRST FALL MEETING OF THE KENTUCKY SECTION

The first fall meeting of the Kentucky Section of the Mathematical Association of America was held at Berea College, Berea, Kentucky, on November 24, 1934. Chairman W. R. Hutcherson presided at all sessions. Sixteen members and friends of the Association from Tennessee were present.

The attendance was seventy-one, including the following twenty-five members of the Association: N. B. Allison, J. G. Black, J. D. Bond, P. P. Boyd, L. W. Cohen, H. H. Downing, L. A. Fair, A. R. Fehn, Georgia M. Haswell, Charles Hatfield, W. R. Hutcherson, J. A. Hyden, E. R. Keller, C. G. Latimer, F. Elizabeth LeStourgeon, Buena C. Mathis, W. L. Miser, W. L. Moore, Mabel I. Nowlan, R. S. Park, J. K. Peterson, D. W. Pugsley, D. E. South, F. L. Wren, H. A. Wright.

Just before adjournment for lunch Professors W. L. Miser, P. P. Boyd and W. R. Hutcherson were appointed a committee to consider the question of forming a Kentucky-Tennessee Section and to report its findings at the afternoon business session. At that time Professor Miser reported that the committee thought it best to let the question rest. He extended an invitation to all to meet with the Tennessee group at Vanderbilt University in Nashville in the fall of 1935. The committee was continued to perfect plans for this meeting. At the suggestion of Professor Boyd, the Kentucky Section expressed its pleasure at having the Tennessee group with us and thanked them for interesting us in a fall meeting.

At a luncheon held at Boone Tavern, President W. J. Hutchins of Berea College welcomed the mathematicians and their friends and invited the group to his home to see the furniture made by Berea students.

On Friday evening Professor W. L. Wren of George Peabody College gave a general address in Science Hall, Berea College, to the Berea Mathematics Group and members and friends of the Association. His subject was "Desirable mathematical preparation of teachers of secondary mathematics." Abstract of this address follows:

In the educational crisis that has arisen during the depression, mathematics, as an integral part of the secondary curriculum, has found criticism rather harsh and severe. While this criticism has not been entirely unjustified, it has been largely misdirected. The justification for such vehemence is not in the shortcomings of mathematics as a subject, which merits intelligent study, but in the deleterious instruction imparted by those teachers of mathematics who are poorly prepared and non-enthusiastic. It is poor teaching whether in the elementary school, secondary school, college, or university, that leaves the impression that mathematics is *merely* a tool subject composed of a conglomerate mass of signs, symbols, and laws of operation. The mathematical preparation of the teacher of secondary mathematics should provide that historical, technical, and liberal information that would enable him so effectively to organize and present the subject matter that those studying under his direction should be confronted

with many opportunities to do constructive thinking, to realize the intrinsic worth of mathematics as an interpreter of their environment and as a contributor to their more efficient functioning as members of a civilized social order, and to come into contact with logical processes that should better enable them to discriminatingly integrate those principles and truths which tend to make living more significant and from which might evolve a wholesome philosophy of life.

At the morning and afternoon sessions the following papers were given:

1. "The Weddle and Kummer surfaces for restricted positions of six base points" by Professor J. A. Hyden, Vanderbilt University.

2. "An approach to the teaching of some applications of triple integration" by Professor J. D. Bond, University of Tennessee.

3. "A preliminary report on a kinetic theory of exterior ballistics" by Professor J. S. Morrel, Vanderbilt University.

4. "A plane Cremona transformation of order seven" by Professor Augustus Sisk, University of Tennessee, introduced by Professor Bond.

5. "Some notes on analytic geometry" by Professor C. G. Latimer, University of Kentucky.

6. "Inequalities" by Professor H. H. Downing, University of Kentucky.

7. "Metric spaces of infinitely many dimensions" by Professor L. W. Cohen, University of Kentucky.

8. "The role of definitions in mathematics" by J. H. D. Teller, University of Kentucky, introduced by Professor Boyd.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

2. Professor Bond presented a method of giving to the student of elementary calculus a more confident approach to the set-up of limits of integration over a solid bounded by elementary surfaces by emphasizing some simple principles of picturization, definite determination of order of procedure, and the meaning of typical elements.

3. Professor Morrel introduced his subject by giving a brief historical review of exterior ballistics, pointing out the failure of ballisticians up to the present to deduce from fundamental mathematical and physical principles a satisfactory law of atmospheric resistance. The theory presented in this paper is based upon the hypothesis that all effects of a gas upon a body in motion through it are the results of collisions between the body and the particles of gas. The investigation was carried through in detail for a spherical body moving at a low velocity, giving satisfactory laws of resistance, and of the Magnus effect. A tentative procedure was suggested applicable to projectiles of any shape moving in any manner.

4. Professor Sisk gave a brief description of a plane involution of degree seven and pointed out the comparative simplicity of the geometric machinery used to generate this involution as developed by Virgil Snyder.

6. In this paper Professor Downing shows how the formula for integration

by parts may be extended to expressions involving several factors; to expressions involving nth order derivatives; and to multiple integrals of expressions which involve powers of the differential of the independent variable. Finally certain of the results obtained were shown to be special cases of the formula for partial integration.

8. Mr. Teller contrasted the theory of definitions in deductive systems as developed by Schmidt (Journal of Philosophy, vol. 30, pp. 645–59) with the nominalistic theory of Russell and Peano. He showed that the former theory resolved some of the paradoxes of defining that Russell encountered. As a preliminary the classical Aristotelian theory of definitions was briefly discussed.

A. R. FEHN, *Secretary*

THE SECOND ANNUAL MEETING OF THE OKLAHOMA SECTION

The second regular meeting of the Oklahoma Section of the Mathematical Association of America was held in Tulsa, Oklahoma, Friday, February 8, 1935, Doctor N. A. Court presiding.

The attendance was seventy-two, including the following fourteen members of the Association: E. F. Allen, J. C. Brixey, N. A. Court, Mildred Dolezal, J. O. Hassler, H. L. Hall, E. E. Heimann, L. W. Johnson, J. E. LaFon, Dora McFarland, D. R. Shreve, W. T. Short, C. E. Springer, Grace West.

The following officers were elected for the next year: Chairman, E. F. Allen, Oklahoma A. and M. College; Vice-Chairman, L. W. Johnson, Central State Teachers College; Secretary, C. E. Springer, University of Oklahoma. The next meeting will be held in February 1936, at Oklahoma City, in connection with the Oklahoma Educational Association meeting.

The following four papers were read:

1. "Sets of Pythagorean numbers" by Professor H. L. Hall, Northwestern State Teachers College.

2. "A geometric locus arising from a problem in number theory" by Professor S. B. Townes, University of Oklahoma, introduced by Professor Court.

3. "Modern tendencies in the teaching of mathematics" by Professor E. E. Heimann, East Central State Teachers College.

4. "A square through four given points" by Professor L. W. Johnson, Central State Teachers College.

Abstracts of the papers follow:

1. Professor Hall stated that the Greek mathematicians had evolved a rule for forming a set of Pythagorean numbers to fit any odd integer, viz.: square the integer, subtract one and add one to the square, and take half the difference and sum. For an even integer, square half the integer, subtract one and add one to the square. If we use the formulas $(n+d)^2-n^2=k^2$ or $d(n+d)=k^2$ for odd integers, where k is any odd integer (known), n any integer more than 2, we deduce that d may be any divisor of k, including 1, but less than k. Each value

of d gives a set to go with k. Using the formula, $(n+d)^2-n^2=(2k)^2$, where $2k$ is any even integer, we see that d is an even divisor of $4k^2$, less than $2k$, and the quotient of $4k$ by d must also be even.

2. Professor Townes gave a geometric description of a problem in number theory. The integers were represented by the vertices of a lattice work.

3. In his paper Professor Heimann pointed out that the greatest problem facing colleges today is that of working out a closer correlation between high school and college mathematics, so that a student may be able to continue his mathematical training in college without any loss of time. College standards must be maintained by stating definitely where college mathematics should begin and by refusing to give credit for work below that level. Finally, the methods of teaching have changed as well as the emphasis on teaching skill.

4. Professor Johnson gave a historical sketch of the problem of constructing a square so each side should pass through a given point. All published solutions to this problem are essentially of two types. It was shown how to construct a particular one of the six solutions which this problem has in general. When there is an infinite number of solutions order may be so introduced into the problem that four of the solutions differ from the remainder, also the locus of the centers of the squares is a circle. Interesting properties of this circle were discussed. If the four given points constitute an orthocentric group the circle is the nine-point circle of the orthocentric group.

E. F. ALLEN, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The nineteenth regular meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado School of Mines, Golden, Colorado, Friday and Saturday, April 19 and 20, 1935.

There were three sessions, Professor J. C. Fitterer presiding at each.The attendance was fifty-two including the following twenty members of the Association: L. A. Aroian, C. F. Barr, Jack Britton, J. R. Everett, J. C. Fitterer, G. W. Gorrell, I. L. Hebel, Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, W. V. Lovitt, S. L. Macdonald, A. S. McMaster, W. K. Nelson, Greta Neubauer, E. D. Rainville, O. H. Rechard, A. W. Recht, C. H. Sisam.

Professor A. W. Recht of the University of Denver was elected chairman for next year. The next meeting was scheduled for some time in April 1936 at the University of Denver. Members and friends of the Association were guests of the School of Mines at a banquet held the evening of April 19. The following eight papers were read:

1. "Teaching science in mathematics" by Professor J. C. Stearns, University of Denver, introduced by Professor Lewis.

2. "The Pearsonian system of frequency curves" by Professor L. A. Aroian, Colorado State College.

3. "Remarks on a boundary value problem of the heat equation" by E. D. Rainville, Junior Engineer, U. S. Bureau of Reclamation.

4. "The use of convergent series in the evaluation of integrals with infinite limits" by Professor J. R. Everett, Colorado School of Mines.

5. "Greatest integers" by Professor W. K. Nelson, University of Colorado.

6. "A graphic solution of complex roots" by Professor C. F. Barr, University of Wyoming.

7. "Conformal mapping in hydrodynamics" by Professor C. A. Hutchinson, University of Colorado.

8. "Waring's problem and Diophantine equations with inequality conditions" by Professor A. J. Kempner, University of Colorado.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

2. Professor Aroian developed the differential equation of the Pearsonian system on the usual assumptions. The main types were discussed and examples of them from actual practise were given.

4. Mr. Rainville's paper discussed two distinct functions, apparently both of them solutions to the same mixed boundary value problem of the one dimensional heat equation. The need for a careful definition of what is meant by a solution to such a problem was noted. A tentative definition was adopted, without, however, any existence or uniqueness theorems to substantiate the conjecture.

5. Professor Nelson discussed greatest integers of bracketed numbers with regard to the interchanging of the operation of applying the bracket with the operations of adding, multiplying and raising to integral powers. The definite integral of a function of x and the definite integral of the bracket of the same function of x were compared. Formulas in terms of greatest integers were given for the Colorado sales tax and for service charges for checking accounts in Denver banks.

6. A brief section of Professor Barr's paper presented a circle method for representing the complex roots of a quadratic equation. Its purpose was to extend the construction given in Dickson's Theory of Equations to include complex roots. The body of the paper was devoted to a graphic construction for the complex roots of cubic equations with but one real root. The method involved only the graph of the cubic and the drawing of two straight lines. One of these lines was a properly selected member of the pencil of lines through the intersection of the curve with the x-axis. The other was the vertical locus of the midpoints of the chord formed by the curve and its other real intersections with the lines of this pencil. The real part of the complex roots is given by the x-position of this vertical line, and the coefficient of the imaginary part by the horizontal distance from this line to the intersection of the curve with this selected line of the pencil. The selection of the line is made by doubling the slope of the tangent to the curve from its real x-intercept.

The major part of the Saturday morning session was given over to a sym-

posium on "The secondary mathematics situation." Professor C. A. Hutchinson of the University of Colorado acted as chairman for the symposium. The leaders were Professor C. H. Sisam, Colorado College; Professor A. E. Mallory, Colorado State College of Education, by invitation; and Mr. H. W. Charlesworth, East High School, Denver, by invitation.

A. J. LEWIS, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section was held at the Texas Technological College, Lubbock, Texas, on April 20, 1935.

Among the forty persons attending the meeting were the following twenty-two members of the Association: J. H. Binney, E. O. Box, H. E. Bray, J. E. Burnam, Nat Edmonson, E. L. Harp, Jr., J. A. Hurry, Roy MacKay, Lida B. May, J. N. Michie, E. D. Mouzon, Jr., C. A. Murray, W. L. Porter, P. K. Rees, C. R. Sherer, F. W. Sparks, Ruth W. Stokes, Jennie L. Tate, E. L. Thompson, F. E. Ulrich, R. S. Underwood, C. N. Wunder.

At noon those attending the meeting were, jointly with the Texas Section of the Society for the Promotion of Engineering Education, the guests of the Lubbock Chamber of Commerce at an old-fashioned barbecue. The Texas Technological College entertained those attending the meeting at a dinner in the evening following the meeting. President Bradford Knapp of the College was the principal speaker.

At the business session following the presentation of papers the following officers were elected for the coming year: Chairman, F. W. Sparks, Texas Technological College; Vice-Chairman, L. R. Ford, Rice Institute; and it was voted to make the office of Secretary-Treasurer a permanent one, with Nat Edmonson, Agricultural and Mechanical College of Texas, continuing in the office. It was also voted to hold the 1936 meeting of the Section at the A. and M. College, College Station, in conjunction with the meeting of the Texas Section of the S.P.E.E.

The following papers were read:

1. "Non-unique solutions of ordinary differential equations" by O. H. Hamilton, San Antonio Junior College, introduced by Professor Sparks.

2. "An expansion for an n-dimensional determinant" by Professor F. W. Sparks, Texas Technological College.

3. "Roots of the derivative of a polynomial" by Professor H. E. Bray, Rice Institute.

4. "Generalized Vandermonde determinants," second paper, by E. R. Heineman, Texas Technological College, introduced by Professor Porter.

5. "Symbolic cubic forms in six variables" by Professor Ruth W. Stokes, North Texas State Teachers College.

6. "Rational fractions" by W. L. Scott, Rice Institute, introduced by Professor Bray.

7. "A grossly neglected convergence test" by Professor R. S. Underwood, Texas Technological College.

8. "On certain meromorphic functions" by Dr. F. E. Ulrich, Rice Institute.

9. "Transforms of Fuchsian groups" by Dr. P. K. Rees, Texas Technological College.

10. "A new proof of Minkowski's theorem on the product of two linear forms" by R. Q. Seale, Southern Methodist University, introduced by Professor Mouzon.

11. "On the generation of conics by certain non-projective pencils of lines" by Professor Hillel Halperin, A. and M. College of Texas.

12. "On curves of bounded turning" by J. T. Hurt, Rice Institute, introduced by Professor Bray.

13. "Developing mathematical power in freshman classes" by Professor E. O. Box, East Texas State Teachers College.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Hamilton discussed the properties of solutions of the first order differential equation, $y'=f(x, y)$, as given by Professor Ettlinger. (See this MONTHLY, vol. 42 (1935), p. 411.) New results obtained by Mr. Hamilton are as follows: The collection of solutions is shown to be continuous with respect to the initial conditions as given by the coordinates of the point P. Every solution is shown to be the limit function of a sequence of solutions converging uniformly to it. Under an additional hypothesis, limit of $f(x, y)$ equal to a constant k (depending on P) as x approaches the abscissa of P along any absolutely continuous curve passing through P, the following results are demonstrated. The solutions through a fixed point p are shown to be continuous monotone increasing functions of $y'=f(x, y)$ for a fixed x. For a fixed point P it is shown that not more than a countable number of values of c exist such that there are two solutions of $f(x, y)+c$ through P. A consequence of this last result is that $y'=f(x, y)$ may be approximated as close as one pleases by a differential equation which has a unique solution through P.

2. An n dimensional determinant $[a_{i_1 i_2 \cdots i_n}]$, $i_1, \cdots, i_n=1, 2 \cdots n$, has n expansions depending on the manner in which the algebraic signs of the individual terms are determined. Professor Sparks exhibited a method by means of which we may pass from one determinant to another by means of a substitution operator.

3. In this paper Professor Bray gave an outline of the proof of the following theorem: If $P(z)$ is a polynomial of degree $n\geqq 2$ whose roots are $z_j=r_je^{i\theta_j}$ $(j=1, 2, \cdots, n)$ and if $r_n>r_j(j=1,2, \cdots, n-1)$, and if ρ_{n-1} represents the greatest of the moduli of the roots of the derivative $P'(z)$, then ρ_{n-1} is not greater than the greatest root of the derivative of $Q(z)$, the real polynomial whose roots are the numbers r_j; except when all the angles θ_j are equal (mod 2π). This theorem has an immediate application in enabling one to cut off the corners of the Gauss-polygon in the problem of locating the roots of the derivative.

4. Mr. Heineman showed that every alternant whose principal diagonal is $|a_1^{p_1}a_2^{p_2} \cdots a_n^{p_n}|$, where the p's are any rational numbers, positive or negative, can be expressed as a rational function of special determinants $V_{n,i}$. The alternant $V_{n,i}$ has indices which are obtained from the matrix $\|0, 1/z, 2/z, \cdots, n/z\|$ by omitting the $(n-i+1)$th symbol. The letter z represents the least common denominator of the p's.

5. Dr. Stokes in this paper shows that every symbolic form of degree three and class six can be reduced to one of the following canonical forms: (1) $u_1u_2u_3+u_1u_4u_5+u_2u_4u_6$, (2) $u_1u_2u_3+u_4u_5u_6$. These types are non-equivalent. Form (2) has linear forms of rank three. Almost simultaneously with her first consideration of this problem, G. B. Gourewitch, independently, obtained the same results.

6. The problem of approximating to real irrational numbers by rational fractions which belong to one of the three types O/E, E/O, O/O, where E is an even integer and O is an odd integer was studied by Mr. Scott by the aid of the geometry of the complex plane. The following results were obtained. There are infinitely many fractions of a given type satisfying $|7\omega - p/q| < k/q^2$ (ω irrational), when $k \geqq 1$. There exist irrational numbers for which only a finite number of fractions of a given type will satisfy this inequality when $k < 3\cdot 10^{-1/2}$ $(=0.9486)$. For $3\cdot 10^{-1/2} \leqq k < 1$, the facts are still in doubt.

7. Professor Underwood calls attention to a highly useful test for the convergence of series which, though a part of the classical theory, has been almost universally overlooked by textbook writers. The test, applied to the series of positive terms $\sum_{n=1}^{\infty} u_n$, is as follows: If, for $n > k$, $nu_n > \epsilon > 0$, the series diverges; if $n^p u$ remains finite for some $p > 1$, the series converges. As an example of the easily applicable nature of the test it was found that, even in a less general form than that stated above, it sufficed to solve 35 of the 48 problems in an exercise following 26 separate theorems in a modern textbook devoted entirely to infinite series.

8. Dr. Ulrich discussed the general problem of the distribution of the points at which a meromorphic function assumes a non-exceptional value. A functional equation satisfied by meromorphic functions of finite, non-zero order with exactly two exceptional values was obtained. He then indicated how this relation might be used to study the distribution of the points in question for that class of meromorphic functions.

9. The Fuchsian group of transformations $S(z) = GTG^{-1}(z)$, G fixed, T variable, was considered by Dr. Rees, The necessary and sufficient condition that the radius r_s, of the isometric circle of the transformation S be a real constant $1/k$ times that of T was found algebraically and a simple geometric significance was shown for $k=1$. The geometric conditions that $r_s < r_t$ and that $r_s > r_t$ were also shown. Furthermore a condition under which r_s is infinite was shown. These conditions were all expressed in terms of the position of the center of the line segment connecting the centers of the isometric circles of T and its inverse T^{-1}.

10. By means of the elementary properties of convergents Mr. Seale proved that

if $\alpha\delta-\beta\gamma=1$, integers x, y always exist such that $|(\alpha x+\beta y-\xi_0)(\gamma x+\delta y-\eta_0)| \leqq 1/4$, where α, β, γ, δ, ξ_0, η_0 are any real numbers. If α/β is irrational he showed also that $|\alpha x+\beta y-\xi_0|$ can be made arbitrarily small at the same time that Minkowski's theorem is satisfied.

11. The usual construction of a point conic by the use of a ruler only, by means of two projective pencils, has, among others, the great disadvantage that only parts of the auxiliary ranges, necessary in the construction, can be utilized and so only parts of the conic can be constructed. The construction, however, by means of two non-projective pencils with vertices at the foci (in the case of the parabola one of the pencils has its vertex at infinity) does not have those disadvantages. These constructions are based on properties of conics which are special cases of the general theorem (believed to be new) that if from any point P on the focal axis of a conic perpendiculars are drawn to the tangents to the conic, they will cut off on the focal radii constant segments F_1P_1 and F_2P_2. Moreover, the algebraic sum of F_1P_1 and F_2P_2 is a constant independent of P.

12. Let θ (mod 2π) be the direction angle of a curve, and let T be the total variation of θ. The turning of the curve is defined as the least such T for all possible determinations of θ. Mr. Hurt discussed the turning of the level curves of the Green's function as compared with the turning of the boundary curve.

13. Professor Box placed emphasis upon the fact that, whereas the aim is often to teach as many mathematical facts as possible and the results have been poor, we should change the aim to that of teaching sound mathematical logic together with an ability to use the facts learned. Mathematical power is defined to mean ability to think logically about mathematical situations. Means of developing, through drill, interest and critical examination, are listed.

NAT EDMONSON, *Secretary*

AN EULER SUMMATION FORMULA

By IRWIN ROMAN, Michigan College of Mining and Technology

Introduction

Several years ago, the writer had occasion to evaluate the sums of numerous infinite series which converged so slowly that direct calculation was not feasible. In order to facilitate these evaluations, he made approximations which led to formulas resembling those usually credited to Euler and Maclaurin. The latter appear in various places in the literature of series, interpolation and differences, but usually are embodied in the text in such a manner as to mask their simplicity and utility. Accordingly, it has seemed desirable to present a direct and elementary development of the formulas, accompanied by a few examples of their usefulness, thus making the results available to many readers who might otherwise miss them.

The derivation here given involves no training beyond that usually included

in a first course in calculus (involving, in particular, Taylor's Theorem), and could be presented to an undergraduate class. By contrast, the usual derivations are not only less direct, but imply a familiarity with the theory of functions, the method of differences or the theory of non-linear interpolation, all of which are frequently missing in the preparation of an undergraduate.

To discuss the convergence or the accuracy of the results would carry us out of the elementary field, but the results are reliable in all ordinary cases. The reader who wishes to pursue the question further may consult the references given, which, in turn, will list other references.

Development of Formulas

Let u_n be the nth term of a series whose sum is to be found. Let $f(x)$ be a continuous function of x, with continuous derivatives of all orders entering the calculations, and such that

$$f(n) = u_n \qquad (n = 0, 1, 2, \cdots, m) \tag{1}$$

where the series consists of the $(m+1)$ terms, $u_0, u_1, \cdots, u_m$. In particular, m may be finite or infinite. If we write

$$x = n + h, \tag{2}$$

Taylor's Theorem leads to the result

$$f(x) = \phi(h) = f(n) + \sum_{k=1}^{\infty} \frac{f^{(k)}(n)}{k!} h^k. \tag{3}$$

Integrating in x from $(n-h)$ to $(n+h)$, equation (3) becomes

$$\int_{n-h}^{n+h} f(x)dx = \int_{-h}^{h} \phi(h)dh = 2hf(n) + 2\sum_{k=1}^{\infty} \frac{f^{(2k)}(n)}{(2k+1)!} h^{2k+1}. \tag{4.1}$$

The terms involving odd order derivatives disappear because they involve even powers of h and have the same value at each limit.

This equation may also be written

$$f(n) = \frac{1}{2h}\int_{n-h}^{n+h} f(x)dx - \sum_{k=1}^{\infty} \frac{f^{(2k)}(n)}{(2k+1)!} h^{2k}. \tag{4.2}$$

Applying equation (4.2) to $f^{(2k)}(n)$, we obtain

$$f^{(2k)}(n) = \frac{1}{2h}\int_{n-h}^{n+h} f^{(2k)}(x)dx - \sum_{j=1}^{\infty} \frac{f^{(2j+2k)}(n)}{(2j+1)!} h^{2j}. \tag{4.3}$$

If we write

$$a_{2k} = -\frac{1}{(2k+1)!} \qquad (k = 1, 2, \cdots, \infty) \tag{5.1}$$

$$\theta_0 = \frac{1}{2h}\int_{n-h}^{n+h} f(x)dx \tag{5.21}$$

$$\theta_{2k} = \frac{1}{2h}\Big[f^{(2k-1)}(x)\Big]_{n-h}^{n+h} \qquad (k = 1, 2, \cdots, \infty) \tag{5.22}$$

it follows that equation (4.2) may be written

$$f(n) = \theta_0 + \sum_{k=1}^{\infty} a_{2k} f^{(2k)}(n) h^{2k} \tag{6.1}$$

and equation (4.3) may be written

$$f^{(2k)}(n) = \theta_{2k} + \sum_{j=1}^{\infty} a_{2j} f^{(2j+2k)}(n) h^{2j}. \tag{6.2}$$

Substituting equation (6.2) in equation (6.1) and repeating the process, we find

$$\begin{aligned} f(n) &= \theta_0 + \sum_{k=1}^{\infty} a_{2k}\theta_{2k} h^{2k} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{2j}a_{2k}\theta_{2j+2k} h^{2j+2k} \\ &\quad + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{2i}a_{2j}a_{2k}\theta_{2i+2j+2k} h^{2i+2j+2k} + \cdots . \end{aligned} \tag{7}$$

By extension, this may be written

$$f(n) = \theta_0 + \sum_{u=1}^{\infty} b_{2u}\theta_{2u} h^{2u} \tag{8}$$

where b_{2u} is the sum of all possible products of the a's, the sums of whose indices is $2u$, and each term having a coefficient which is the number of distinct permutations possible among the indices of that term. Thus

$$b_{2u} = \sum_{v=1}^{u} a_{2v} b_{2u-2v} = -\sum_{v=1}^{u} \frac{b_{2u-2v}}{(2v+1)!} \qquad (u > 0). \tag{9.1}$$

(9.2) For convenience, we define $b_0 = 1$.

Combining equations (5.1) and (9), the b's may be computed progressively. The first six values are

$$\begin{aligned} &b_0 = 1, && b_2 = -1/6, && b_4 = 7/360, \\ &b_6 = -31/15120, && b_8 = 127/604800, && b_{10} = -73/3421440. \end{aligned} \tag{10}$$

These numbers are closely associated with the Bernoulli numbers. Specifically, on the system $B_2 = 1/6$, $B_4 = 1/30$, $B_6 = 1/42$, etc., it can be shown that, for r a positive integer,

$$B_{2r} = (-1)^{r-1}\frac{(2r)!}{2^{2r}}\sum_{u=0}^{r} \frac{b_{2u}}{(2r-2u)!} \tag{11.1}$$

$$(11.2) \qquad b_{2r} = (-1)^r \frac{2^{2r}-2}{(2r)!} B_{2r}.$$

Combination of these two relations leads to the recursion relation

$$(11.3) \quad B_{2r} = \frac{(-1)^{r-1}}{2^{2r}-1}\left\{\frac{1}{2} + \sum_{u=1}^{r-1} (-1)^u(2^{2u-1}-1)\frac{(2r)!}{(2u)!(2r-2u)!} B_{2u}\right\}$$

which holds for $r>1$. Since the values of $B_{2r}/(2r)!$ can be shown to ultimately approach a geometric series of ratio $1/(4\pi^2)$, as r becomes infinite, and since the ratio of $(2^{2u}-2)$ to $(2^{2u-2}-2)$ approaches the limit 4, it follows that b_{2r} will ultimately approach a geometric series of ratio $-1/\pi^2$.

If we write

$$(12) \qquad Z(x) = \frac{1}{2h}\left[\int f(x)dx + \sum_{u=1}^{\infty} b_{2u}h^{2u}f^{(2u-1)}(x)\right],$$

equation (8) may be written

$$(13.1) \qquad f(n) = Z(x)\Big]_{n-h}^{n+h}.$$

If $n=k+2hi$, this leads, on combining overlapping terms, to

$$(13.2) \qquad \sum_{i=s}^{t} f(k+2hi) = Z(x)\Big]_{k+(2s-1)h}^{k+(2t+1)h}.$$

Combining equations (12) and (13.1) and rearranging terms, we obtain

$$(13.3) \qquad \int_{n-h}^{n+h} f(x)dx = 2hf(n) - \sum_{u=1}^{\infty} b_{2u}h^{2u}\left[f^{(2u-1)}(x)\right]_{n-h}^{n+h}.$$

If the interval from a to b is divided into $2m$ equal parts and equation (13.3) is applied to each part, we obtain

$$(13.41) \quad \int_a^b f(x)dx = 2h\sum_{k=1}^{m} f[a+(2k-1)h] - \sum_{u=1}^{\infty} b_{2u}h^{2u}\left[f^{(2u-1)}(x)\right]_a^b$$

where

$$(13.42) \qquad h = \frac{b-a}{2m}.$$

The interpretations of equations (13) are important, each having its own field of utility. Equation (13.1) expresses the value of a function $f(x)$ for the value $x=n$ in terms of the definite integral of $f(x)$ over a range $x=n-h$ to $x=n+h$ and of the odd order derivatives of the function for the limits of that same range. The value of h is arbitrary, beyond the simple restriction that the function and its derivatives must be continuous over the range. Equation (13.2)

expresses the sum of the values of $f(x)$ at equally spaced values of the argument in terms of the definite integral of $f(x)$ over a range extended half the spacing of the arguments at each end of the range of the terms, and of the odd order derivatives of the function at the ends of this extended range. This form is especially useful in summing series. Equation (13.3) expresses the value of a definite integral in terms of the value of the integrand at the midpoint of the interval and of the odd order derivatives of the integrand at the limits of integration. Equation (13.41) is a formula for approximate integration. The interval is subdivided into an even number of parts. Starting with the first internal point of division, the values of the function for alternate points of division are added and this sum is corrected by a term involving the odd order derivatives of the integrand at the limits of integration. Usually the accuracy increases with the number of subdivisions.

In particular, if $k=0$ and $h=1/2$, equation (13.2) becomes

$$\sum_{i=s}^{t} f(i) = Z_1(x)\Big]_{s-1/2}^{t+1/2} \tag{14.11}$$

where

$$Z_1(x) = \int f(x)dx + \sum_{u=1}^{\infty} \frac{b_{2u}}{2^{2u}} f^{(2u-1)}(x). \tag{14.12}$$

If $k=0$ and $h=1$, equation (13.2) becomes

$$\sum_{i=s}^{t} f(2i) = Z_2(x)\Big]_{2s-1}^{2t+1} \tag{14.21}$$

where

$$Z_2(x) = \frac{1}{2}\int f(x)dx + \frac{1}{2}\sum_{u=1}^{\infty} b_{2u} f^{(2u-1)}(x). \tag{14.22}$$

If $k=1$ and $h=1$, equation (13.2) becomes

$$\sum_{i=s}^{t} f(2i+1) = Z_2(x)\Big]_{2s}^{2t+2} \tag{14.3}$$

where $Z_2(x)$ is defined by equation (14.22).

Equations (14) frequently enable us to find the sum of a finite series as the difference of two values of an auxiliary function, or to replace an infinite series by a more rapidly convergent infinite series, if the necessary integral and derivatives do not increase too rapidly with the order. The relations are sometimes useful in evaluating divergent series in which the error made in stopping at a specific term can be shown to lie within the permissible tolerance.

The geometrical significance of the method is simple. The area under a curve and that under the corresponding step polygon are nearly equal. Either area may be expressed as approximately equal to the other, if the step polygon

has altitudes which are the values of the function for the midpoints of the bases of the polygon, and if the function is sufficiently smooth. The difference between the two areas can then be treated in the same way, successively. If we start with the function, we have merely to divide the range into equal sections and build the series whose terms are the values of the function at the midpoint of each range. If we start with the series, we replace the series by a function such that the functional value reduces to the terms for integral values of the variable. The selection of the series corresponding to the function is unique. The selection of the function corresponding to the series is not unique, but for practical purposes, there is usually little choice. The selected function is the polynomial of lowest degree possible, in which case the derivatives ultimately vanish and the series is finite. If a polynomial cannot be selected, the law of the terms will usually furnish the simplest function to select. If the derivatives of the selected function do not increase too rapidly with order, the new series will converge with fair rapidity, as may be seen by inspecting the values of b_{2u} in equations (10).

In applying the formulas to specific cases, the constant terms may be dropped from the auxiliary function Z, Z_1 or Z_2, since the difference of two values is always to be taken. To illustrate the formulas, several examples will be given. The first few are included because of their simplicity. The later ones will introduce results not ordinarily available.

Examples

EXAMPLE 1. *Consecutive Integers.* As a simple example, let us find the sum of n consecutive integers. For this case, we may select $f(x)=x$. Then $\int f(x)dx=x^2/2$, $f'(x)=1$ and all higher derivatives vanish; so that, by equation (14.12), dropping the constant term,

$$Z_1(x) = \tfrac{1}{2}x^2.$$

By equation (14.11), the sum of the consecutive integers from s to t is

$$\sum_{i=s}^{t} i = \tfrac{1}{2}x^2\Big]_{s-1/2}^{t+1/2} = \tfrac{1}{2}\left[(t+\tfrac{1}{2})^2 - (s-\tfrac{1}{2})^2\right] = \tfrac{1}{2}(t+s)(t-s+1).$$

In particular, the sum of the first n integers is

$$\sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1).$$

EXAMPLE 2. *Consecutive Odd Integers.* To find the sum of consecutive odd integers, we may use equation (14.1) or equation (14.3). In the former case, we may select $f(x)=2x+1$ so that $Z_1(x)=x^2+x$ and

$$\sum_{i=s}^{t} (2i+1) = \left[x^2+x\right]_{s-1/2}^{t+1/2} = (t+s+1)(t-s+1).$$

In the latter case, $f(x)=x$ and $Z_2(x)=x^2/4$ so that

$$\sum_{i=s}^{t} (2i + 1) = \left[\tfrac{1}{4}x^2\right]_{2s}^{2t+2} = (t + s + 1)(t - s + 1).$$

In particular, the sum of the first n odd integers is

$$\sum_{i=0}^{n-1} (2i + 1) = n^2.$$

EXAMPLE 3. *Arithmetic Series.* Consider the arithmetic series whose first term is a, common difference d and number of terms n. For this case, we may select $f(x) = a + dx$ and use equation (14.1) so that we need $Z_1(x) = ax + dx^2/2$ and it follows that

$$\sum_{i=0}^{n-1} (a + di) = \left[ax + \tfrac{1}{2}dx^2\right]_{-1/2}^{n-1/2} = an + \tfrac{1}{2}dn(n - 1) = \tfrac{1}{2}n[2a + (n - 1)d].$$

EXAMPLE 4. *Consecutive Squares.* To find the sum of the first n squares, we may take $f(x) = x^2$ and use equation (14.1) so that

$$Z_1(x) = \frac{1}{3}x^3 - \frac{1}{24}(2x) = \frac{1}{3}x^3 - \frac{1}{12}x$$

and

$$\sum_{i=1}^{n} i^2 = \left[\frac{1}{3}i^3 - \frac{1}{12}i\right]_{1/2}^{n+1/2} = \frac{1}{6}n(n + 1)(2n + 1).$$

EXAMPLE 5. *Powers of Consecutive Integers.* To find the sum of the pth powers of the first n natural numbers, we may apply equation (14.1). In this case, for $p > 0$.

$$f(x) = x^p, \qquad \int f(x)dx = \frac{1}{p + 1}x^{p+1},$$

$$f^{(k)}(x) = p(p - 1)(p - 2)(\cdots)(p - k + 1)x^{p-k} = \frac{p!}{(p - k)!}x^{p-k}.$$

Accordingly $f^{(k)}(x)$ is constant if $k \geqq p$. Hence in $Z_1(x)$ we may make the restriction $2u - 1 \leqq p - 1$. If p is even, $u \leqq p/2$, while if p is odd $u \leqq (p - 1)/2$. Hence we may restrict the range of u from i to w where

$$w = p/2 \text{ if } p \text{ is even}$$

$$w = (p - 1)/2 \text{ if } p \text{ is odd}$$

and it follows that

$$Z_1(x) = \frac{x^{p+1}}{p + 1} + \sum_{u=1}^{w} \frac{b_{2u}}{2^{2u}} \frac{p!}{(p - 2u + 1)!} x^{p-2u+1}.$$

For $u=0$, the indicated summand reduces to the isolated term, so that

$$Z_1(x) = p!\sum_{u=0}^{w} \frac{b_{2u}}{2^{2u}(p-2u+1)!} x^{p-2u+1}$$

and

$$\sum_{x=1}^{n} x^p = p!\sum_{u=0}^{w} \frac{b_{2u}}{2^{2u}(p-2u+1)!}\left[(n+\tfrac{1}{2})^{p-2u+1} - (\tfrac{1}{2})^{p-2u+1}\right].$$

For numerical calculations, this form is sufficient, but the result may also be expressed as a polynomial in n, if desirable. The summation is over a finite number of terms depending on the value of p, if p is a positive integer. If p is not a positive integer, the series is infinite. By combining the present case results, we may find the sum of any series whose general term is a polynomial in the counter. The value of n appears only in the first term within the bracket.

EXAMPLE 6. *Multiple Sines and Cosines.* If $f(x) = \sin ax$, we have

$$\int f(x)dx = -a^{-1}\cos ax$$

$$f^{(2u-1)}(x) = (-1)^{u-1}a^{2u-1}\cos ax$$

so that

$$Z_1(x) = -a^{-1}\cos ax - \sum_{u=1}^{\infty}(-1)^u\frac{b_{2u}}{2^{2u}}a^{2u-1}\cos ax$$

$$= -\frac{\cos ax}{a}\sum_{u=0}^{\infty}(-1)^u(a/2)^{2u}b_{2u}.$$

Thus

$$\sum_{n=s}^{t}\sin na = \frac{1}{a}\left\{\sum_{u=0}^{\infty}(-1)^u(a/2)^{2u}b_{2u}\right\}\left\{\cos(s-\tfrac{1}{2})a - \cos(t+\tfrac{1}{2})a\right\}$$

$$= \frac{2}{a}\left\{\sum_{u=0}^{\infty}(-1)^u(a/2)^{2u}b_{2u}\right\}\left\{\sin(s+t)(a/2)\sin(t-s+1)(a/2)\right\}.$$

In particular, for $s=t=n$, this becomes

$$\sin na = \frac{2}{a}\left\{\sum_{u=0}^{\infty}(-1)^u(a/2)^{2u}b_{2u}\right\}\left\{\sin na \sin(a/2)\right\}.$$

Since this holds for all values of n, we have two results of interest:

$$\csc(a/2) = \frac{2}{a}\sum_{u=0}^{\infty}(-1)^u(a/2)^{2u}b_{2u}$$

$$\sum_{n=s}^{t}\sin na = \csc(a/2)\sin(s+t)(a/2)\sin(t-s+1)(a/2).$$

If we replace a by $2x$, the former becomes

$$\csc x = \frac{1}{x} \sum_{u=0}^{\infty} (-1)^u b_{2u} x^{2u}.$$

In particular,

$$\csc 1 = \sum_{u=0}^{\infty} (-1)^u b_{2u} = \sum_{u=0}^{\infty} | b_{2u} | = 1.188396.$$

Similarly, it may be shown that

$$\sum_{n=s}^{t} \cos na = \csc (a/2) \cos (s+t)(a/2) \sin (t-s+1)(a/2).$$

EXAMPLE 7. *Exponentials.* If $f(x) = e^x$, it follows from equation (12) that

$$Z(x) = \frac{e^x}{2h}\left[1 + \sum_{u=1}^{\infty} b_{2u} h^{2u}\right] = \frac{e^x}{2h} \sum_{u=0}^{\infty} b_{2u} h^{2u}$$

and by equation (13.1), that

$$e^n = \frac{e^{n+h} - e^{n-h}}{2h} \sum_{u=0}^{\infty} b_{2u} h^{2u} = \frac{e^n}{h} \sinh h \sum_{u=0}^{\infty} b_{2u} h^{2u}.$$

Hence

$$\sum_{u=0}^{\infty} b_{2u} x^{2u} = x \operatorname{csch} x$$

and

$$\sum_{u=0}^{\infty} b_{2u} = \operatorname{csch} 1 = 0.850919.$$

By studying $f(x) = xe^{ax}$, it may be shown that

$$\sum_{u=0}^{\infty} u b_{2u} x^u = \tfrac{1}{2} x(1 - x \coth x) \operatorname{csch} x$$

whence

$$\sum_{u=0}^{\infty} u b_{2u} = \frac{1 - \coth 1}{2 \sinh 1} = -0.1332.$$

EXAMPLE 8. *Probability Integral.* Let $f(x) = e^{-x^2}$. Then each derivative is of the form $f^{(u)}(x) = -e^{-x^2} P_u$ where P_u is a polynomial in x. Differentiation shows that

$$f^{(u+1)}(x) = -e^{-x^2}\left[\frac{dP_u}{dx} - 2xP_u\right]$$

$$f^{(u+2)}(x) = -e^{-x^2}\left[\frac{d^2P_u}{dx^2} - 4x\frac{dP_u}{dx} + (4x^2-2)P_u\right].$$

Hence

$$P_{u+2} = \frac{d^2P_u}{dx^2} - 4x\frac{dP_u}{dx} + (4x^2-2)P_u.$$

In particular,

$$P_1 = 2x \qquad \frac{dP_1}{dx} = 2 \qquad \frac{d^2P_1}{dx^2} = 0.$$

$$P_3 = 8x^3 - 12x \qquad \frac{dP_3}{dx} = 24x^2 - 12 \qquad \frac{d^2P_3}{dx^2} = 48x$$

$$P_5 = 32x^5 - 160x^3 + 120x$$

$$\frac{dP_5}{dx} = 160x^4 - 480x^2 + 120 \qquad \frac{d^2P_5}{dx^2} = 640x^3 - 960x$$

$$P_7 = 128x^7 - 1344x^5 + 3360x^3 - 1680x.$$

Setting $n=h=w/2$ in equation (13.3) we obtain

$$\int_0^w e^{-x^2}dx = we^{-w^2/4} + \sum_{u=1}^{\infty} b_{2u}(w/2)^{2u}\left[e^{-x^2}P_{2u-1}\right]_0^w.$$

But $P_{2u-1}=0$ for $x=0$, so that

$$\int_0^w e^{-x^2}dx = we^{-w^2/4} + e^{-w^2}\sum_{u=1}^{\infty} b_{2u}(w/2)^{2u}P_{2u-1}(w),$$

where $P_{2u-1}(w)$ is calculated for $x=w$. Explicitly,

$$\int_0^w e^{-x^2}dx = we^{-w^2/4} - w^3e^{-w^2}\left\{\frac{1}{12} + \frac{7}{480}w^2 - \frac{79}{13440}w^4 - \frac{7253}{1935360}w^6 + \cdots\right\}.$$

For small values of w, we have the first approximation

$$\int_0^w e^{-x^2}dx = we^{-w^2/4},$$

and the second approximation

$$\int_0^w e^{-x^2}dx = we^{-w^2/4} - \frac{1}{12}w^3e^{-w^2}.$$

A table of exponentials thus furnishes the value of the integral with conveniently rapid convergence. It is possible to get a better approximation by using equation (13.4) but there will be an isolated term for each interval so that there is no gain unless great accuracy is desired. The probability integral is the present integral multiplied by the factor $2/\sqrt{\pi}$.

EXAMPLE 9. *Harmonic Series.* If $f(x)=1/x$, then $\int f(x)dx=\log x$ and $f^{(k)}(x)=(-1)^k k!/x^{k+1}$ so that

$$Z_1 = \log x - \sum_{u=1}^{\infty} \frac{(2u-1)!b_{2u}}{(2x)^{2u}}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i} &= \log \frac{n+\frac{1}{2}}{\frac{1}{2}} - \sum_{u=1}^{\infty} (2u-1)!b_{2u}[(2n+1)^{-2u} - 1] \\ &= \log n + \log\left(2+\frac{1}{n}\right) + \sum_{u=1}^{\infty} (2u-1)!b_{2u}[1-(2n+1)^{-2u}].\end{aligned}$$

Hence

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{i} - \log n\right] = \log 2 + \sum_{u=1}^{\infty}(2u-1)!b_{2u}.$$

A more rapidly convergent series may be obtained by isolating r terms. Then

$$\begin{aligned}\sum_{i=r+1}^{n}\frac{1}{i} &= \log n + \log(2+1/n) - \log(2r+1) \\ &\quad - \sum_{u=1}^{\infty}(2u-1)!b_{2u}[(2n+1)^{-2u} - (2r+1)^{-2u}].\end{aligned}$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{i} - \log n\right] &= \sum_{i=1}^{r}\frac{1}{i} + \log 2 - \log(2r+1) \\ &\quad + \sum_{u=1}^{\infty}(2u-1)!(2r+1)^{-2u}b_{2u}.\end{aligned}$$

This is the "Eulerian Constant." If r is selected as 100, the value of $\sum_{i=1}^{r}(1/i)$ is 5.18737752 while log 2 is 0.69314718 and log 201 is 5.30330492. In the last summation, the term for $u=2$ has a value of about 10^{-9} so that we may replace the sum by $b_2/(201)^2$ which has the value 0.000004124. Hence the Eulerian Constant has the value 0.57721566, with an error not exceeding one unit in the last digit.

Comments and References

The formulas and examples given in this paper are not intended to constitute an exhaustive compilation but are offered to show the convenience of the Euler-

Maclaurin relations, either in the forms presented or in the customary forms. Few, if any, of the results are new. Similarly, the bibliography is meager but will serve as a guide to the reader interested in following the subject further. No attempt has been made to establish priority of the material cited. It is believed that at least some of the references will be available to each reader, even in cases where library facilities are limited.

In equations (11), the proofs have been omitted. The proof of equation (11.1) may be made by transforming the sum in example 5, by means of the binomial theorem, into a polynomial in the number of terms and comparing the result with the customary expansions. Equation (11.2) follows from a comparison of the expansion of csc x in example 6 with the customary expansions. While the writer has not found equation (11.3) in the literature, he doubts its novelty. The usual relations involve the Euler relations.

The references will assist in extending the discussion given in this paper. Closely associated results may be found under a variety of titles, as examination of the references will show. The first three examples are included in most algebra texts, while the fourth appears in many advanced algebra texts and in most elementary texts on the method of differences or summation of series.

From the many possible references, the following have been selected, on the basis of availability rather than merit:

A. Adams, Edwin P., *Smithsonian Mathematical Formulae*, Smithsonian Publication 2672, Washington, 1922.

Bo. Boole, George, *Calculus of Finite Differences*, Reprint of the third edition, Stechert, New York, 1931.

Br. Bromwich, T. J. I., *Infinite Series*, Macmillan, London, Second Edition, 1931.

C. Carr, G. S., *Synopsis of Pure Mathematics*, Hodgson, London, 1886.

F. Freeman, Harry, *Actuarial Mathematics*, Cambridge Press, Cambridge, 1931.

H. Hall, H. S., and Knight, S. R., *Higher Algebra*, Macmillan, London, Fourth Edition, 1920.

J. Jolley, L. B. W., *Summation of Series*, London, Chapman and Hall, 1925.

P. Peirce, B. O., *A Short Table of Integrals*, Boston, Ginn and Co., 1899.

Sca. Scarborough, James B., *Numerical Mathematical Analysis*, Baltimore, Johns Hopkins Press, 1930.

Sch. Schwatt, I. J., *Operations with Series*, Philadelphia, Univ. of Pennsylvania Press, 1924.

St. Steffensen, J. F., *Interpolation*, Baltimore, Williams and Wilkins, 1927.

W. Whittaker, E. T., and Watson, G. N., *Modern Analysis*, Cambridge Univ. Press, Second Edition, 1915.

M. American Mathematical Monthly.

U. U. S. Bureau of Mines, Technical Paper 502, *How to Compute Tables for Determining Electrical Resistivity of Underlying Beds and their Application to Geophysical Problems*, Washington, 1931.

While a detailed compilation is beyond the scope of this paper, the following will serve to show the importance of the topics discussed:

1. For the definitions, numerical values and recursion formulas of the Bernoulli and associated numbers, see the above references as follows:

A. arts. 6.902, 6.905; Bo. pp. 90–91 and ch. VI; C. arts. 1539–40; H. par. 406; J. series 162, 163, 616; P. p. 90; Sch. ch. XV; St. par. 134; W. ch. VII; M. vol. 35 (1928), pp. 424–8 (Underwood).

2. For examples of expansions of functions involving the Euler-Maclaurin formulas, the Bernoulli or associated numbers and for examples of series whose sums have been evaluated by the use of these, see:

A. pars. 1.874–6, 3.61, 6.400, 6.43, 6.470, 6.602, 7.18; Bo. pp. 91–2, and ch. VI; Br. pp. 34, 324–5; C. arts. 276, 800–1, 1543–6, 2744, 2773, 2916–8, 2939–42, 3821–2; H. pars. 68, 69, 405; J. series 15–22, 59, 71a, 99, 100, 104, 164–6, 184, 200, 201, 204, 205, 377, 378, 416, 418, 420, 424, 433, 442, 465, 470–3, 494, 498–500; P. formulas 764, 774–7, 782–4, 792–5; Sch. pp. 86–8 and ch. XV; St. par. 101; W. ch. VII; M. vol. 35 (1928), pp. 77–80 (Ginsburg); M. vol. 35 (1928), pp. 424–8 (Underwood); U. pp. 11–14.

3. For examples of the evaluation of definite integrals by methods similar to those of this paper, see:

A. art. 6.700; Bo. p. 47 ff; C. arts. 2263–6, 2395–7, 2991–7, 3772–80; F. ch. XVII; J. series 530a; Sca. arts. 40–4.

4. For the customary Euler-Maclaurin formula, for other transformations intended to reduce the calculations involved in evaluating infinite series, and for other formulas resembling those in this paper, see:

A. art. 1.86; Bo. p. 90; Br. p. 328; C. art. 3820; F. p. 300; J. series 615; Sca. art. 43; St. pars. 141–154; W. ch. VII; M. vol. 40 (1933), pp. 226–9 (Shohat); M. vol. 40 (1933), pp. 537–42 (Camp); M. vol. 41 (1934), pp. 563–5 (Ward).

WARING'S PROBLEM[1]

By P. M. BATCHELDER, University of Texas

1. *Introduction.* Waring's problem in its general sense is the problem of representing positive integers as sums of like powers of other positive integers. The theorems with regard to the representation of integers by sums of squares are classical. A necessary and sufficient condition that a positive integer can be expressed as the sum of two squares (positive or zero) is that each of its prime factors of the form $4m+3$ occurs an even number of times; a positive integer can be expressed as the sum of three squares if and only if it is not of the form $4^n(8m+7)$; *every* positive integer can be expressed as a sum of four squares. These results were known to Fermat about 1640, and proofs were published by Euler, Lagrange, and Legendre in the eighteenth century.

2. *Waring's Conjecture.* With regard to the representation of integers by sums of cubes and higher powers our knowledge is still far from complete. In 1770 Edward Waring, Lucasian Professor of Mathematics at Cambridge, a good though not great mathematician, asserted in his *Meditationes Algebraicae* that every positive integer is the sum of at most 4 squares, 9 cubes, 19 fourth powers, and so on. He made no attempt to prove his statement, which is probably merely a plausible conjecture based on a little computation. His assertion is usually interpreted as meaning that in general for every power k there exists a finite integer N_k such that every positive integer is the sum of at most N_k kth

---

[1] Read before the Texas Section of the Association, May 5, 1934.

powers. The name "Waring's problem" has often been used in a restricted sense to denote the problem of proving the existence of N_k for every power; in this sense it was completely solved by Hilbert in 1909. He demonstrated by an ingenious process the existence of N_k for every power, but his proof does not serve to determine its numerical value. We shall understand by N_k the *least* integer which has the property stated; since zero terms are allowed, it is obvious that any larger integer has the same property.

3. *Cubes*. Apart from squares, for which $N_2=4$ by a theorem quoted in §1, the only value of k for which N_k has been definitely determined is $k=3$. Maillet proved in 1895 that every positive integer is the sum of 21 cubes, and that every integer exceeding 115,872 is the sum of 17 cubes. Since integers not exceeding 115,872 can be readily shown to require fewer than 17 cubes (see §8), it follows that $N_3 \leqq 17$. Fleck reduced the value to 13 in 1906, and finally in 1909 Wieferich proved that every positive integer is the sum of 9 cubes. (A gap in his proof was filled by A. J. Kempner in 1912.) Two integers are known which actually require 9 cubes, namely

$$23 = 2\cdot 2^3 + 7\cdot 1^3,$$

$$239 = 2\cdot 4^3 + 4\cdot 3^3 + 3\cdot 1^3 = 5^3 + 3\cdot 3^3 + 4\cdot 2^3 + 1^3;$$

hence $N_3=9$.

4. *Fourth Powers*. The first verification of Waring's conjecture for fourth powers was due to Liouville, who proved shortly before 1859 that $N_4 \leqq 53$. This value was reduced to 47 by Réalis in 1878, to 41 by Lucas in the same year, to 39 by Fleck in 1906, to 38 by Landau in 1907, to 37 by Wieferich in 1909, and to 35 recently by Mrs. Emily C. Pixley in her doctor's dissertation at the University of Chicago. No integer is known which requires more than 19 fourth powers to represent it; 79 requires that many: $79=4\cdot 2^4+15\cdot 1^4$, and so do 159, 239, 319, 399, 479, and 559. Tables giving minimum decompositions of integers into sums of fourth powers have been computed to 28,561; these tables and deductions based on them show that every positive integer $<10^{26}$ is the sum of 19 fourth powers. Hence it is a very plausible guess that the true value of N_4 is 19, as Waring said. Thus there remains ample opportunity for future mathematicians to improve on Mrs. Pixley's value.

5. *Higher Powers*. For higher powers the following results have been proved to date. Every positive integer is the sum of 54 fifth powers, of 160 sixth powers, of 258 seventh powers, of 575 eighth powers, of 981 ninth powers, of 2421 tenth powers, and of 10,711 twelfth powers.

6. *Euler's Conjecture*. About 1772 J. A. Euler, the son of Leonhard Euler, pointed out that N_k cannot be less than the number

$$I = q + 2^k - 2,$$

where q is the greatest integer $<(3/2)^k$, and in 1933 L. E. Dickson published results which tend to confirm the suggestion that I is the true value of N_k. This gives $N_3=9$, $N_4=19$, $N_5=37$, $N_6=73$, $N_7=143$, $N_8=279$, $N_9=558$, $N_{10}=1079$,

$N_{11}=2132$, $N_{12}=4223$, etc. Euler's statement is easily proved. Consider the number $q\cdot 2^k-1$; this is less than 3^k, since q is the integral part of the quotient $3^k/2^k$, hence in expressing it as a sum of kth powers only powers of 1 and 2 are available. Only $q-1$ terms 2^k can be used, so 2^k-1 terms 1^k are required to complete the representation.

7. *Representation of Large Integers.* The tables which have been computed for various values of k indicate that the integers which require the greatest numbers of terms are certain ones near the beginning which have to be built up from powers of 1, 2, 3, etc., like the number $q\cdot 2^k-1$ above, and that as we go on to larger integers the number of terms required tends to decrease. Von Sterneck published a table in 1903 showing the number of cubes required for all integers up to 40,000; in this no integer >239 requires 9 cubes, none >454 requires 8, and none >8042 requires 7. This fact suggests the possibility of finding numbers $G_k<N_k$ such that all sufficiently large integers can be expressed as sums of G_k kth powers. Results of this sort are called asymptotic theorems. In what follows G_k denotes the *least* integer which will serve.

Since no integer of the form $4^n(8m+7)$ can be represented by a sum of three squares (§1), it follows that $G_2=N_2=4$. In 1909 Landau proved that all sufficiently large integers are sums of 8 cubes, using what is called analytic number theory, in which the methods and results of the theory of functions of a complex variable are employed. In an important series of papers from 1919 to 1927 Hardy and Littlewood applied analytic methods to various problems in number theory. Among their results is the theorem that for every k

$$G_k \leqq (k-2)2^{k-1}+5,$$

so that all sufficiently large integers are sums of 9 cubes, 21 fourth powers, 53 fifth powers, etc. They also obtained for $k\geqq 4$ the sharper inequality

$$G_k \leqq (k-2)2^{k-2}+k+5+\zeta_k,$$

where ζ_k is the greatest integer contained in

$$\frac{(k-2)\log 2-\log k+\log (k-2)}{\log k-\log (k-1)}.$$

This gives 19 fourth powers, 41 fifth powers, 87 sixth powers, 193 seventh powers, 425 eighth powers, 949 ninth powers, 2113 tenth powers, etc. R. D. James has recently (1933) reduced these limits for odd values of k, proving that $G_5\leqq 35$ and that for $k\geqq 7$ and odd

$$G_k \leqq (k-3)2^{k-2}+k+9+\zeta_k;$$

this gives 164 seventh powers and 824 ninth powers.

Extensive calculations have been made by A. E. Western to determine empirically the value of G_3. His work makes the following conclusions probable: only a finite number of integers require five or more cubes to represent them,

so $G_3=4$; the largest one which requires six is less than 2,000,000, and the largest one which requires five is between 10^{12} and 10^{14}.

It is easy to prove that $G_3 \geqq 4$. Every integer has one of the forms $3m$, $3m\pm 1$, and by cubing these we see that every cube has one of the forms $9m$, $9m\pm 1$. Hence the sum of three cubes cannot have either of the forms $9m+4$, $9m+5$, so numbers of these forms require at least four cubes.

The square of an odd number has the form $8m+1$, since

$$(2k+1)^2 = 4k(k+1)+1$$

and either k or $k+1$ is divisible by 2; hence its fourth power has the form $16m+1$. The fourth power of an even number has the form $16m$. It follows that every number of the form $16m+15$ requires at least 15 fourth powers to represent it, so $G_4 \geqq 15$. Moreover, it is readily proved that every integer of the form $31\cdot 16^m$ requires 16 fourth powers. Suppose it were possible to express it as the sum of 15 or fewer:

$$31\cdot 16^m = x_1^4 + x_2^4 + \cdots + x_{15}^4 \qquad (m \geqq 1).$$

Since the right-hand side is divisible by 16, each x_i must be even, so that

$$31\cdot 16^{m-1} = \left(\frac{x_1}{2}\right)^4 + \left(\frac{x_2}{2}\right)^4 + \cdots + \left(\frac{x_{15}}{2}\right)^4.$$

By repetition of this process we could express 31 itself as the sum of 15 fourth powers; but 16 are required: $31=2^4+15\cdot 1^4$. Hence $G_4 \geqq 16$.

Kempner proved in 1912 that $G_6 \geqq 9$, and that for $k=2^q$ $(q>1)$ $G_k \geqq 4k$; this gives $G_8 \geqq 32$. The only general result of this type is that for every k $G_k \geqq k+1$, which was proved by Maillet and Hurwitz in 1908.

The number G_k appears to have greater fundamental significance than N_k. The fact that 23 and 239 require 9 cubes to represent them, while all others (presumably) can be expressed by fewer than 9, is an interesting idiosyncrasy of those particular integers, but not a truth of any general importance; it is merely "an entertaining arithmetical fluke," as Hardy expresses it. On the other hand, if it turns out that $G_3=4$, a profound property of integers in general will be established.

8. *Dickson's Method.* In stating that the asymptotic theorems hold for "sufficiently large" integers, we mean that there exists in each case a constant C such that they are true for all integers $n>C$. This constant is extremely large for most of the asymptotic theorems so far proved. Thus for many of the results of Hardy and Littlewood C is a number of billions or trillions of digits. In spite of the enormous size of these constants, Dickson has recently devised methods by which, starting from a table which shows that all the integers between two given limits are sums of h kth powers, it can be proved that all the integers in a larger interval are sums of $h+1$ kth powers, all those in a still larger interval are sums of $h+2$, and so on until the C is reached beyond which one of the

asymptotic theorems applies. He has been able in this way to find smaller limits for N_k in several cases than had been previously obtained.

The general nature of Dickson's method is indicated by the following discussion (taken from one of his papers)[2] for $k=3$. As already mentioned (§7), von Sterneck's table shows that all integers between 8042 and 40,000 are sums of 6 cubes. It follows from this fact that every integer N between 40,000 and 67,000 is a sum of 7 cubes; for subtract $30^3=27,000$ from each N; since the difference $N-30^3$ lies between 13,000 and 40,000, it can be expressed by 6 cubes, and so N itself is a sum of 7 cubes. Similarly, every integer M between 67,000 and 94,872 is a sum of 7 cubes, since $M-38^3=M-54,872$ lies between 12,128 and 40,000. This argument may be repeated until we have finally subtracted 103^3, and shows that all integers between 454 and 1,132,727 are sums of 7 cubes. A similar argument based on this result shows that all integers between 239 and 232,604,691 are sums of 8 cubes, and a third application shows that all integers less than 4 billion are sums of 9 cubes. Wieferich's proof that every positive integer is the sum of 9 cubes, as simplified by Dickson,[3] applies directly only to integers $>9\cdot 5^{12}$ (about 2.2 billion), and the above argument serves to bridge the gap between this value and the largest one in the table.

9. *Representation of "Almost All" Integers.* Another question which has been considered in connection with Waring's problem is that of determining for each power k the least integer G_k' such that "almost all" positive integers can be expressed as sums of G_k' kth powers. By that we mean that if $B(n)$ denotes the number of positive integers $\leqq n$ which can *not* be so expressed, then

$$\lim_{n\to\infty} \frac{B(n)}{n} = 0.$$

Hardy and Littlewood proved that $G_4' \leqq 15$ and that for $k=3$ and $k>4$

$$G_k' \leqq (k-2)2^{k-2}+3,$$

whence "almost all" positive integers are sums of at most 5 cubes, 15 fourth powers, 27 fifth powers, 67 sixth powers, etc. For $k=4$ the value can be determined exactly, for we proved in §7 that all numbers of the form $16m+15$ require at least 15 fourth powers, and since these constitute 1/16 of all integers, the ratio $B(n)/n$ for them does not approach 0. Hence $G_4' \geqq 15$, and comparison with Hardy and Littlewood's result shows that $G_4' = 15$. Numbers of the form $31\cdot 16^m$, which require 16 fourth powers, are farther and farther apart as n increases, and for them the limit of $B(n)/n$ is 0.

10. *Generalizations.* Waring's problem can be generalized in various ways. In the first place we may introduce coefficients and prove such theorems as these: every odd positive integer can be expressed in the form $x^2+y^2+2z^2$;

---

[2] Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 549–553. See also this MONTHLY, vol. 41 (1934), pp. 547–555.

[3] Transactions of the American Mathematical Society, vol. 30 (1928), pp. 1–18.

every positive integer can be expressed in the form $x^2+2y^2+3z^2+6w^2$; every positive integer is the sum of 8 cubes plus t times a ninth cube, if t is any of the numbers 1, 2, 3, $\cdots$, 23 except possibly 20, but not if $t>23$. More generally, the expression of integers in the form x^2+y^2 leads us to consider representations of the form $ax^2+bxy+cy^2$, where a, b, and c are integers; we come here to the extensive theory of binary quadratic forms. Similarly $x^2+y^2+z^2$ leads naturally to ternary quadratic forms.

To generalize in another direction, we may look on a sum of kth powers as a sum whose terms are values of the function x^k for certain integral values of x; replacing x^k by any polynomial $f(x)$ which takes on integral values when x is an integer, we are led to consider the problem of representing integers as sums of values of $f(x)$ for integral values of x. As a simple example, consider the numbers 1, $1+2$, $1+2+3$, $\cdots$, which the Greeks named triangular numbers. The general triangular number is $x(x+1)/2$, where x is a positive integer. Every positive integer of the form $8m+3$ can be expressed as the sum of three squares:

$$8m + 3 = a^2 + b^2 + c^2;$$

furthermore, a, b, and c must all be odd, since otherwise their sum would not have the remainder 3 when divided by 8, so write

$$a = 2x + 1, \qquad b = 2y + 1, \qquad c = 2z + 1.$$

Substituting these values in the equation and simplifying, we get

$$m = \frac{x(x + 1)}{2} + \frac{y(y + 1)}{2} + \frac{z(z + 1)}{2}.$$

Since m may be any positive integer, we have proved that every positive integer is the sum of at most three triangular numbers. Triangular numbers and squares are particular cases of *polygonal* numbers; the general expression for a polygonal number of order m (or m-gonal number) is $x+(m-2)(x^2-x)/2$. Fermat discovered the theorem that every positive integer is the sum of at most three triangular numbers, four squares, five pentagonal numbers, and in general of m polygonal numbers of order m; this was proved by Cauchy about 1813.

Waring conjectured that every positive integer "of the proper form" is the sum of a finite number of values of a given polynomial, the integer being a multiple of 3 if the polynomial is $3x^4+6x^3+24$. Considerable work has been done in recent years on the representation of integers by sums of values of a polynomial. I will conclude by quoting a theorem proved by Kamke in 1921, which establishes in precise form the truth contained in Waring's rather vague statement. If $f(x)$ is a polynomial of degree $n\geqq 2$ with rational coefficients, that of x^n being positive, which takes on integral values $\geqq 0$ for every integral value $\geqq 0$ of x, then every positive integer is the sum of a finite number of 1's and a finite number of values of $f(x)$ for integral values $\geqq 0$ of x.

Bibliography. A good brief bibliography of Waring's problem is given by Dickson in the Bulletin of the American Mathematical Society, vol. 39, pp.

725–727 (October 1933), and a complete one (to 1919) in Volume 2 of his *History of the Theory of Numbers*.*

INTEGRAL FUNCTIONS ASSOCIATED WITH CERTAIN BINOMIAL COEFFICIENT SUMS

By GEORGE RUTLEDGE and R. D. DOUGLASS, Massachusetts Institute of Technology

1. *Introduction.* In a recent paper in this MONTHLY† the writers defined a sequence of constants $A_2, A_4, A_6, \cdots$. These constants are the Maclaurin coefficients of an even integral function

$$Q(x) = 1 + A_2x^2 + A_4x^4 + A_6x^6 + \cdots, \tag{1}$$

presently to be defined.

The functions

$$Q(n, z) = \frac{P_n(1-2z) + P_{n-1}(1-2z)}{2}, \tag{2}$$

$$M(n, z) = \frac{P_n(1-2z) - P_{n-1}(1-2z)}{2}, \tag{3}$$

where

$$P_n(\zeta) = F\left(n+1, -n, 1, \frac{1-\zeta}{2}\right) \tag{4}$$

is the generalized Legendrian polynomial, n not being restricted to integer values, may be written, with x replacing n,

$$Q(x, z) = F(x, -x, 1, z), \tag{5}$$

$$M(x, z) = \frac{z}{x}\frac{\partial}{\partial z}Q(x, z), \tag{6}$$

where (6) is derived from (2) and (3) by application of known properties of the derivatives of the Legendrian polynomials.

Furthermore, (5) and (6) are Stirling interpolation series, namely,

$$Q(x, z) = 1 - \frac{z}{1^2}x^2 - \frac{z^2}{2^2}x^2\left(1 - \frac{x^2}{1^2}\right) - \frac{z^3}{3^2}x^2\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) - \cdots, \tag{7}$$

$$M(x, z) = -\frac{z}{1}x - \frac{z^2}{2}x\left(1 - \frac{x^2}{1^2}\right) - \frac{z^3}{3}x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) - \cdots. \tag{8}$$

---

* Since this paper was written, Vinogradow has proved the inequality $G_k < 3h^2 + 4k$, where h is the least integer which exceeds $k \log 8k$. With the aid of this, James and Zuckerman have proved the exact result $N_{15} = 33{,}203$. See Bulletin of the American Mathematical Society, vol. 41 (1935), p. 197. For still later results, see *ibid.*, pp. 795, 801.

† Vol. 41 (1934), pp. 29–36.

Now the necessary and sufficient condition* for convergence of the Stirling interpolation series

$$\lambda_1 x + \lambda_2 x\left(1 - \frac{x^2}{1^2}\right) + \lambda_3 x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) + \cdots, \tag{9}$$

for other than integer values of x is convergence of the series $\sum \lambda_n$. If this latter series is convergent, then (9) is convergent for all values of x and defines an integral function.†

We shall be particularly concerned with the integral functions defined by (7) and (8) when z has the value -1. We shall denote these functions by $Q(x)$, $M(x)$, respectively, thus:

$$Q(x) = 1 + \frac{x^2}{1^2} - \frac{x^2}{2^2}\left(1 - \frac{x^2}{1^2}\right) + \frac{x^2}{3^2}\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) - \cdots, \tag{10}$$

$$M(x) = \frac{x}{1} - \frac{x}{2}\left(1 - \frac{x^2}{1^2}\right) + \frac{x}{3}\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) - \cdots. \tag{11}$$

2. *The Integers $Q(n)$, $M(n)$.* When x is an integer n, we have from (7) and (8)

$$\begin{aligned} Q(n, z) &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{n-1+s}{s} z^s \\ &= (1 - z) \sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} \binom{n+s}{s} z^s, \end{aligned} \tag{12}$$

$$M(n, z) = \sum_{s=0}^{n-1} (-1)^{s+1} \binom{n}{s+1} \binom{n+s}{s} z^{s+1}. \tag{13}$$

Then, placing $z = -1$, we obtain‡

$$Q(n) = \sum_{s=0}^{n} \binom{n}{s} \binom{n-1+s}{s} = 2 \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{n+s}{s}, \tag{14}$$

$$M(n) = \sum_{s=0}^{n-1} \binom{n}{s+1} \binom{n+s}{s}. \tag{15}$$

---

* Rutledge, Jour. Math. and Phys. of Mass. Inst. Tech., vol. 9 (1930), p. 262.

† Nörlund, *Differenzenrechnung*, p. 209.

‡ This problem originated in an attempt to evaluate

$$\sum_{s=0}^{n} \binom{n}{s} \binom{n-1+s}{s}.$$

The corresponding sum with alternating signs is well known. This latter sum may be evaluated here by noting that

$$Q(x, 1) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x}$$

and hence

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{n-1+s}{s} = \frac{\sin \pi n}{\pi n} = 0 \qquad \text{for } n = 1, 2, 3, \cdots.$$

Thus $Q(n)$, $M(n)$ are integers. These integers are readily computed by means of recursion formulas to be derived in the next section.

3. *Recursion Formulas for* $Q(n)$, $M(n)$. If the known formulas

$$nP_n(3) = 3(2n-1)P_{n-1}(3) - (n-1)P_{n-2}(3), \tag{16}$$

$$(n-1)P_{n-1}(3) = 3(2n-3)P_{n-2}(3) - (n-2)P_{n-3}(3), \tag{17}$$

are multiplied, respectively, by $2n-3$, $2n-1$, we obtain

$$(2n-3)nP_n(3) = (12n^2-24n+9)P_{n-1}(3) - (2n^2-5n+3)P_{n-2}(3), \tag{18}$$

$$\begin{aligned}(2n-3)nP_{n-1}(3) = {} & (12n^2-24n+9)P_{n-2}(3) - (2n^2-5n+3)P_{n-3}(3) \\ & - P_{n-1}(3) + P_{n-3}(3).\end{aligned} \tag{19}$$

If we now note that

$$-Q(n-1) + M(n-1) + Q(n-2) + M(n-2) = 0, \tag{20}$$

$$-Q(n-1) - M(n-1) + Q(n-2) - M(n-2) = -P_{n-1}(3) + P_{n-3}(3), \tag{21}$$

we may add the right-hand side of (20) to the right-hand side of (18), and in (19) we may substitute the left-hand side of (21) for the right-hand side. Taking half the sum and half the difference of (18) and (19) as thus amended we have the recursion formulas

$$(2n-3)nQ(n) = (12n^2-24n+8)Q(n-1) - (2n^2-5n+2)Q(n-2), \tag{22}$$

$$(2n-3)nM(n) = (12n^2-24n+10)M(n-1) - (2n^2-5n+2)M(n-2). \tag{23}$$

By use of (14), (15) and (22), (23) we obtain the values tabulated as follows:

n	$Q(n)$	$M(n)$
1	2	1
2	8	5
3	38	25
4	192	129
5	1002	681
6	5336	3653
7	28814	19825
8	15718 4	10854 5
9	86414 6	59841 7
10	47800 08	33174 45
11	26572 086	18474 633
12	14832 1344	10327 4625
13	83076 4794	57916 8825
14	46668 90936	32569 57317

4. *Asymptotic Relations.* By use of the known asymptotic values*

$$P_n(3) \sim \frac{1}{\sqrt{2n\pi}} \frac{e^{(n+1/2)\alpha}}{\sqrt{\sinh \alpha}}, \tag{24}$$

$$P_{n-1}(3) \sim \frac{1}{\sqrt{2n\pi}} \frac{e^{(n-1/2)\alpha}}{\sqrt{\sinh \alpha}}, \tag{25}$$

where $\alpha = 2 \sinh^{-1} 1 = \log (3+2\sqrt{2})$, we have from (2) and (3),

$$Q(n) \sim \sqrt[4]{2} \frac{\cosh \alpha n}{\sqrt{\pi n}}, \tag{26}$$

$$M(n) \sim \frac{1}{\sqrt[4]{2}} \frac{\cosh \alpha n}{\sqrt{\pi n}}. \tag{27}$$

5. *The Coefficients of* $Q(x)$. We append the values to eleven decimals of the coefficients $A_2, \cdots, A_{16}$ of the function $Q(x)$ as computed from the defining series together with Euler's transformation for increasing the rapidity of convergence.

$$\begin{aligned} Q(x) = \; & 1.00000\ 00000\ 0 \\ & +0.82246\ 70334\ 2\ x^2 \\ & +0.16265\ 46674\ 0\ x^4 \\ & +0.01416\ 48977\ 6\ x^6 \\ & +0.00069\ 13694\ 7\ x^8 \\ & +0.00002\ 15581\ 1\ x^{10} \\ & +0.00000\ 04663\ 4\ x^{12} \\ & +0.00000\ 00074\ 1\ x^{14} \\ & +0.00000\ 00000\ 9\ x^{16} \end{aligned}$$

Checks on the computation are the values $Q(1) = 2$, $Q(2) = 8, \cdots$, and the elliptic integral

$$Q\left(\frac{1}{2}\right) = \frac{2\sqrt{2}}{\pi} E\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right). \tag{28}$$

The latter relation is derived from

$$F\left(\frac{1}{2}, -\frac{1}{2}, 1, z\right) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{1 - z \sin^2 \phi}\, d\phi \tag{29}$$

by a slight transformation.

6. *Application of a Fundamental Identity for de la Valleé-Poussin Summation.* If the Stirling interpolation series (9), which for any sequence of λ's defines a function $G(x)$ at least for integer values of x, is given, then we have the identity†

* Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 79.

† Rutledge, loc. cit., p. 262.

$$(30)\qquad \sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} (-1)^{j-1} \frac{2G(j)}{j} = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Now from (14) we have

$$(31)\qquad \begin{aligned} Q(n) &= 2\sum_{s=0}^{n-1} \binom{n-1}{s}\binom{n+s}{s} \\ &= \frac{2}{n}\left\{n - n\left(1 - \frac{n^2}{1^2}\right) + n\left(1 - \frac{n^2}{1^2}\right)\left(1 - \frac{n^2}{2^2}\right) - \cdots\right\}. \end{aligned}$$

Therefore

$$(32)\qquad \sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} (-1)^{j-1} Q(j) = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$$

Similarly, from (11),

$$(33)\qquad \sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} (-1)^{j-1} \frac{2M(j)}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}\cdot$$

Since

$$(34)\qquad u_0 + \lim_{n\to\infty} \sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} u_j$$

defines the de la Valleé-Poussin sum* of the series $\sum u_n$, we see that (32) and (33) illustrate respectively non-existence and existence of the limit in question for rapidly oscillating series† $\sum u_n$, $\sum v_n$, where, from (26), (27),

$$(35)\qquad \frac{u_n}{v_n} \sim \frac{n}{\sqrt{2}}\cdot$$

---

* See Gronwall, Journal für die reine und angewandte Mathematik, vol. 147 (1917), p. 16, and *Summation of series and conformal mapping*, Annals of Math., series 2, vol. 33 (1932), p. 101; Rutledge, *The inverse matrix for de la Valleé-Poussin summation*, Journal of Math. and Phys. of Mass. Inst. Tech., vol. 11 (1932), p. 73.

† For convenience we define $u_0=0$, $v_0=0$.

The integral function $Q(x)$ derives its principal interest for us from the fact expressed by equation (32), namely, that

$$\sum_{n=1}^{\infty} (-1)^{n-1} Q(n)$$

is that series whose Valleé-Poussin transform is

$$\sum_{n=1}^{\infty} (-1)^{n-1}.$$

In a paper* now in course of publication it is proved that the function $Q(n)$ defines the range of de la Valleé-Poussin summation in the same sense that the function n^k defines the range of Cesàro summation of order k.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE DIOPHANTINE EQUATION $m^2 = n!+1$

By HANSRAJ GUPTA, Govt. College, Hoshiarpur

1. *Introduction.* In 1876 H. Brocard† asked if $n!+1$ was a perfect square and S. Ramanujan‡ in 1913, asked for values n, other than 4, 5, 7, which make it a square. In 1885, Brocard repeated his question§ but in 1887 he suggested|| that the only solutions are 4, 5, and 7. In 1906, A. Gérardin remarked¶ that if

$$n! + 1 = m^2, \tag{1}$$

has solutions other than

$$n = 4, 5, 7; \qquad m = 5, 11, 71;$$

then m has at least 20 digits.

In this note I show that the equation (1) has no solutions in the range $7 < n \leqq 50$. This enables us to substitute 34 for 20 in Gérardin's remark.

To reach this result I used a table giving the values of $n!$ for $1 \leqq n \leqq 50$. (See J. Peters, *Zehnstellige Logarithmentafel*, vol. 1, table 4a.) The table for the square root of $n!+1$ has been calculated by me and my younger brother and

* Rutledge and Douglass, *The range of de la Valleé-Poussin summation*, Journal of Math. and Phys. of Mass. Inst. Tech., vol. 14 (1935), pp. 191–194.

† Nouvelle Correspondance Math., vol. 2.

‡ Journal of the Indian Math. Soc., vol. 5 (1913), p. 159; also Ramanujan, *Collected Papers* (1927), p. 327.

§ Nouvelles Annales de Math., ser. 2, vol. 4, p. 391.

|| Mathesis, vol. 7 (1887), p. 280.

¶ Nouvelles Annales de Math., ser. 4, vol. 6 (1906), p. 222.

pupil Dinanath. From the many checks that I have applied, I feel certain that the results stated are correct.

2. THEOREM. *$n!+1$ is a square, if and only if $n!/8$ is a triangular number.*

Since $n!+1$ is odd, when $n>1$;
Let .

$$n! + 1 = (2h + 1)^2 = 4h^2 + 4h + 1.$$

Then

$$\frac{n!}{8} = \frac{h(h + 1)}{2}.$$

Conversely, if

$$\frac{n!}{8} = \frac{h(h + 1)}{2},$$

then

$$n! + 1 = (2h + 1)^2.$$

Hence (1) has solutions, if and only if

$$\frac{n!}{4} = N \cdot M, \quad \text{where} \quad N - M = 1.$$

This gives

$$m = N + M = 2N - 1 = 2M + 1.$$

Moreover,

$$m \equiv \pm 1 \pmod{p^\alpha},$$

where

$$\frac{n!}{4} \equiv 0 \pmod{p^\alpha}, \quad \text{but} \quad \not\equiv 0 \pmod{p^{\alpha+1}};$$

p being a prime $\geqq 2$. In particular,

$$m \equiv \pm 1 \pmod{10}, \quad n > 4.$$

Hence

$$N \equiv 1, 5, \text{ or } 6 \pmod{10},$$

while

$$M \equiv 0, 4, \text{ or } 5 \pmod{10}.$$

These conditions are very helpful in determining values of n which can possibly make $n!+1$ a square.

In the first instance, I tried to find a value of $n>7$ satisfying the above conditions. Failing to find any up to 50, and to be doubly sure, I began the laborious work now presented in the following table.

It was easily seen that $n!+1$ is not a square, when $n=p-1$, where p is a prime >7 and <50, for within these limits, I found that

$$n!+1\equiv 0 \pmod{p}, \quad \text{but} \quad \not\equiv 0 \pmod{p^2}.$$

n	square root of $n!+1$	remainder
8	200	321
9	602	477
11	6317	12312
13	78911	74880
14	295259	414120
15	1143535	2071776
17	18859677	11551672
19	348776576	442548225
20	1559776268	1960632177
21	7147792818	2657058877
23	16 0785623545	13 0518272976
24	78 7685471322	147 8154932317
25	393 8427356614	544 6454455005
26	2008 2117944245	3861 0655379976
27	10434 9745809073	20403 3398880672
29	297351 0046012910	380915 5729331901
31	9067986 9067935485	5260740 2757814776
32	51296280 2680363491	26059996 8274292920
33	294674695 5341073478	494634430 3884983517
34	1718233974 2875652406	1126840400 3386411165
35	1 0165209277 9175702171	4969301985 2305886760
37	37 0995324650 1409085690	55 5321088842 0637223901
38	228 6968774309 3501007951	82 2117240043 8234781600
39	1428 2115417961 5294686593	664 7790725377 8706052352
41	57838 1592144527 0815783609	43039 7437300339 5286935120
43	2457951 6484946125 8960674062	3537883 0211538986 6600420157
44	16304206 7417843078 8228519563	9694741 7086840604 9326289032
45	109371943 7815202197 0306618007	114938795 2536649157 7783347952
47	5085501366 7402369565 8451670185	5171087816 0896326970 3982065776
48	3 5233386996 6202265350 5900576721	3 3456933974 1803952511 9592888160
49	24 6633708976 3415857454 1304037050	15 9587510876 7898528361 2227297501
50	174 3963680863 6061169620 9329639024	247 8469531667 2521333008 1856327425

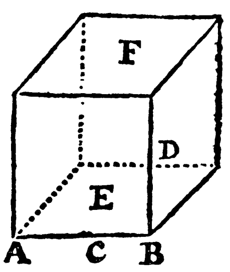
WALLIS ON THE TRINITY

By R. C. ARCHIBALD, Brown University

In *Scripta Mathematica* (vol. 2, p. 197), Professor E. H. Neville, of the University of Reading, drew attention to a statement that John Wallis (1616–1703), Savilian professor of geometry at Oxford for 54 years, had described the parallelism between the Trinity and three dimensions, and asked where such a discussion might be found. An answer to this question has been made possible by the generosity of Mrs. E. P. Carling of London, England, in presenting to Brown University a Wallis volume which does not seem to be in the British Museum or in any other of the larger libraries of America. This volume contains eight separately-paged items (varying in length from 10 to 102 pages), each with its own title-page. The first, entitled *The Doctrine of the Blessed Trinity Briefly Explained In a Letter to a Friend*, is dated London, 1690, but all of the other items are dated 1691. The first seven items are seven letters on the same subject. The eighth item is entitled *Three Sermons Concerning the Sacred Trinity*,* sermons delivered at Oxford in 1664 and 1691. The "advertisement" and "additions" with which the work ends, show that the eight items were originally published together as here.

It is the first item of 20 pages, dated 11 Aug. 1690, which has particular interest for us in furnishing an answer to Professor Neville's query. And from this item we quote from the second paragraph of page 11 through the second paragraph on page 15. The cut is an exact reproduction of the one in Wallis's work. The quotation is as follows:

"I will begin with what concerns the most gross of Finite Beings, that is, Material Bodies.



"Suppose we then a Cubical Body, (which what it is, every one knows, that knows a Dy.) In this are *Three Dimensions*, (Length, Breadth, and Heighth) and yet but *One Cube*. Its Length (suppose between East and West) *A. B.* Its Breadth (suppose between North and South) *C. D.* Its Heighth (between Bottom and Top) *E. F.* Here are *Three* Local Dimensions, truly Distinguished each from other, (not only imaginarily:) The distance between East and West

* A Latin translation of these three sermons was printed in J. Wallis, *Opera Quædam Miscellanea*, Oxford, 1699, p. 301–332.

(whether we think or think not of it) is not that between North and South; nor be either of these that between Top and Bottom. The Length is not the Breadth, or Heighth; the Breadth is not the Length, or Heighth; and the Heighth is not the Length, or Breadth: But they are *Three* dimensions,* truly distinct each from other: Yet are all these but *One* Cube: And if any one of the *Three* were wanting it were not a Cube. There is no Inconsistence therefore, that what in one regard are *Three* (three Dimensions) may, in another regard, be so united as to be but *One*, (one Cube.) And if it may be so in Corporeals, much more in Spirituals.

"Suppose we further, Each of these Dimensions infinitely continued; the Length infinitely Eastward and Westward, the Breadth infinitely Northward and Southward, the Heighth infinitely Upward and Downward: Here are *Three* infinite Dimensions, and but *One* infinite Cube; and these *Three* Dimensions (though distinct) are equal each to other (else it were not a Cube;) and though we should allow, that a Cube cannot be infinite (because a Body, and therefore a finite Creature): Yet a Spirit may; such as is the Infinite God. And therefore no Inconsistence; that there be *Three* Personalities (each infinite, and all equal), and yet but *One* Infinite God, essentially the same with those *Three Persons*.

"I add further, That such Infinite Cube, can therefore be but *One*, and those *Three* Dimensions can be but Three, (not more nor fewer:) For, if Infinite as to its Length (Eastward and Westward), and as to its Breadth (Northward and Southward), and as to its Heighth (Upward and Downward); it will take up all imaginary space possible, and leave no room either for more Cubes or more Dimensions: And if this infinite Cube were (and shall be) Eternally so, its Dimensions also must be Infinite and Co-eternal.

"I say further, If in this (supposed) Cube, (we suppose in Order, not in Time) its first Dimension, that of Length, as *A. B.*, and to this Length be given an equal Breadth (which is the true generation of a Square) as *C. D.*, which compleats the square Basis of this Cube; and to this Basis (of Length and Breadth) be given (as by a further Procession from Both) an equal Heighth *E. F.*, which compleats the Cube; and all this eternally, (for such is the Cube supposed to be), here is a fair Resemblance (if we may *parvis componere magna*) of the *Father*, (as the Foutain or Original;) of the *Son*, (as generated of him from all Eternity); and of the *Holy-Ghost*, (as eternally Proceeding from Both): And all this without any Inconsistence. This *longum*, *latum*, *profundum*, (Long, Broad, and Tall), is but *One* Cube; of *Three Dimensions*, and yet but *One Body*: And this *Father*, *Son*, and *Holy-Ghost*; *Three Persons*, and yet but *One God*. And as, there, the Dimensions are not (in the Abstract) predicated or affirmed each of other, or the Cube of either, (the Length is not the Breadth or Heighth, nor either of these a Cube); but (in the Concrete) Cube is affirmed of all; this *longum*, *latum*, *profundum*, is a *Cube*, and the *same Cube*: So here, (in the Ab-

* In the printed work the word "Divisions" was given here, but this is crossed out and the word "dimensions" is written in, apparently by Wallis's hand.

stract) the Personality of the *Father* is not that of the *Son*, nor either of these that of the *Holy-Ghost*, nor the *Deity* of *Godhead* any of these; but (in the Concrete) though the Personalities are not, yet the Persons are, each of them *God* and the *same God*.

"If it be objected, that those Concretes are Affirmed or Predicated each of other; (that *longum* is also *latum* and *profundum*, (this *Long* is *Broad* and *Tall*;) but not so here, the *Father* is not the *Son* or *Holy-Ghost*: I answer, That, if the words be rightly considered, the Analogy holds here also: For when we say, this *Long* is *Broad* and *Tall* (where *Cube* or *Body* is understood) the full meaning is plainly thus; This Body, which, as to one Dimension (that of Length), is said to be a *long* Body, is the same Body, which, as to another Dimension (that of Breadth), is said to be a *broad* Body, and which, as to a third Dimension (that of Heighth), is said to be a *tall* Body. So here, That God, which (as to one Personality) is *God the Father*, is the same God, which (as to another Personality) is *God the Son*, and which (as to a third Personality) is *God the Holy-Ghost*. So the Analogy holds every way, nor is there any Inconsistence in either Case.

"I proceed to the Consideration of somewhat more Spiritual, and less Material than that of a Body locally extended."

It may be appropriate, in conclusion, to draw attention to a recent thoroughly documented work, of over 320 pages, by Joseph Iwanicki, *Leibniz et les Démonstrations mathématiques de l'Existence de Dieu*, Strassburg, Librairie Universitaire d'Alsace, 1933.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Reihentwicklungen in der mathematischen Physik. By Josef Lense. Berlin and Leipzig, Walter de Gruyter and Co., 1933. 178 pages.

The object of this book is to briefly present the essential properties of Bessel functions, spherical harmonics and Lamé's ellipsoidal harmonics. The reader is assumed to have an elementary knowledge of differential equations and of the theory of functions of a complex variable.

The first section deals with general systems of orthogonal functions, and, as illustrations, the Laguerre and Hermite functions are defined. There follow three long sections, dealing in turn with Bessel functions, spherical and ellipsoidal harmonics. Section five, on asymptotic series, includes the Euler-Maclaurin sum formula which is used to calculate Euler's constant. This leads up to the discussion of the gamma function in section six, which among others de-

rives all the properties of these functions which were needed in the section on Bessel functions.

The exposition is always clear, and many of the derivations distinctly elegant. The book should be useful to students of mathematical physics, as its treatment of the special functions is less sketchy than that in the typical text in advanced calculus, and less formidable than such encyclopedic treatises as those of Watson or Hobson.

The title is somewhat misleading, as no theorems on the validity of series expansions are proved, though several are stated without proof. Moreover, although several problems in potential theory are solved by using series expansions; in each case the author merely points out that his solution holds if the series converge uniformly and not necessarily otherwise.

PHILIP FRANKLIN

An Elementary Treatise on Pure Mathematics. By N. R. C. Dockeray. London, G. Bell and Sons, 1934. xiv+566 pages. $5.00.

This is an excellent textbook covering about the same range of topics and at about the same level as our courses in Advanced Calculus. This is evident from the list of chapter headings which follow: I. Convergence and divergence of series. II. Vandermonde's theorem and the binomial theorem. III. Complex numbers and De Moivre's theorem. IV. Limits, continuity and differentiation. V. The exponential theorem and logarithmic series. VI. Convergence of series of complex terms; circle of convergence. VII. Functions of two or more independent variables. VIII. Expansions of sin $n\theta$ and cos $n\theta$ in series and products. IX. Methods of integration. X. Definite integrals. XI. Taylor's theorem. XII. Applications of the calculus to curves and curve tracing. XIII. Further theorems on convergence. XIV. The theory of uniform convergence. XV. The expansion of trigonometric functions as infinite products.

While there is a bit more attention to rigor than is usual in most such courses and the point of view of the author is pure mathematical analysis with little bother about application, this book could serve admirably as a textbook or reference book in Advanced Calculus courses. Of course the length of the book and its wealth of problems would require some selection of topics by the instructor. A more serious objection is the lack of heavy type for theorems and formulae. This makes the use of the book for reference somewhat difficult unless the reader has marked all the important results.

The exposition leaves little to be desired in clarity and conciseness of proofs, and at the same time each new idea is introduced carefully and with several illustrations. The first chapter covering limits of sequences and convergence tests of series is the outstanding example of excellent exposition and forms perhaps the best presentation of these topics in the English language. The proofs of the properties of continuous functions in Chapter IV is another example of elegance. This high level of excellence sags badly in the same chapter on the topic of differentials. In common with many writers on this subject, no

clear distinction is kept between the differential of the dependent and independent variables, so that we have the "proof" that $dx=\Delta x$ and $d^2x=0$.

The content of Chapter V will prove interesting and novel to many students. The infinite series $\sum x^n/n!$ and the function of x defined by it are investigated for properties. The author carefully avoids e^x and uses the symbol "exp x" to fix the attention on the definition and to prevent the use of information known but not yet proved about the function. A neat proof of the irrationality of e is given (page 148), but this is marred by a typographical error in that "q" instead of "$q!$" appears as the factor outside the bracket throughout the proof. While no close search for typographical errors was made, very few were noticed and the few observed caused no difficulty.

Again in Chapter X the author keeps the reader's attention fixed on the definition by using the symbol $I_{ab}\{f(x)\}$ for the integral defined as the limit of a sum. The usual symbol for the definite integral is not used until the relation of this definition to integration as inverse to differentiation is shown.

Homogeneous coordinates are used in Chapter XII without any introduction or reference. However this will not prove much of a stumbling block even to those students unacquainted with these coordinates as only the idea and very simplest properties are used.

One of the fine features of the book is its excellent collection of problems (about 1500 by the author's count) ranging from the simplest to rather difficult ones introducing bits of new theory. The summation of series by successive differences, cross ratio and its properties, and Leibniz's rules for integration and differentiation of an integral with respect to a parameter are examples of points of new theory given through the problems.

W. L. AYRES

Plane and Spherical Trigonometry. By L. M. Kells, W. F. Kern and J. R. Bland. New York and London, McGraw-Hill Book Company, Inc., 1935. xiv+269 pages+tables, 115 pages. $2.50.

This is an exceptionally well-written textbook on trigonometry. By skilful arrangement of the contents the authors succeed in presenting in the first four chapters (fifty-four pages) enough material for a brief course in plane trigonometry. The various topics are presented interestingly and clearly. Each topic is followed by numerous well-selected and graded exercises, answers to which are provided at the end of the text. Special attention is given to the study of rectilinear figures, a feature that should be particularly valuable in the instruction of engineering students. Special features of the book are a chapter on the slide rule, a chapter on the applications of spherical trigonometry to navigation and astronomy, and an elaborate set of proportional parts tables which will eliminate much of the drudgery involved in the computational side of trigonometry.

Good typography and clear diagrams enhance the physical attractiveness of an excellent textbook.

J. M. FELD

MATHEMATICS CLUBS

EDITED BY F. W. OWENS and HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with referenres, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

CLUB TOPICS

Twenty years ago this month, the AMERICAN MATHEMATICAL MONTHLY became the official journal of the Mathematical Association of America and the first article, immediately following the announcement of the new organization was by L. J. Richardson on "*Digital Reckoning Among the Ancients.*" This article is worthy to stand first in a compilation of topics presented in the MONTHLY and especially suited for presentation in our mathematics clubs.

The following list of references, all in the MONTHLY, is not intended to be exhaustive but rather to serve as an incentive to review the volumes in a search for the best possible topics.

Digital reckoning, vol. 23, p. 7.
Geometrography, vol. 25, p. 37.
Arithmetrical prodigies, vol. 25, p. 91.
Ptolemy's theorem, vol. 25, p. 94.
Russian peasant multiplication, vol. 25, p. 141.
Women as mathematicians, vol. 25, p. 136.
Polygon of seventeen sides, vol. 27, p. 322.
Mathematics in America, vol. 27, p. 443.
Life insurance and mathematics, vol. 27, p. 291.
Algebra of correlation, vol. 31, p. 110.
Trigonometry of correlation, vol. 31, p. 275
A note on knots, vol. 31, p. 237.
Mathematics and music, vol. 31, p. 1.
Early history of division by zero, vol. 31, p. 387.
On a transformation by paper folding, vol. 31, p. 433.
Fundamental mathematical requirements for a biologist, vol. 36, p. 179.

To quote from a recent letter concerning needs of our clubs, we sometimes need "dessert after the more solid fare of a mathematical feast." To judge from programs of other years, mathematical plays have been popular. This department will welcome manuscripts of such plays. Four plays have been printed in earlier volumes.

Flatlanders, a mathematical play in one act, vol. 26, p. 264.
Evolution of numbers, an historical drama, vol. 35, p. 146.
Discord in mathematics land, a play, vol. 37, p. 151.
Mechanics, a dramatic skit, vol. 38, p. 42.

The following Mathematical Romance was first used at a Christmas party of the Cornell Parabola Club in 1919. It has since been used in several states but has not been published.

The Romance was mimeographed. Each guest received a copy and attempted to substitute a mathematical term in each numbered blank.

THE MATHEMATICAL ROMANCE OF POLY —1— AND RAY—2—

Poly was a —3— yet rather —4— girl for her features were —5— though —6— and her —7— —8— while Ray was a —9— faced young man of —10— build and —11— purpose and —12— opinions. They lived in —13— —14— and their —15— often —16—.

Poly was a —17— young woman of —18— temper, who trusting her —19— —20— over Ray, kept his hopes —21— from —22— to —23—. But at last she became so —24— that he reached the —25— and flew off at a —26—, a very —27— Ray —2—. Still his love was —28— and the —29— of their estrangement was of —30— duration.

One day a —31— escaped her lips, and Ray who was far from —32—, noticed the —33— and asked, "Will you give me a —34— of your love by a sweet —35—?" She was an —36— girl, but so —37— were their —38— that she did not —39— when she was —40— by his arms.

Mr. I. Cosa —1— was a —41— business man —42— —43— in affairs of the heart, to whom the —44— —45— of life were —46—. He and Ray held —47— opinions, so when Ray came to the —48— of the matter he received at a —49— answer and Polly's father, growing —50—, called him a —51— for stealing a girl's affections before he had —52— the —53— of the H.C.L. However, Ray inherited a valuable —54— of land and Mr. I. Cosa —1—, who could —55— between —56— and —57— wealth, gave his consent.

A large —58— of —59— attended the wedding. There was the uncle whose —60— nearly —61— his —62— and the cousin who was a —63— favorite and the —64— minded old maid saying "prunes and —65—." The bride was the —66— of all attention as she appeared in a gown —67— on —68— —69— with a veil falling —70— to her —71—. As Ray slipped the —72— on her third —73—, he whispered, "The world is —74— for me with you as the —75— of my existence." Throwing her bouquet in a —76— —77— to her attendants, the happy —78— sped to the land of the —79— for a honeymoon, far from the icy —80— as they had previously —81—. They lived happily since they had —82— dispositions and Polly stayed in her proper —83—.

Answers for the Mathematical Romance. 1, Hedron; 2, Show (=ratio); 3, plane; 4, a-cute; 5, regular; 6, angular; 7, form; 8, symmetrical; 9, round; 10, square; 11, set; 12, fixed; 13, adjacent; 14, regions; 15, paths; 16, intersected; 17, complex; 18, variable; 19, increasing; 20, power; 21, oscillating; 22, zero; 23, infinity; 24, arbitrary; 25, limit; 26, tangent; 27, cross; 28, constant; 29, period; 30, infinitesimal; 31, loci (=low sigh); 32, obtuse; 33, sign; 34, demonstration; 35, osculation; 36, independent; 37, similar; 38, inclinations; 39, object; 40, encircled; 41, solid; 42, 43, everywhere dense; 44, 45, ideal elements; 46, unknown; 47, opposite; 48, point; 49, negative; 50, irrational; 51, lo-cus; 52, solved; 53, problem; 54, section; 55, differentiate; 56, imaginary; 57, real; 58, number; 59, relations; 60, perimeter; 61, equaled; 52, height; 63, prime; 64, vacuous; 65, prisms; 66, focus; 67, cut; 68, straight; 69, lines; 70, perpendicularly; 71, feet; 72, circle; 73, digit; 74, π; 75, center; 76, 77, parabolic arc; 78, pair; 79, pyramids; 80, poles; 81, projected; 82, complementary; 83, sphere.

Club Reports

1934–35

In an effort to reach all the mathematics clubs in the colleges and universities, a questionnaire was sent to all organized clubs of which there was a record available. Sixty reports have been received. These will be published as rapidly as possible. Any group doing club work which has not received a copy of the questionnaire is requested to communicate with this department.

Unless otherwise stated, all officers mentioned in these reports are for the school year 1934–35 and all programs were given during that year.

Wellesley Mathematics Club

Martha Hathaway, President; Ella Peck, Vice President; Martha Morrow, Treasurer and Secretary; Lennie P. Copeland, Faculty Adviser. There were forty-three members, of whom seven

were of the faculty. The six meetings included: open house with a "Scrambled mathematical term contest"; discussion of D. E. Smith's "Number stories of long ago" by Martha Curtis, Jean Bieberbach, Elma Van Artsdalen, and Caraline Meaker, and "Reminiscences of the author" by Helen A. Merrill and Clara E. Smith; "Linkages, with models" by R. K. Morley, Worcester Polytechnic Institute, with a formal dinner for the speaker; a patchwork program, "Browsing shelf" by Madeline Dunn, "Slide rule" by Virginia Lincoln, "Lehmer's factor machine" by Martha Morrow; guessing contest, with pictures and objects representing mathematical terms; annual supper and election.

Miss Helen A. Merrill has written a number of mathematical songs which are a valuable and traditional part of all these meetings.

Sigma Phi Mu, New Jersey State Teachers College

Harold Jordan, President; Dorothy Schmitt, Vice President; John Feltman, Treasurer; Marie Mulcare, Secretary, H. F. Fehr, Faculty Adviser.

Ten meetings were held with the following programs: "Teaching mathematics to slow moving groups" by V. S. Mallory; "Introduction to non-Euclidean geometry" by Jacob Orenstein; "Geometric and algebraic fallacies" by William Fillas; "The use of mathematics in business" by Mr. Nelson; "Postulational theory" by Samuel Gordon; a social meeting; "Applications of Ptolemy's theorem" by H. F. Fehr; "Mathematics of finance" by Dr. Hildebrandt; election of officers; supper hike with the Science Club.

The Mathematics Club of Butler University

John Batchelor, President; Mary Lucille Trager, Vice President; Dorothy Settles, Secretary; Robert Fohl, Treasurer.

With twenty-five members the club held nine meetings with programs as follows: "Fallacies" by Harriet Summers, Mary Katherine Mangus and Robert Fohl; "The four-color problem" by Helen Patrick and Mildred Rugenstein; Christmas party with farce, "The case of Matthew Mattix," presented by the members; "Elementary vector analysis and its applications to Euclidean geometry" by Douglas Ewing; "Greek geometry" by Mary Lucille Trager; "Mathematics in America" by Reginald Riley; "Leibniz" by John Batchelor; "Euler" by Garry Bolin; "Gauss" by Anvilla Fail; "Steiner" by Mary Lucille Trager; "The fourth dimension" by Juna Lutz; "Viete" by Alfred Meunier; "Napier" by Dorothy Settles; "Descartes" by Robert Fohl; "Newton" by David Knapp; annual picnic on the Butler campus.

Officers for 1935–36 are Robert Fohl, President; Mary Katherine Mangus, Vice President; Helen Patrick, Secretary; Harold Carson, Treasurer.

The Mathematics Club of the North Texas State Teachers College

A. M. Timberlake, President; W. Lee Cox, Vice President; Dorothy Jane Powers, Secretary-Treasurer; Vernadine Cooper, Reporter; Mary Ruth Cook, Historian; Dr. Ruth Stokes, Faculty Adviser.

Nine meetings were held including an election of officers and new members, a picnic, an initiation and party at College Club House, and a Christmas party at home of the Faculty Adviser. The programs for various meeting were as follows: "Mathematical puzzles" by Dorothy Jane Powers; "The infinite in mathematics" by Dr. W. H. Bruce, President Emeritus of the College; "Mathematics in secondary schools of foreign countries," "In France" by Lawrence Boyd, "In England" by Ellen Marie Francis, "In Italy" by Ray Karnes, "In Germany" by W. E. Cook; "Some applications of determinants" by W. Lee Cox; "Mathematical societies in the United States" by Morris Childers"; "A few types of summable series" by Addie Millican; "The three famous problems of antiquity" by Lloyd Shawn; "The mathematics of physics" by A. M. Timberlake; "The mathematics of chemistry" by James Spurlock; "Modern concepts in geometry" by Bernice Woods.

Pi Mu Epsilon of Washington Square College, New York University

Three business and social meetings were held with the following programs: "Contact Transformation" by Dean P. L. Thorne; "Geometry from a Transformation Viewpoint" by Morris Kline; "Bernoulli Polynomials."

At the annual dinner Professor Selby Robinson spoke on "Dimension Theory, "Professor R. Courant became an honorary member, and ten new members were initiated. Officers for 1935–36 are: Louis Barron, Director; Sol Feith, Vice Director; J. Herbert Moss, Secretary; David Gurinsky, Treasurer; Louise Hajek, Librarian.

The second Annual Pi Mu Epsilon Interscholastic Mathematics Contest sponsored by this chapter was held on May 4, 1935. A total of 395 contestants from 108 schools took the examination. Of the 108 high schools, 88 were represented by teams of four contestants. The papers were scored on a range from 0 to 500. The median score was 186; the first quartile score was 218; and the third quartile score was 156. The highest score was 368 and the lowest score was 85.

The awards were:

Gold medal: Richard P. Feynman of the Far Rockaway High School of New York City. Score, 368.

Silver medal: John Bratt of the Hackensack High School of Hackensack, N. J. Score, 336.

Bronze medal: Phil H. Weiss of the Evander Childs High School of New York City. Score, 335.

Pi Mu Epsilon Interscholastic Mathematics Contest Cup: Boys High School of New York City.

Cup, Section J: Millburn High School of Millburn, N. J.

Cup, Section S: Central High School of Valley Stream, Long Island.

Cup, Section W: A. B. Davis High School of Mount Vernon, N. Y.

Cup, Section Y: Boys High School of New York City.

The teams attaining the second and third places in the general contest were:

Second: Evander Childs High School.

Third: Townsend Harris High School

The teams attaining the second and the third places in each section were as follows:

Section J: Second, Eastside, Paterson, N. J.; Third, Weequahic, Newark, N. J.

Section S: Second, Lynbrook High School, Lynbrook, N. Y.; Third, Great Neck High School, Great Neck, N. Y.

Section W: Second, Pelham Memorial High School, Pelham, N. Y.; Third, New Rochelle High School, New Rochelle, N. Y.

Section Y: Second, Evander Childs High School, New York City; Third, Townsend Harris High School, New York City.

The members of Pi Mu Epsilon also assisted in the preparation and explanation of two mathematical exhibits given in December and May. The purpose of these exhibits was to make clear and attractive the principles of mathematics. This was done by means of charts and apparatus. There were about 500 visitors at each time.

The Mathematics Club of Washington Square College, New York University

Simon Arnold, President; Estelle Rosenberg, Vice President; Saul Gold, Secretary; Herman Spector, Editor; Seymour Rothstein and Abraham Silverstein, Directors of Coaching; C. K. Payne, Faculty Adviser.

The club held three socials and one boat ride besides sponsoring a series of lectures dealing with the fundamentals of mathematics and logic. Among these were "Mathematics and logic" by Mr. Kagey; "Non-Euclidean geometry" by D. A. Flanders; "History and fundamentals of differential and integral calculus" by R. Courant; "Transformations" by M. Kline. *Math X*, the official publication of the club, had a successful year.

We believe the greatest achievement of the club is its system of free coaching to all students in all mathematics courses, at all hours of the day. These classes have proved of benefit to both

student and teacher, the student gaining valuable information while the tutor secured teaching practice.

Officers for 1935–36 are: Norman Steinman, President; Selma Siegel, Vice President; Helen Rice, Secretary; Seymour Rothstein, Treasurer; Stanley Mandlebaum, Editor; Naomi Rosenstein, Director of Coaching; C. K. Payne, Faculty Adviser.

White Mathematics Club of the University of Kentucky

Seven meetings were held, including a social evening and a picnic. The programs were as follows: "The evolution of the number system" by C. G. Latimer; "The three classical geometrical problems;" "Trisection of angles" by Henry Spragens; "The squaring of the circle" by Annabelle Davis; "The duplication of the cube" by James Distler; "Astronomy" by H. H. Downing; "The fundamental theorem of integral calculus" by Anne Jeane Blackburn; "A locus problem in analytics" by Lucile Dodson; "Converted interests" by Marcells Holtzclaw; "The discovery of calculus" by Fletcher Donaldson; "Peculiarities of mathematics" by Pauline Thompson.

Officers for 1935–36 are: Fletcher Donaldson, President; Lucille Dodson, Secretary; Pauline Thompson, Annabelle Davis, and William Pell, Program Committee; M. G. Brown, Faculty Adviser.

Chi Upsilon Zeta of Elmira College for Women

Six meetings were held, including a picnic. Programs were as follows: "A, B, C" by Eleanor Schrader; "An excursion in numbers" by Clare Daley; "Map projection" by Violet Parker; "Math tricks" by Jean White; "Boarding house geometry" by Betty Cole; "Leonardo da Vinci" by Mary English; "Witch of Agnesi" by Mary Copley; "Slide rule" by Elizabeth Merrell; "History of logarithms" by Marjorie Lee; "Lewis Carroll" by Margaret Kelly.

Harvard Mathematical Club

Officers for the thirty-first year of this club were, Z. I. Mosesson, President; Coleman Herpel, Secretary-Treasurer; H. Whitney, Faculty Adviser. There was a membership of twenty-six active members, chiefly graduate students and seven sustaining members from the faculty. The programs for the thirteen meetings were as follows: "Mathematical centers here and abroad" by G. D. Birkhoff; "Interpolation to a complex function" by J. H. Curtiss; "Real non-analytic functions" R. P. Boas, Jr.; "Zermelo's theorem" by H. M. MacNeille; "Kronecker's theorem" by H. Whitney. "The structure of mathematical proofs" by S. MacLane; "Laws of the mean" by W. E. Sewell; "Toeplitz's theorem" by M. H. Heins; "Some triangle and circle geometry" by T. D. Woolsey, Yale '36; "Gauss's date rule" by T. B. Rosenthal, Yale '35; "Certain remarkable curves" by Wilfred Kaplan; "A unified calculus of propositions, classes and relations" by W. V. O. Quine; "Topics in the foundations of mathematics" by H. E. Robbins; "Harmonic functions" M. Morse.

At the final business meeting of the year the club elected W. F. Osgood, Perkins Professor of Mathematics in Harvard University, Emeritus, an Honorary Member in recognition of his constant interest in the organization.

Louisiana State University Mathematics Club

Frank A. Rickey, President; J. A. Ward, Vice President; Earl Thomas, Secretary-Treasurer.

Six meetings were held with the following programs: "Mathematicians in America before 1875" by I. C. Nichols. "On a method of solving a cubic" by J. A. Ward, "The correlation of mathematics and chemistry" by A. R. Choopin; "A practical problem in curve fitting" by Major P. Cole; "Mathematical statistics in the field of agricultural economics" by R. J. Saville; "On the Farey series" by Marcelina White.

Mathematics Society of the New Jersey College for Women

Carol Eichler, President; Virginia Lane, Vice President; Janet Thompson, Secretary; Elsie Matheke, Treasurer; A. E. Meder, Jr.; Faculty Adviser.

Eleven meetings were held, including a Christmas party and a general problem day when members presented problems and the whole club worked on them. In addition, the members were guests of the Rutgers Mathematics Club when B. H. Brown of Dartmouth College spoke on "Map Making," and ten members attended the December meeting of the Philadelphia Section of the Mathematical Association of America. The programs for the year included the following: "A mathematical basis for apportioning members in congress" by Gertrude Hirsch; "Archimedes' investigation of mathematical methods by means of mechanics" by Elsie Matheke; "This simple universe" by B. H. Brown of Dartmouth College; "All rational fractions can be integrated" by Mildred Ringel; "Discontinuous functions" by Julliette Marshall, Carol Eichler, Virginia Lane and Mary Wells; "Applications of the methods of mathematics to the problems of economics" by A. E. Meder, Jr.; "Imaginary roots of a polynomial and the real roots of its derivative" by Janet Thompson; "Theory of functions of a complex variable" by Gertrude Hirsch.

This last paper was prepared by Miss Hirsch as part of her requirements for "graduation with distinction in mathematics."

PROBLEMS AND SOLUTIONS

Edited by Otto Dunkel, H. L. Olson, and W. F. Cheney, Jr.

Elementary Problems

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 187. *Proposed by A. Gloden, Luxembourg.*

Find the smallest square whose last six digits follow the pattern *aabaab*.

E 188. *Proposed by N. A. Court, University of Oklahoma.*

A variable sphere (P) passes through a fixed point and has its center on a fixed sphere (A). Find the envelope of the radical plane of the two spheres (P) and (A).

Note. The analogous problem relative to two circles was solved analytically in the *Nouvelles Annales de Mathématiques*, ser. 2, vol. 15 (1876), p. 284, Q. 1198.

E 189. *Proposed by V. Thébault, Le Mans, France.*

What is the smallest positive integer base which may be used for a system of enumeration, such that in this system there exists a perfect square written with five digits, all the same?

E190. *Proposed by Fred Discepoli, New York City.*

If a and n are positive integers greater than the positive integer b, then if $a^n+b^n=c^n$, c can never be an integer.

E 191. *Proposed by E. P. Starke, Rutgers University.*

Three points on a central conic are joined to form a triangle. The tangents

at these points form a second triangle. Prove that the center of the conic and the centroids of the triangles either all coincide or are all distinct.

Corrected statement of E 183 [1935, 621] *Proposed by V. Thébault, Le Mans, France.*

In triangle PQR, on PQ lay off $PA=PR$, on PR lay off $PB=PQ$, on QR lay off $QC=QP$, on QP lay off $QD=QR$, on RP lay off $RE=RQ$, and on RQ lay off $RF=RP$. Now if the areas of the triangles PQR, PDE, QFA and RBC, with proper signs, are denoted by k, l, m and n respectively, show that $1/k = 1/l + 1/m + 1/n$.

The department editor extends apologetic regrets for his error in transcribing the orignal statement of this problem.

SOLUTIONS

E 159 [1935, 392]. *Proposed by F. A. Lewis, University of Alabama.*

If N is a positive integer composed of n successive 9's, and M is any positive integer less than $N+2$, then the sum of the digits in the product MN is $9n$.

Solution by E. P. Starke, Rutgers University

For $M=N+1=10^n$, the result is obvious. For $10^{m-1}\leqq M<10^m$, $m\leqq n$, we have $MN=10^n(M-1)+(10^n-M)$. Thus the first m digits of MN are those of $(M-1)$, and the last m digits are those of the arithmetical complement of M. The sum of these digits in pairs form m 9's. The middle $n-m$ digits of MN are already 9's. Thus we have $9n$ as the total sum of digits.

Also solved by B. Le F. Brown, C. E. Buell, W. E. Buker, W. L. Porter, M. O. Reade, C. W. Trigg, Simon Vatriquant and the proposer.

E 160 [1935, 392]. *Proposed by Daniel Finkel, Brooklyn, New York.*

If v, w, x, y and z are different digits, and $P=vwx$, $Q=wxv$, $R=xyz$, $S=zxy$ [note that these are not products], and if further, $4R=P$, $3R=Q$, $3S=P$; determine the values of the letters and show that the solution is unique.

Solution by William Douglas, Courtenay, British Columbia

x can not be 0, since $3(zxy)=(vwx)$.

x must be 1 or 2 since $4(xyz)=(vwx)$, a three-place number.

But $4(xyz)$ is even, so x is even, and we have $x=2$.

Then $4(2yz)=(vw2)$, so that v is 8 or 9.

Also $3(2yz)=(w2v)$, so that z is correspondingly 6 or 3.

But since $3(zxy)=(vw2)$, z can not be 6, so that $z=3$.

From this it follows that $v=9$.

Now $3(zxy)=(vwx)$ becomes $3(32y)=(9w2)$, so that $y=4$ and $w=7$.

Consequently, $P=972$, $Q=729$, $R=243$ and $S=324$. The uniqueness of this solution follows from its method of derivation.

The given statement that $3R=Q$ is superfluous information, as was pointed

out in the solutions of C. E. Buell, R. A. Johnson, Sidney Kaplan, Leon Recht, M. Y. Woodbridge, B. C. Zimmerman and the proposer.

Also solved by W. J. Berry, W. E. Buker, Mary L. Constable, H. L. Juditz, F. L. Manning, W. L. Porter, M. O. Reade, Domina E. Spencer, E. P. Starke, C. W. Trigg and Simon Vatriquant.

E 161 [1935, 393]. *Proposed by W. B. Clarke, San Jose, California.*

Construct a triangle whose circumcenter lies on its orthic triangle.

Solution by Simon Vatriquant, Brussels, Belgium.

Let ABC be the required triangle, and PQR its orthic triangle, where P, Q, and R are the feet of the altitudes from A, B, and C respectively. If we take P as origin and BC as the x-axis, the respective coordinates of A, B and C may be written as $(0, h)$, $(m, 0)$ and $(n, 0)$.

The equation of AB is $hx+my=mh$.

The slope of CR is therefore m/h, and its equation is $hy=m(x-n)$.

Since R is on both these lines, its coordinates must be

$$x = m(h^2 + mn)/(h^2 + m^2), \qquad y = mh(m - n)/(h^2 + m^2).$$

From this we deduce the equation of PR as $y=xh(m-n)/(h^2+m^2)$.

Now the equations of the perpendicular bisectors of BC and AB are respectively $2x=m+n$ and $2y-h=(2x-m)m/h$, so that the coordinates of the circumcenter, which lies at their intersection, are

$$x = (m + n)/2, \qquad y = (h^2 + mn)/2h.$$

Now if this circumcenter lies on PR, we obtain the condition

$$(h^2 + mn)^2/h^2 = m^2 - n^2.$$

Here the left side is the square of twice the ordinate of the circumcenter, and this equals the distance from A to the orthocenter. Consequently, if the circumcenter lies on the orthic triangle, then the difference of the squares of the projections of two sides on the third (which equals the difference of the squares of those two sides) equals the square of the distance from the orthocenter to the vertex opposite the third side.

The construction is now easy. Choose two points, B and C on any circle and draw the radius perpendicular to the one from B. Let this radius cut the chord BC at P, and there erect a perpendicular to BC cutting the circle at A_1 and A_2, the two possible positions for the third vertex of the triangle ABC, having the given circle as circumcircle, and having the center of that circle on its orthic triangle.

Also solved by R. A. Johnson and the proposer.

E 162 [1935, 393]. *Proposed by J. M. Feld, New York City.*

Factor as far as possible.

$$(kx - y + z)(x + ky - z)(x - y - kz) - (kx + y - z)(x - ky - z)(x - y + kz),$$

Solution by Walter Penney, Union City, N. J.

Since the given expression reduces to zero if we set $x=y$, or $y=z$, or $z=x$, we know that the original expression must equal $L(x-y)\ (x-z)\ (y-z)$, where L is a constant to be determined.

We evaluate L by letting $x=2$, $y=1$ and $z=0$, whence $L=2(k^2-1)$. Hence the complete factorization of the given expression is

$$2(k+1)(k-1)(x-y)(x-z)(y-z).$$

Also solved by W. E. Buker, Sidney Kaplan, M. O. Reade, E. P. Starke, J. F. Thomson, C. W. Trigg, Simon Vatriquant, M. Y. Woodbridge and the proposer.

E 163 [1935, 393]. *Proposed by W. A. Carver, Lakewood, Ohio.*

A man purchased at a post-office some one-cent stamps, three-fourths as many two's as one's, three-fourths as many five's as two's, and five eight-cent stamps. He paid for them all with a single bill, and there was no change. How many stamps of each kind did he buy?

Solution by C. C. Richtmeyer, Mt. Pleasant, Michigan.

Let $16x$, $12x$ and $9x$ be the respective numbers of one-cent, two-cent and five-cent stamps purchased. Then we have $85x+40=100k$, or

$$x=\frac{20k-8}{17}=k+\frac{3k-8}{17},$$

where k may have the possible values 1, 2, 5, 10, 20, 50, 100, 1000 or 10,000. On testing these values, $k=1000$ is found to be the only one for which x is an integer. Hence our gentleman purchased 18,816 one's, 14,112 two's, 10,584 five's and 5 eight's, totaling just \$1000.

Also solved by L. J. Adams, W. E. Buker, W. B. Clarke, Mary L. Constable, Wm. Douglas, Daniel Finkel, Sidney Kaplan, Sim Lasher, F. L. Manning, W. L. Porter, M. O. Reade, Domina E. Spencer, E. P. Starke, C. W. Trigg, Simon Vatriquant, M. Y. Woodbridge, E. N. Yeager, and B. C. Zimmerman.

E 164 [1935, 393]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A problem in the extraction of square root was worked out in the customary fashion and then a different letter was assigned to each different digit and substituted for that digit wherever it occurred. The result was:

$$\begin{array}{cccccc|l}
w & o & o & d & e & n & b\ e\ d \\
w & s & & & & & \\ \cline{1-2}
 & d & o & d & & & \\
 & d & e & s & & & a\ w\ e \\ \cline{2-4}
 & & s & n & e & n & \\
 & & s & n & e & n & a\ w\ s\ d \\ \cline{3-6}
\end{array}$$

Determine the numerical value of each letter and show that the solution is unique.

Solution by Leon Recht, College of the City of New York.

Since $2b = aw$, w is even and $a = 1$. Since $2(be) = aws$, s is even. The only perfect square that consists of two even digits is 64. Hence $w = 6$, $s = 4$, and $b = 8$. But $2(be) = aws$, or $2(8e) = 164$, so $e = 2$. Furthermore, $2(162) = des$, so $d = 3$. Hence the square root is uniquely equal to 823, from which its square is 677,329, and the complete reconstruction of the original problem is

$$\begin{array}{llllll|l}
6 & 7 & 7 & 3 & 2 & 9 & 8\ 2\ 3 \\
6 & 4 & & & & & \\
\hline
 & 3 & 7 & 3 & & & 1\ 6\ 2 \\
 & 3 & 2 & 4 & & & \\
\hline
 & & 4 & 9 & 2 & 9 & 1\ 6\ 4\ 3 \\
 & & 4 & 9 & 2 & 9 & \\
\hline
\end{array}$$

Also solved by W. E. Buker, W. B. Clarke, Mary L. Constable, Wm. Douglas, Daniel Finkel, H. L. Juditz, Sidney Kaplan, Evelyn M. Kennedy, F. L. Manning, M. O. Reade, Domina E. Spencer, E. P. Starke, C. W. Trigg, Simon Vatriquant, M. Y. Woodbridge, B. C. Zimmerman and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3764. *Proposed by J. S. Frame, Brown University.*

Prove that

$$\sinh^{n+1} nx + \cosh^{n+1} nx < \cosh^n (n+1)x,$$

for all real positive values of x, and for all real values of n greater than unity. Show that the inequality is reversed when $0 < n < 1$ and $x > 0$; whereas equality holds either if $n = 0$ or 1, or if $x = 0$.

3765. *Proposed by R. P. Agnew, Cornell University.*

The number x, $0 < x < 1$, is not representable in the form $k/2^n$ where k and n are positive integers. For each n there is a unique integer k_n such that

$(k_n-1)/2^n<x<k_n/2^n$. Show that for an infinite set of values of n, x is a point inside the middle third of this interval, i.e., for an infinite set of values of n

$$(3k_n - 2)/(3\cdot 2^n) < x < (3k_n - 1)/(3\cdot 2^n).$$

3766. *Proposed by Morris E. Levenson, New York.*

Evaluate

$$\int_0^\infty e^{-x} \log^2 x dx.$$

3767. *Proposed by N. A. Court, University of Oklahoma.*

Construct four spheres such that the sphere of similitude* of any two shall pass through the centers of the remaining two spheres.

3768. *Proposed by N. A. Court, University of Oklahoma.*

Construct a tetrahedron given in position the mid-points of the six edges.

SOLUTIONS

3686 [1934, 394]. *Proposed by Elmer Schuyler, Brooklyn, New York.*

Construct the triangle ABC, given the centers of the escribed circles corresponding to the sides BC and AB and the point of contact of BC with the inscribed circle.

Note by Harry Langman, Brooklyn, New York.

This problem is impossible of solution with compasses and straight edge.

Let O be the center of the inscribed circle, tangent to BC at S and AB at T. Let D be the center of the escribed circle corresponding to BC, tangent to BC at P and AB at V, and F the center of the one corresponding to AB. Draw AOD, a straight line. Draw OB, and let G be the projection of S on FD. Then FBD is straight, $OB \perp FD$, and $FA \perp AD$.

The points D, F and S are given. Hence G is given. Let $BG=x$, and set $SG=a$, $FD=b$, $GD=c$. Then

$$\overline{BS}^2 = x^2 + a^2. \tag{1}$$

The right triangles BSO and SGB are similar. Hence

$$\frac{OB}{BS} = \frac{BS}{SG} = \frac{SO}{GB},$$

whence

$$OB = \frac{x^2 + a^2}{a}, \qquad OS = \frac{x}{a}\sqrt{x^2 + a^2}. \tag{2}$$

---

* See, for instance, *Modern Pure Solid Geometry* by the proposer, 1935, The Macmillan Company, p. 153.

Similarly, $BD\cdot BG=BP\cdot BS$, from which

$$BP = \frac{(x + c)x}{\sqrt{x^2 + a^2}}. \tag{3}$$

Also, since $PD/BD=SG/BS$,

$$PD = \frac{a(x + c)}{\sqrt{x^2 + a^2}}. \tag{4}$$

Now

$$\overline{OD}^2 = \frac{(x^2 + a^2)^2 + a^2(x + c)^2}{a^2}. \tag{5}$$

Since $AD/FD=BD/OD$,

$$\overline{AD}^2 = \frac{a^2b^2(x + c)^2}{(x^2 + a^2)^2 + a^2(x + c)^2}. \tag{6}$$

Now $OT=OS$ and $DV=DP$. Since $OS/PD=AO/AD$, $(PD-OS)/PD=(AD-AO)/AD=OD/AD$. But

$$PD-OS = \frac{a(x + c)}{\sqrt{x^2 + a^2}} - \frac{x}{a}\sqrt{x^2 + a^2} = \frac{a^2(x + c) - x(x^2 + a^2)}{a\sqrt{x^2 + a^2}} = \frac{a^2c - x^3}{a\sqrt{x^2 + a^2}},$$

and this expression must be positive. Hence

$$\frac{a^2c - x^3}{a\sqrt{x^2 + a^2}}\cdot\frac{\sqrt{x^2 + a^2}}{a(x + c)} = \frac{(x^2 + a^2)^2 + a^2(x + c)^2}{a^2b(x + c)},$$

or

$$b(a^2c - x^3) = (x^2 + a^2)^2 + a^2(x + c)^2. \tag{7}$$

If we set

$$x = ya, \qquad c = pa, \qquad b = qa, \tag{8}$$

equation (7) becomes

$$y^4 + qy^3 + 3y^2 + 2py + p^2 - pq + 1 = 0. \tag{9}$$

For the root to be constructible, (9) must be reducible, with constructible coefficients. Depressing the third power, we obtain a reducing cubic in the square of one of the factor coefficients; and this cubic may be put into the form

$$u^3 + Au + B = 0, \tag{10}$$

where A and B are polynomials in p and q. The roots of (10), and hence those of (9), are not constructible if A and B are independent; and A and B are in-

dependent unless their Jacobian with respect to p and q vanishes identically. Examination of this Jacobian for particular numerical values of p and q shows that it does not vanish identically, and hence the problem is impossible of construction by compasses and straightedge.

3687 [1934, 394]. *Proposed by Melvin Dresher, Graduate Student, Yale University.*

If $S(i, j)$ denotes the sum of the divisors common to i and j, show that:

$$\begin{vmatrix} S(1,1) & S(1,2)\cdots S(1,n) \\ S(2,1) & S(2,2)\cdots S(2,n) \\ \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ S(n,1) & S(n,2)\cdots S(n,n) \end{vmatrix} = n!.$$

I. *Solution by M. A. Heaslet, Stanford University*

LEMMA: Let $n = p_1^{\alpha_1} p_2^{\alpha_2}, \cdots, p_t^{\alpha_t}$ then

$$\begin{aligned}(1)\qquad S(r, n) - \sum_{k=1}^{t} S\left(r, \frac{n}{p_k}\right) + \sum_{k,l=1}^{t} S\left(r, \frac{n}{p_k p_l}\right) \\ - \sum_{k,l,m=1}^{t} S\left(r, \frac{n}{p_k p_l p_m}\right) + \cdots = \begin{matrix} 0 \text{ when } r < n, \\ n \text{ when } r = n. \end{matrix}\end{aligned}$$

The proof of the Lemma follows from the following theorem (See P. Bachmann, *Niedere Zahlentheorie*, 1st Part, p. 96): If two functions $f(n)$, $\psi(n)$ exist such that for every integer $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}, \cdots$, and all its divisors $1, d, d' \cdots, n$

$$(2)\qquad f(n) = \psi(1) + \psi(d) + \psi(d') + \cdots + \psi(n)$$

then the function $\psi(n)$ can be expressed by

$$(3)\qquad \psi(n) = f(n) - \sum f\left(\frac{n}{p_1}\right) + \sum f\left(\frac{n}{p_1 p_2}\right) - \sum f\left(\frac{n}{p_1 p_2 p_3}\right) + \cdots$$

where the first summation extends over all different prime factors of n, the second over all combinations of two, and so on.

Setting $f(n) = S(r, n)$, r being fixed, and

$$\psi(k) = \begin{matrix} k, \text{ if } k \text{ divides } r, \\ 0, \text{ if } k \text{ does not divide } r, \end{matrix}$$

proves the Lemma.

PROOF OF THEOREM: From the last column of the determinant subtract each of the columns numbered n/p_k where p_k runs through all different primes dividing n, then add all columns numbered $n/p_k p_l$, subtract all numbered

$n/p_kp_lp_m$, etc. By the above Lemma the elements in the last column will then be

$$0, 0, 0, \cdots, n.$$

Expanding the determinant by means of the elements of the last column we obtain a determinant of order $n-1$ of the form of the given determinant multiplied by n.

By induction we may then conclude the original determinant has the value $n!$

II. *Solution by Harry Langman, Brooklyn, New York*

We observe that

(a) The determinant is symmetric.

(b) Since the common factors of k and $mk+h$, $h<k$, are the same as those for k and h, the first k elements in the kth row (or column) keep repeating.

The first column consists of 1's. Subtract this column from each of the others, then the first row from each of the others. The resulting determinant is still symmetric. The leading element is 1, the aligned elements 0. Each element (i, j) is now the sum of all factors of i and j, excluding 1. The principal minor of order 2 is now

$$\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix},$$

the second row of which keeps repeating horizontally and the second column vertically.

We next subtract the second column from all others of order $2k$. Now all the elements not in column 2 contain all the common factors of i and j, except 1 and 2. Now subtract the second row from all others of order $2k$. By (b), all the other elements in the first 2 rows or columns are 0. The resulting determinant is still symmetric and has the principal third-order minor

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix},$$

the third row of which keeps repeating horizontally.

Similarly, we next subtract column 3 from all others of order $3k$. The elements of these columns now exclude the factors 1, 2, 3. Subtracting the third row from all others of order $3k$, we obtain a symmetric determinant with

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

as a principal minor. By (b), all the other elements of the first three rows or columns are 0, and the fourth row keeps repeating horizontally.

Suppose we have reduced the determinant to the form where the principal minor of order k is

$$\begin{vmatrix} 1 & 0 \cdots 0 \\ 0 & 2 \cdots 0 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 \cdots k \end{vmatrix},$$

with all other elements of the same rows or columns 0, and such that all the remaining elements (i, j) have the possible common factors of i and j from 1 to k excluded. Then the $(k+1)$st element of the principal diagonal must be $k+1$, since all other possible smaller factors have been excluded. It should be observed that in any of the steps we have applied (b) is not vitiated. Hence the $(k+1)$st row or column consists in the number $k+1$ in every $(k+1)$st position, the other elements being 0. If we subtract the $(k+1)$st column from all others of order $m(k+1)$, the elements of these columns now exclude the factor $k+1$. Since no elements of other columns have this factor, all the elements not in the $(k+1)$st column now have all possible factors from 1 to $k+1$ subtracted. Subtracting now the $(k+1)$st row from all others of order $m(k+1)$, we have a symmetric determinant of the same form as the previous, with $k+1$ instead of k.

The determinant thus ultimately reduces to one where each element (i, i) is i and all others 0. The determinant thus has the value $n!$

Solved also by Frank Ayres, Jr., M. Coral, C. Denbow, Hansraj Gupta, and E. P. Starke.

Editorial Note. The theorem in I in regard to (2) and (3) was proved and used in the solution of 232 [1931, 539]. If we denote by $C(r, n)$ the expression in (1), then $C(r, n)$ is equal to n, if r is divisible by n, and it is zero if r is not divisible by n. This is easily proved directly. For $n=1$ the theorem is obvious. Each divisor in the expressions for S is a divisor of n; let d be such a divisor less than n. Then d contains at least one prime factor of n to a lower power than it occurs in n. For example, let $p_1, p_2, \cdots, p_k$ be the primes of n which appear to a lower power in d, including zero powers. In the $(j+1)$th term of (1) (the jth summation) d occurs precisely ${}_kC_j$ times with the sign $(-1)^j$. Hence d is eliminated, since

$$\sum_{j=0}^{k} (-1)^j {}_kC_j = 0.$$

Thus every divisor d in (1) is eliminated, with one possible exception; if r is divisible by n, the divisor n appears only once and in the first term. This proves the theorem.

It is easily shown that, if n_1 and n_2 are relatively prime integers

$$C(r, n_1n_2) = C(r, n_1)C(r, n_2). \tag{4}$$

There is an analogous theorem. Let d_{ij} denote the greatest common divisor of the integers i and j. For a given integer r set

$$\Delta_{rn} = d_{rn} - \sum_1 d_{rs} + \sum_2 d_{rs} - \sum_3 d_{rs} + \cdots, \tag{5}$$

where the integer under the Σ, for example 2, means that s runs through all the integers of the type n/p_1p_2. Then

$$\begin{aligned} \Delta_{rn} &= \phi(n), \text{ if } r \text{ is divisible by } n, \\ &= \quad 0 \quad, \text{ if } r \text{ is not divisible by } n, \end{aligned} \tag{6}$$

where $\phi(n)$ means the number of integers prime to n and less than n, including unity. There is also an equation corresponding to (4). By the use of (6) it easily follows that the determinant of the nth order with d_{ij} as the element in the ith row and jth column is equal to

$$\phi(1)\phi(2) \cdots \phi(n).$$

Proofs are given on pages 97–99 in the volume of Bachmann cited in I.

The solution by Starke used $C(r, n)$ in the same manner as in I. Starke made no use of (2) and (3), and in his proof of the part for which r is divisible by n he states that this may be proved by induction on r, or by a simple device due to H. J. S. Smith which is given in Dickson's *History of the Theory of Numbers*, vol. 1, p. 289. Starke remarks also that the problem is a special case of a theorem by Smith which is abstracted on page 123 in the above volume of Dickson.

Returning to the theorem in (2) and (3) of I, let $\psi(d_i)$ be uniquely defined for the range

$$R: \quad n = d_N > d_{N-1} > \cdots > d_2 > d_1 = 1,$$

consisting of all the divisors of n. There are N equations such as (3) for which the determinant of the system, in the above order, is unity. Hence there is a unique determination of the values of $f(d_i)$ expressed linearly in terms of the values of ψ. The coefficients are integers, and $f(d_i)$ involves ψ with arguments which do not exceed d_i numerically, and the coefficient of $\psi(d_i)$ is unity. The coefficients do not depend upon the nature of f and ψ; and any special case for which no $\psi(d_i)$ is zero may be used to find these coefficients. In the work by Smith in the first reference, his simple algebraic process shows that, if $\psi(d_i) = d_i^s$, where s is any given number, and $f(d_i) = \sigma_s(d_i)$, where $\sigma_s(d_i)$ means the sum of the s powers of the divisors of d_i, then (3) is satisfied by these functions for the system in which n runs through R. This shows that the unique determination of $f(n)$ must be (2).

Conversely, the system of N equations (2) may also be regarded as having the determinant unity. There is a unique solution for $\psi(n)$ expressed linearly in terms of $f(d_i)$; and the coefficients are integers independent of the nature of f and ψ. We know already that (3) is a solution; and, since the values of f are arbitrary, this solution is unique in the linear form.

In the volume of Bachmann cited, pages 90–96, it is shown without use of the formula for $\phi(n)$ in terms of n, i.e., using merely the definition of $\phi(n)$, that (2) is true for $\psi(n)=\phi(n)$ and $f(n)=n$. Hence (3) gives the usual formula for $\phi(n)$ after replacing in it $f(d_i)$ by d_i.

3688 [1934, 394]. *Proposed by W. D. Baten, University of Michigan.*

Show that

$$I = \sum_{r=0}^{n-1} (n-1)^{(r)} n^{(-r-1)} = 4^{n-1} B(n, n),$$

where

$$a^{(s)} = a(a-1)(a-2)\cdots(a-s+1), \qquad a^{(0)} = 1,$$

and

$$a^{(-s)} = \frac{1}{a(a+1)(a+2)\cdots(a+s-1)},$$

and $B(n, n)$ is the familiar beta function.

Solution by F. Underwood, University College, Nottingham, England.

Using the given definitions we may write

$$\begin{aligned} I &= \sum_{r=0}^{n-1} \frac{(n-1)!}{(n-r-1)!} \frac{(n-1)!}{(n+r)!} = \frac{[(n-1)!]^2}{(2n-1)!} \sum_{r=0}^{n-1} \binom{2n-1}{n-r-1} \\ &= B(n, n) \sum_{i=0}^{n-1} \binom{2n-1}{i} = \frac{B(n,n)}{2} \sum_{i=0}^{2n-1} \binom{2n-1}{i} \\ &= \frac{B(n,n)}{2} (1+1)^{2n-1} = 4^{n-1} B(n, n). \end{aligned}$$

Solved also by Roy MacKay, E. P. Starke and the proposer.

3689 [1934, 395]. *Proposed by Maud Willey, Long Beach, Miss.*

If the points $A_i (i=1, 2, 3, 4)$ are the vertices of a tetrahedron T, O is any point, B_i is the trace of line A_iO on the face of T opposite A_i, and C_i is the harmonic conjugate of O with respect to A_i and B_i; then each edge of T is coplanar with a line through two of the points C_i. If O is the centroid of T, it is also the centroid of the tetrahedron with the vertices C_i.

Solution by S. Vatriquant, Brussels, Belgium

The first part of the statement is obvious, for, since A_jC_j and A_kC_k intersect at O, A_jA_k and C_jC_k are coplanar.

We know that, if a point P divides a segment MN in the ratio $k:1$, then we have $P=(M+kN)/(1+k)$, and for Q the conjugate of P with respect to MN, $Q=(M-kN)/(1-k)$, denoting by the point itself the vector-coordinate from an arbitrary origin.

If O is the centroid of the A's, then it is also the centroid of the B's, for $B_1=(A_2+A_3+A_4)/3$, and then $\sum B_i=\sum A_i$.

We have also, since the centroid divides A_iB_i in the ratio 3:1, $O=(A_i+3B_i)/4$ and consequently $C_i=(A_i-3B_i)/(-2)$.

Thus $\sum C_i=(3\sum B_i-\sum A_i)/2=\sum A_i$, and O is also the centroid of the C's.

Solved also by J. W. Clawson, Harry Langman, Roy MacKay, and F. Underwood.

3690 [1934, 395]. *Proposed by W. M. Whyburn, University of California at Los Angeles.*

Solve the functional equation

$$f(x)f(-x) = c^2 = [f(0)]^2,$$

subject to the single restriction that $f(x)$ be a single valued, positive, real function of the real variable x.

Solution by Aaron Herschfeld, Columbia University.

We may define the general solution of the functional equation as the set $\{f\}$ of all functions satisfying the equation, and subject to the given restrictions of being single valued, positive, etc. This criterion enables us to verify that the following two solutions are each completely general.

1. We construct the general solution as follows: For $x>0$, $f(x)$ is single valued and positive, but otherwise arbitrary; $f(0)=|c|>0$; for $x<0$, $f(x)$ is defined by the relation

$$f(x) = [f(0)]^2/f(-x) = c^2/f(-x).$$

2. An apparently more explicit formulation of the general solution may be derived thus: Since $f(x)$ is positive and single valued, its real logarithm is real and single valued. Hence

$$F(x) = \log f(x) - \log f(0) = -[\log f(-x) - \log f(0)] = -F(-x).$$

Therefore

$$\log f(x) = \log f(0) + F(x), \qquad f(x) = f(0)e^{F(x)},$$

where $F(x)$ is an arbitrary odd, real, single valued function. Hence our general solution in terms of the arbitrary real constant c is

$$f(x) = |c| e^{F(x)}.$$

Solved also by E. B. Escott, J. M. Feld, H. Halperin, Harry Langman, E. C. Kennedy, Irving Segal, E. P. Starke, S. Vatriquant, and the proposer.

Editorial Note. Escott states that the problem is similar to Exercise 12, page 313, of Boole's *Calculus of Finite Differences*. Using Babbage's method, described on page 307 of that text, he found that $f(x) = c\phi(-x)/\phi(x)$, where $\phi(x)$ is an arbitrary positive function of x.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Volume I of the collected works of George Abram Miller has recently been published by the Illinois University Press, Urbana, Illinois. This volume contains reprints of articles published before 1900, as well as three articles relating to the history of group theory, which were written for this volume to exhibit the place of the author's work in the general development of the subject.

Part I of E. H. Moore's General Analysis has just been published in the Memoirs of the American Philosophical Society. It is a book of vi+251 pages devoted to the algebra of matrices in a form especially adapted to general analysis. The three parts to follow are entitled *The Fundamental Notions of General Analysis*, *Generalized Fourier Series and Modular Spaces*, and *The Characteristic Value Problem of General Analysis.*

Word has been just received that at its meeting held recently in Warsaw Professor J. D. Tamarkin, of the Department of Mathematics at Brown University had been elected a regular member of the class of mathematical and physical sciences of the Warsaw Society of Sciences. This is a high honor which has been conferred outside of Poland on only three other mathematicians among the thirty members in the class.

Harvard University expects to appoint two Benjamin Peirce Instructors in mathematics for the academic year 1936–37. These instructorships are intended to give promising young men the opportunity to combine their research activities with a moderate amount of teaching, including some advanced instruction. Applications should be sent to Professor W. C. Graustein, Chairman of the Division of Mathematics, by February 15, 1936.

Professor Harlow Shapley has resigned from the Harvard council of the faculty of arts and sciences. He will be replaced by Dr. Julian Lowell Coolidge, professor of mathematics and Master of Lowell House.

Professor Felix Bernstein has been awarded a research fellowship at Columbia University.

Associate Professor Tobias Dantzig has been promoted to a professorship in mathematics and has been appointed Executive Officer in the department of mathematics at the University of Maryland.

Dr. L. A. Dye of Cornell University has been appointed assistant professor of mathematics at The Citadel, Charleston, S. C.

S. E. Field, instructor in mathematics at Ironwood, Michigan, Junior College, has recently been appointed Registrar of the college.

Dr. C. C. Grove of the College of the City of New York has been promoted to an assistant professorship.

E. H. Hanson has been appointed professor of mathematics and director of the department at the North Texas Teachers College.

Assistant Professor L. M. Kells of the United States Naval Academy has been promoted to an associate professorship.

W. F. Kern of the United States Naval Academy has been promoted to an assistant professorship.

Dr. G. B. Lang of the University of Illinois has been appointed professor of science and mathematics at West Georgia College.

Dr. Hans Lewy of Brown University has been appointed a lecturer at the University of California.

G. A. Lyle, of the United States Naval Academy, has been promoted to an assistant professorship.

W. O. Menge of the University of Michigan has been promoted from assistant professor to associate professor.

Assistant Professor T. E. Mergendahl has been promoted to a professorship of mathematics at Tufts College.

Dr. Deane Montgomery, of Harvard University, has been appointed an assistant professor of mathematics at Smith College.

Professor G. T. Whyburn, of the University of Virginia, has been appointed chairman of the department of mathematics there.

Associate Professor L. T. Wilson, of the United States Naval Academy, has been promoted to a professorship.

The following appointments to instructorships in mathematics are announced:

A. and M. College of Texas: Dr. E. C. Klipple
University of Arizona: Dr. E. J. Purcell
Brown University: Dr. B. C. Getchell
College of the City of New York (evening session) Dr. H. E. Vaughan, Dr. D. S. Nathan
Colgate University: J. F. Wardwell
University of Florida: E. S. Quade
Johns Hopkins University: Dr. J. H. Curtiss
Kansas State College: Paul Cramer
Lafayette College: Dr. R. P. Bailey
Lehigh University: Dr. J. L. Vanderslice
University of Maryland: C. B. Tompkins
University of Minnesota: Dr. O. K. Sagen
Mt. Holyoke College: Dr. Frances E. Baker
University of Nebraska: Dr. Anna A. Stafford, M. S. Webster
University of Nevada: H. C. Ayres
Purdue University: Dr. A. H. Smith
University of Rochester: Dr. Wladimir Seidel
University of Texas: O. H. Hamilton
United States Naval Academy: Dr. A. E. Currier, J. R. Hammond
University of Wisconsin: Katherine S. Arnold, Dr. S. C. Kleene, Dr. K. W. Wegner

Dr. George Gailey Chambers, professor of mathematics at the University of Pennsylvania, died October 24, 1935, at the age of sixty-two. He had been a member of the University faculty for twenty-nine years. He was a charter member of the Mathematical Association.

Dr. A. B. Dinwiddie, since 1918 president of Tulane University, died on November 21, 1935, at the age of sixty-four. He had taught mathematics at Tulane University since 1906. He was a charter member of the Mathematical Association.

Associate Professor Raymond Garver, of the University of California at Los Angles, died on Thursday, November 7, 1935, after a brief illness. He had been a member of the Mathematical Association for about ten years.

Professor J. I. Hutchinson of Cornell University died on December 1, 1935, at the age of sixty-eight. He had been a teacher at Cornell since 1894.

Dr. Charles Dunham Rose, professor emeritus of mathematics and astronomy at Nebraska Wesleyan University, died September 30, 1935 at the age of seventy-six.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N. Y.

BOOKS FOR REVIEW should be addressed to R. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Summer Meeting, Harvard University, Aug. 31, 1936.

Twenty-first Annual Meeting, Washington, D.C., Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN.
ILLINOIS, May 8–9.
INDIANA, North Manchester, May.
IOWA, April.
KANSAS, March.
KENTUCKY, May, November.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 4.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, May.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Mar.
SOUTHERN CALIFORNIA, Mar. 7.
TEXAS, College Station, April.
WISCONSIN, May 2.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 2, FEBRUARY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE TWELFTH ANNUAL MEETING OF THE NEBRASKA SECTION

The twelfth annual meeting of the Nebraska Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Nebraska Academy of Sciences at the University of Nebraska, Lincoln, Nebraska, Friday afternoon, May 3, 1935, with Professor A. L. Candy of the University of Nebraska as chairman.

The attendance was thirty-one, including the following fifteen members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. L. Candy, J. A. Daum, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, A. L. Hill, J. M. Howie, R. M. McDill, T. A. Pierce, Lulu L. Runge, R. B. Thompson.

Officers for the ensuing year were elected as follows: Chairman, J. M. Howie, Nebraska Wesleyan University; Secretary-Treasurer, T. A. Pierce, University of Nebraska; Member of Executive Committee, R. M. McDill, Hastings College.

The following program was presented:

1. "A summation formula for alternating series" by Professor C. C. Camp, University of Nebraska.

2. "Some easy methods of drawing curves in polar coordinates" by Professor A. K. Bettinger, Creighton University.

3. "The interdependence of high school and college mathematics." A round table discussion led by Professor A. R. Congdon, University of Nebraska, Professor R. M. McDill, Hastings College, and Professor J. M. Howie, Nebraska Wesleyan University.

4. A demonstration of the new and improved model of Father Rigge's machine for tracing harmonic curves, by Professor M. G. Gaba, University of Nebraska.

Abstracts of the papers follow:

1. A very powerful method for summing finite or infinite alternating series is the analogue of the Euler-Maclaurin summation formula for series with positive signs. Thus, if $f(x)$ is analytic, $a \leqq x \leqq b$, then

$$f(a+\omega) - f(a+3\omega) + f(a+5\omega) - \cdots \pm f(b-\omega) = \tfrac{1}{2}[f(a) \pm f(b)] \\ - \tfrac{1}{2}E_1\omega^2[f^{\mathrm{II}}(a) \pm f^{\mathrm{II}}(b)]/2! + \tfrac{1}{2}E_2\omega^4[f^{\mathrm{IV}}(a) \pm f^{\mathrm{IV}}(b)]/4! - \cdots,$$

where $E_1=1$, $E_2=5$, $E_3=61$, etc., are Euler's numbers. If $f^{2n+1}(x)$ is of constant sign, $a \leqq x \leqq b$, the remainder is numerically less than the last term used; i.e., $\frac{1}{2}(-1)^n E_n\omega^{2n}[f^{2n}(a) \pm f^{2n}(b)]/(2n)!$. Professor Camp illustrated the method on the series $\pi/4 = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots$ by adding the first ten terms arithmetically and applying the formula to the rest.

2. Professor Bettinger showed that any curve of the type

$$r = \frac{(a + b\sin n\theta + c\cos n\theta)^\alpha}{(d + e\sin n\theta + f\cos n\theta)^\beta}$$

where a, b, c, d, e, f may be real numbers, α, β rational numbers, and n any positive rational number, can be sketched easily by the following device. If n be a positive integer draw radial lines dividing each quadrant into n equal sections. If b be of the form $1/m$ where m is a positive integer use m quadrants to form one section. If n be of the form p/q where p and q are positive integers, the quadrants are divided into p equal subsections and q of them are grouped to form a major section. Calculate the four values of r when $n\theta = 0°, 90°, 180°, 270°$. Starting then on the line $\theta = 0°$ mark off successively on each radial line in counter-clockwise direction one value of r, repeating the set of four values as often as necessary. This method is particularly effective for the curves one meets ordinarily in the calculus.

3. Professor Congdon pointed out that the marked decrease in the percentage of high school pupils studying mathematics means a corresponding decrease in college enrollment in freshman mathematics. More states are passing laws eliminating mathematics as a requirement for high school graduation. Colleges should graduate prospective teachers who are capable of appealing to the interests of the pupils and who can convince them that a study of mathematics is worth while and will improve their efficiency.

In considering how mathematics can get and hold the place it deserves in modern education, Professor McDill expressed the thought that the teacher should be an all-round man, in touch with the church, the political community and the business world. He suggested methods of getting in touch with prospective freshmen and of organizing the state sections of the Association so as to make them more effective in dealing with local questions.

Professor Howie feels that the mortality in first year algebra with the decreasing number of pupils who study geometry and third and fourth year mathematics in the high school is due in a great extent to the failure of the teachers to create interest in their subject, failure to show the value and importance of mathematics in life. Since these teachers have been our students, it is evident we have failed to inspire them. More time should be given to college algebra, thus laying a better foundation for future work in analytics and calculus. If colleges prepare better high school teachers, they in turn will receive better prepared freshmen.

J. M. HOWIE, *Secretary*

THE NINETEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The nineteenth annual meeting of the Kentucky Section of the Mathematical Association of America was held in joint session with the Kentucky Academy of Science at the University of Kentucky, Lexington, on Saturday, May 4, 1935. Professor W. R. Hutcherson, chairman of the Section, presided at both sessions.

The attendance was forty-seven, including the following nineteen members of the Association: N. B. Allison, M. C. Brown, L. W. Cohen, H. H. Downing,

A. R. Fehn, Charles Hatfield, W. R. Hutcherson, F. Elizabeth Le Stourgeon, Buena C. Mathis, W. L. Miser, W. L. Moore, Sister Charles Mary Morrison, R. S. Park, D. W. Pugsley, Guy Stevenson, D. E. South, Sister M. Domitilla Thuener, H. A. Wright, H. M. Yarbrough.

Because mathematicians of the A.A.A.S. from the North Central States have scheduled a meeting for next fall at the University of Kentucky, it was agreed that we plan no meeting of our own.

Professor Miser's invitation to join with the Tennessee members and have a joint meeting in the Fall of 1936 at Vanderbilt University, Nashville, was accepted.

Chairman Hutcherson announced that the annual spring meeting of the Kentucky Section in 1936 will be held jointly with the Kentucky Academy of Science at Eastern State Teachers College, Richmond, Kentucky.

Professor R. S. Park, Eastern State Teachers College, was elected chairman, and Professor A. R. Fehn, secretary for the year 1935–1936.

The question of the advisability of this Section taking a definite and active part in stressing the importance of and in stimulating interest in secondary mathematics was freely discussed. It was suggested that those members who so desired should get in touch with teachers of secondary mathematics in their own and surrounding towns, and that the chairman should arrange to meet teachers of secondary mathematics next fall when they come to the meeting of the Central Kentucky Educational Association at Richmond.

The following papers were given:

1. "On matrices and their minimum equations" by N. B. Allison, University of Kentucky.
2. "The equation of the straight line" by Professor H. A. Wright, Transylvania College.
3. "The number and reality of the self-symmetric quadrilaterals in and circumscribed to the triangular-symmetric rational quartic" by Sister M. Domitilla Thuener, Villa Madonna College.
4. "A comparison of methods of determining the meridian by observations of the sun and Polaris" by Professor D. W. Pugsley, Berea College.
5. "Excluded regions in tracing algebraic curves" by Professor W. L. Moore, University of Louisville.
6. "Report concerning the National Council of Teachers of Mathematics" by Buena C. Mathis, Bryant Station High School.
7. "Some applications of mathematics to genetics" by Professor J. S. Bangson, Berea College, introduced by Professor Hutcherson.
8. "Law of mortality" by Professor D. E. South, University of Kentucky.
9. "On the theorem of Hadamard" by Dr. R. V. Bennett, Kentucky Wesleyan College, introduced by Professor Hutcherson.
10. "Descartes method of tangents" by Professor W. L. Miser, Vanderbilt University.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

2. Professor Wright pointed out that the slope-intercept form of the equation of the straight line may be used to advantage for all sets of conditions rather than to use all four forms usually listed in texts. It tells as much as any of the other forms, is as economical to use as any or all forms, and most important of all it portrays functional relationships more clearly than any other and affords a fine opportunity to introduce, in a simple way, the concept "rate of change" in anticipation of the calculus.

3. Sister Domitilla determined the number and reality of the self-symmetric quadrilaterals in-and-circumscribed to the triangular-symmetric rational quartic. The parameter was chosen in such a way that points on the curve, symmetric with respect to one of the three lines of symmetry, have reciprocal values. A correspondence was set up between these reciprocal values which resulted in a twelfth degree equation in sigma, where sigma is equal to the sum of the parameter and its reciprocal. After elimination of all extraneous factors, the equation determining the number of quadrilaterals proved to be of the fourth degree. Since each root of the equation gives one pair of opposite vertices of a quadrilateral, there are two quadrilaterals for each line of symmetry. The reality of the quadrilaterals was determined by a consideration of the relation between sigma and rho, the latter being the characteristic constant that determines the type of the quartic curve. It was found that one of the two quadrilaterals is always imaginary; the other is real for a limited range of values of rho in the case of the trioval-shaped quartic and acnodal-shaped quartic, and for the entire range of values in the case of the loop-shaped quartic.

5. This paper by Dr. W. L. Moore was expository in nature and dealt with a chapter in Frost's Curve Tracing. It was introduced because the author felt that this method could profitably be used to trace curves met with in elementary calculus and because it is not found in the texts on analytic geometry that he has read.

6. As a delegate from the Kentucky Section of the Association, Miss Mathis attended the annual meeting of the National Council of Teachers of Mathematics held in Atlantic City in February in connection with the N.E.A. Approximately two hundred persons attended the sessions during the two days. "Transfer of training" was the theme of the opening program but the principal program concerned "Arithmetic," the subject of the tenth yearbook. In the discussion following this report the Section voiced its support of the activities of the Council and promised cooperation with any endeavor to improve the status of high school and elementary mathematics.

7. Professor Bangson stated that without the application of mathematics, genetics would be greatly handicapped. In the study of variation, quantitative particularly, and qualitative also, the mean, standard deviation and coefficient of variation, together with their probable errors, are constantly used. The expanded binomial indicates the normal curve of distribution, and provides theo-

retical ratios in the second generation produced by inbreeding and it explains aberrant sex ratios. The applicability of Mendelian ratios are determined by the employment of methods like goodness of fit, "student's," and the chi square. Comparisons of experimental treatments and their significance are determined through mathematical procedures, as well as correlation of characters and linkage intensities.

A. R. FEHN, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Teresa, Winona, Minnesota, on Saturday, May 11, 1935. One session was held at 2:00 o'clock with a luncheon at 12:30 o'clock.

In the absence of Sister Thomas à Kempis, chairman of the Section, Professor R. W. Brink presided. Ninety-six persons attended the meeting including the following nineteen members of the Association: R. W. Brink, L. E. Bush, W. H. Bussey, C. S. Carlson, Elizabeth Carlson, Sister M. Claudette, Gladys Gibbens, C. H. Gingrich, W. L. Hart, Dunham Jackson, C. M. Jensen, W. H. Kirchner, Marie M. Ness, M. G. Scherberg, R. R. Shumway, F. J. Taylor, A. L. Underhill, Marion B. White, G. L. Winkelmann.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of the College of St. Teresa, and the efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman, Inez Rundstrom, Gustavus Adolphus College; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: Sister Thomas à Kempis, College of St. Teresa; C. S. Carlson, St. Olaf College; G. L. Winkelmann, St. Johns University.

The following eight papers were presented:

1. "On the generalization of the principle of reflection" by Professor H. P. Thielman, College of St. Thomas, introduced by Professor Bush.

2. "Some remarks on angle trisection" by Professor W. H. Bussey, University of Minnesota.

3. "Classification of conic sections" by Professor A. H. Diamond, College of St. Scholastica, introduced by the Secretary.

4. "Biorthogonal matrices" by Professor Dunham Jackson, University of Minnesota.

5. "Crude approximations to an amortization rate" by Professor W. L. Hart, University of Minnesota.

6. "The mathematics curriculum in the small college" by Professor L. E. Bush, College of St. Thomas.

7. "A note on the cross ratio of four fixed tangents to a conic" by Professor C. S. Carlson, St. Olaf College.

8. "Mathematics preparation of students in pre-business statistics" by Professor W. L. Hart and Joseph Daoust, University of Minnesota.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. The image with respect to a given curve of a point z_1 of the complex plane z is defined as a point z_2 which is such that if the plane z be transformed conformally into a plane w in such a way that the given curve goes over into the axis of reals (or a part of it), the two points z_1 and z_2 go over into conjugate points, say $w_1=a+bi$, $w_2=a-bi$. This definition of images, and hence that of the concept of reflection, reduces to that of inversion in case the given curve is a circle, and in general is equivalent to that of electrical and light images. Transformations were given by Professor Thielman for the reflection with respect to any straight line and any conic section. With certain restrictions on the type of conformal representations employed, this reflection is unique.

2. Professor Bussey's "Remarks on angle trisection" called attention to the fact that the trisection of an angle by means of a carpenter's square published by H. T. Scudder in this MONTHLY for May, 1928 (vol. 35, p. 230), is related to the trisection by means of an auxiliary curve which was published by P. Delanges in Verona, in 1783, under the title *La trisegante nuova curva.* The curve is described by Gino Loria in his *Spezielle algebraische und transzendente ebene Kurven* (vol. 1, pp. 231–232). He calls the curve by the name "trisekante" and gives its equations in polar and cartesian forms as

$$\tfrac{1}{2}a = \rho \cos \tfrac{1}{2}\omega; \qquad (a^2 - y^2)(x^2 + y^2) - \tfrac{1}{4}a^4 = 0.$$

To connect this curve with Scudder's trisection, Bussey used an alternative polar form, $a/2=\rho \sin(\omega/2)$. For $a=4$, this becomes $2=\rho\sin(\omega/2)$, which will be the equation of the locus of the point F of Scudder's diagram, as the angle ABC varies in size, if B be taken as the origin and the line BA as the polar axis. Loria notes that this "trisekante" curve was discovered independently by an American mathematician, W. Hillhouse, a hundred years after its publication by Delanges (*On a new curve for the trisection of an angle*, *Analyst*, vol. 9, 1882, pp. 181–185).

3. Professor Diamond considered the analogue for conic sections of the discriminating cubic for quadric surfaces and showed how it leads to the usual rules for determining the nature of the second degree equation written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

4. Professor Jackson's paper is concerned with the formal expansion of a matrix $\phi(x) \equiv [f_{ij}(x)]$, $i, j=1, 2$, in series of the form

$$\sum_{-\infty}^{\infty} \alpha_n \gamma_n(x), \qquad \gamma_n(x) = \begin{vmatrix} \cos nx & -\sin nx \\ \sin nx & \cos nx \end{vmatrix}$$

and the coefficients α_n are two-rowed matrices of constants. The result is merely the matrix of the Fourier series for the four functions $f_{ij}(x)$ respectively.

5. Suppose that a debt of $\$A$ is to be discharged, principal and interest included, by payments of $\$R$ at the end of each interest period, for n periods. Then, the rate of interest, i, per conversion period satisfies the equation $A = R(a_{\overline{n}|} \text{ at } i)$. Professor Hart called attention to the fact that in some texts on elementary arithmetic indefensibly crude approximations to the rate i are obtained without the theory of annuities. He recommended use of the following approximate rate r which can be sensibly introduced without annuity theory and which, fortunately, bears a reasonable resemblance to the true value of i:

$$r = \left(R - \frac{A}{n}\right) \Big/ \left[\frac{1}{2}\left(A + \frac{A}{n}\right)\right]. \tag{1}$$

The denominator in (1) may be described as the *average* outstanding principal and the numerator as the *average* interest payment; A/n is the average payment of principal. It is found that r is the sum of an infinite series in ascending powers of i:

$$r = i + \frac{n-1}{6} i^2 + \cdots . \tag{2}$$

6. Professor Bush spoke of the difficulty the small college has to adjust the first-year mathematics curriculum to the varying amount and quality of preparation of the entering students. There is always a large number of misfits when segregation is based only on high school grades and special tests. At the College of St. Thomas freshmen are classified into the three following groups (1) all students offering one and one-half units of algebra and who intend to major in the physical sciences, mathematics, or engineering or who make exceptionally high scores on their entrance test; (2) all other students offering one and one-half units of algebra; (3) all students offering only one unit of algebra.

Algebra sections are scheduled at the same hour for each of the three groups. (1) meets six times per week, (2) and (3) four times per week. (1) and (2) are given the regular course in college algebra while (3) takes a non-credit course preparatory to college algebra. Weekly quizzes are given in all groups, and at any time during the first semester that a student in (1) or (2) falls behind in his work he is transferred to (2) or (3). Since all three groups meet at the same hour these transfers can be made without disturbing the rest of the student's schedule. In this way we have almost completely eliminated failures in groups (1) and (2), and many students who almost certainly would have failed if they had remained in the group to which they were originally assigned are able to pass creditably in the lower group. It is to be noted that instead of holding back the better prepared students and pushing forward those with less preparation so as to bring about uniformity of progress for the class as a whole, we push ahead the better prepared group in a six hour per week section, while holding back the less well prepared (who also usually show less aptitude for mathematics) to a pace which enables them to get a thorough training in the fundamentals of algebra before attempting more advanced courses.

7. Professor Carlson showed how a different choice of sequence of available theorems provides a direct proof, without the use of the theorem on projective properties of the cross-ratio or the principle of duality, of the theorem that the cross-ratio of the range of points formed by a variable fifth tangent intersecting four fixed tangents to a conic section is constant.

8. A required first course in statistics, slightly mathematical in its viewpoint, is offered by the Department of Economics at the University of Minnesota for pre-business students in their sophomore year. This course has no mathematical prerequisites beyond two units of high school mathematics. Professor Hart discussed the reasons which led him to initiate a study of the performance of 400 students who took this course in statistics during 1933–1934; his main object was to ascertain what effect, if any, on achievement in statistics was produced by mathematical training beyond the allowable minimum. Mr. Daoust described the data available and presented some of the results of a statistical investigation which he is performing in regard to the specified students. On the basis of his incomplete results, it appears that the final conclusions will probably be favorable from the viewpoint of one who is interested in elevating the mathematical plane of courses in elementary statistics.

A. L. UNDERHILL, *Secretary*

A PROOF OF THE CORNER CONDITIONS IN THE CALCULUS OF VARIATIONS

By J. D. MANCILL, University of Alabama

The simplest type of parametric problem of the calculus of variations is that of minimizing an integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y')dt$$

in a class of admissible curves

$$(1) \qquad x = x(t), \qquad y = y(t) \qquad (t_1 \leqq t \leqq t_2)$$

joining two fixed points 1 and 2 in the xy-plane.* We shall suppose that the integrand function $F(x, y, x', y')$ satisfies the usual continuity and homogeneity properties for (x, y) in a region R of the plane and all $(x', y') \neq (0, 0)$.† A curve (1) will be called *admissible* if it lies in the region R, is continuous, and has a continuously turning tangent except possibly at a finite number of corners.

The usual derivation of the conditions that must be satisfied at each corner of the minimizing curve, which state that

* We shall denote all derivatives with respect to the independent variable t by primes and all others by subscripts.

† See, e.g., Bolza, *Lectures on the calculus of variations*, University of Chicago, p. 117.

$$
\begin{aligned}
F_{x'}(x, y, \bar{x}', \bar{y}') - F_{x'}(x, y, x', y') &= 0 \\
F_{y'}(x, y, \bar{x}', \bar{y}') - F_{y'}(x, y, x', y') &= 0,
\end{aligned} \tag{2}
$$

where the elements $(x, y, \bar{x}', \bar{y}')$ and (x, y, x', y') are those of the minimizing curve at the corner, is made to depend upon the vanishing of the first variation.* Such a derivation of conditions (2) is not applicable to arcs of the minimizing curve lying along the boundary of the region R, along which the first variation need not vanish. The proof of these conditions to be given here is independent of the extremal property of the minimizing curve and thus is valid for corners of arcs of the type just mentioned at which the construction used in the proof is admissible.† The method of proof is simple and may be applied in any number of dimensions and to non-parametric problems.

Suppose that the minimizing curve has a corner at the point 3, and let the arc preceding the corner be denoted by

$$E_{13}: \quad x = x(t), \qquad y = y(t) \qquad\qquad (t_1 \leqq t \leqq t_3),$$

and the arc beyond the corner by

$$E_{32}: \quad x = \bar{x}(t), \qquad y = \bar{y}(t) \qquad\qquad (t_3 \leqq t \leqq t_2).$$

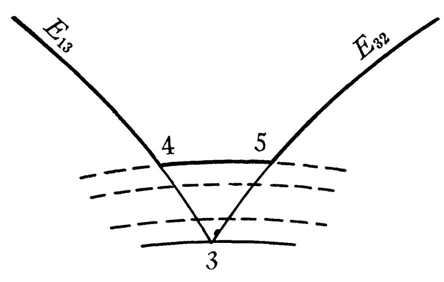
Let

$$x = \phi(s, a), \quad y = \psi(s, a)$$

be the equations of a one parameter family of curves C_a, intersecting E_{13} for $s = s_4(a)$, $t = t(a)$ and intersecting E_{32} for $s = s_5(a)$, $t = \bar{t}(a)$. The incidence relations of the curves are expressed by the equations

$$
\begin{aligned}
x[t(a)] &= \phi[s_4(a), a], \qquad & \bar{x}[\bar{t}(a)] &= \phi[s_5(a), a] \\
y[t(a)] &= \psi[s_4(a), a], \qquad & \bar{y}[\bar{t}(a)] &= \psi[s_5(a), a].
\end{aligned} \tag{3}
$$

Now consider the admissible comparison curve $L_a = E_{14} + C_{45} + E_{52}$, of the type indicated by the heavy line in the figure.



Let

* See, e.g., Bolza, loc. cit., p. 126.

† See, Mancill, *The minimum of a definite integral with respect to unilateral variations*, Dissertation, University of Chicago, pp. 46–53.

$$I(a) \equiv I(L_a) = \int_1^4 F(x, y, x', y')dt + \int_4^5 F(\phi, \psi, \phi_s, \psi_s)ds + \int_5^2 F(\bar{x}, \bar{y}, \bar{x}', \bar{y}')dt.$$

Then by means of the well known homogeneity properties of the function $F(x, y, x', y')$, the four relations (3) and their first derivatives with respect to a, and the fact that $I(a_3) = I(E_{12})$ is supposed to be a minimum value of I, we find that

$$I_a(a_3) = t_a(a_3)E(\phi, \psi, \phi_s, \psi_s, x', y')^3 - \bar{t}_a(a_3)E(\phi, \psi, \phi_s, \psi_s, \bar{x}', \bar{y}')^3 \leqq 0.$$

We may suppose that the function $t(a)$ increases with a. Consequently, we may choose the parameter t so that $t_a(a_3) = 1$ and $\bar{t}_a(a_3) = -1$. Let the direction (ϕ_s, ψ_s) of the curve C_{a_3} through the point 3 approach the direction (x', y') of the arc E_{13} at 3. Then we have

$$I_a(a_3) = E(x, y, x', y', \bar{x}', \bar{y}')^3 \leqq 0. \tag{4}$$

While the condition contained in the inequality (4) is weaker than the familiar form of the corner conditions (2), its derivation does not depend upon any of the other conditions necessary for a minimum. If we combine the inequality (4) with that contained in the condition of Weierstrass,* we obtain

$$I_a(a_3) = E(x, y, x', y', \bar{x}', \bar{y}')^3 = 0.$$

This equality and the inequality $E(x, y, x', y', p, q)^3 \geqq 0$ for all $(p, q) \neq (0, 0)$ are equivalent to the two conditions (2). For, it follows that

$$E_p(x, y, x', y', \bar{x}', \bar{y}') = F_{x'}(x, y, \bar{x}', \bar{y}') - F_{x'}(x, y, x', y') = 0$$

and similarly for $E_q(x, y, x', y', \bar{x}', \bar{y}')$.

RESTRICTED THEORY OF RELATIVITY IN TERMS OF HYPERBOLIC FUNCTIONS OF RAPIDITIES

By VLADIMIR KARAPETOFF, Cornell University

The purpose of this article is primarily expository; it is intended to give a simple and unified presentation of the principal results of the restricted theory of relativity, already known from a somewhat different point of view. The author directly postulates such properties of a space-time manifold that the velocity of light comes out the same for all observers. The corresponding fundamental equations for the transformation of coordinates contain hyperbolic functions of a real variable. While an occasional use of hyperbolic functions in the theory of restricted relativity is not new (see, for example, Bibl. 3, pp. 58 and 73), it is believed that a systematic treatment on this basis gives a much more concise picture of the subject. Of course, the reader is assumed to be willing to study kinematic relations in a non-Euclidean time-space continuum.

---

* Graves, *A proof of the Weierstrass condition in the calculus of variations*, this MONTHLY, vol. 41 (1934), pp. 502–504.

The present analytical treatment is alternative to the graphical one previously given by the author in terms of oblique coordinates in a pseudo-Euclidean space (see References 5–10). The analytical treatment is more elegant and simple, and closely parallels non-relativistic mechanics in our "common-sense" time-space manifold. The reader will find the hyperbolic manifold particularly clear and satisfying in the theory of the relativistic dynamics of a material particle.*

We shall assume, as usual, two observers, S and S', moving relatively to each other along a straight line XX', at a constant velocity q. As viewed by the reader, let S move to the left and S' to the right. The problem is to correlate various simple events, involving moving bodies and radiations, as measured by the two observers. These events cannot be correlated in the usual "common-sense" or non-relativistic manner, because in the world under consideration the velocity of light in vacuum comes out the same no matter what the velocity of the observer is relative to the source of light. Thus, the time-space relationship in the S-S' world is different from ours and must be postulated anew. If we know a kinematic event, as described say by the S observer, then to find the same event as it would be described by S', it is necessary to know the relationship between the space-time coordinates of the two observers.

Let an event consist in some non-uniform motion of a material point along the direction XX' of the relative motion of the two observers. Let the S observer choose a point at rest to himself as origin and let him make a series of instantaneous observations of distances of the moving point from this origin. Taking these distances, x, as abscissas, and the corresponding instants of time, t, on his clock as ordinates, he can plot a curve which will graphically represent the motion of the point and thus be his record of the event. If we knew how to transform these coordinates into the corresponding ones, x' and t', of the other observer, we could predict the record of the same event by the latter. We shall introduce the following postulate whose justification lies in the fact that all the known results of restricted relativity deduced from it agree with those deduced by the other methods: Thus, we shall write (c being the velocity of light in vacuum):

$$x' = x \cosh u - ct \sinh u \tag{1}$$

$$ct' = ct \cosh u - x \sinh u. \tag{2}$$

In the world of the two observers the transformation of time-space coordinates from one observer to the other, equations (1) and (2), obeys formulas similar to those which obtain in a transformation from one two-dimensional orthogonal XY system to another ($X'Y'$), except that hyperbolic functions are used in place of circular ones. The angle u between the two systems of coordi-

---

* Throughout the article, when Euclidean or hyperbolic space is mentioned, *space-time* is understood. By the hyperbolic unidimensional space-time is meant a continuum which satisfies eqs. (1) and (2).

nates will be shown to be a simple function of the relative velocity of the two observers. Throughout the article, all motions are assumed to take place in the direction of the XX' axis, so that the y and z coordinates of the two observers remain the same. Except that a minus sign enters in both formulas and that hyperbolic sines and cosines are used in place of circular ones, the foregoing two formulas are the same as will be found for transformation of orthogonal coordinates in any elementary book on analytic geometry. The clocks of the two observers are so set that both indicate zero when the two origins are in coincidence. The material point under observation is supposed to pass through the common origin at zero hour. In other words, the foregoing equations satisfy the condition that $x=x'=0$ when $t'=0$. This added condition simplifies the theory without making it less general.

In the foregoing equations x and ct are expressed in the same units of length. We may think of ct as the distance covered by a fictitious point moving at the velocity of light during the corresponding interval of time. For example, $ct=6000$ km. means an interval of time equal to 0.02 sec., for it takes that length of time for a beam of light in vacuum to cover 6000 km. Accordingly, the condition $x=ct$ signifies a point moving at the velocity of light. Squaring the two equations and subtracting one from the other, gives

$$(x')^2 - (ct')^2 = x^2 - (ct)^2. \tag{3}$$

If the S observer finds a point to move at the velocity of light, he records $x=ct$. But (3) then gives $x'=ct'$; in other words, the S' observer also finds the same point moving at the velocity of light, in spite of the relative motion between the two observers. Since this condition, namely that the velocity of light should be the same for all observers, is initially prescribed for the S-S' world, we see that equations (1) and (2) correctly characterize the space-time under consideration.

Solving equations (1) and (2) for x and t, we find

$$x = x' \cosh u + ct' \sinh u \tag{4}$$

$$ct = ct' \cosh u + x' \sinh u. \tag{5}$$

It is sometimes more convenient to use these equations than equations (1) and (2) from which they have been derived.

In order to see the significance of the hyperbolic angle u, consider a very simple "event," namely the observer S' staying at his origin. This event is for him expressed by $x'=0$ and the time t' is indefinite. Substituting $x'=0$ in (1) we find that the same event is characterized by the S observer as

$$x/t = c \tanh u. \tag{6}$$

But in this particular case x/t is the rate at which the S' origin is moving away from S; in other words, x/t is the relative velocity q of the two observers. Hence

$$q = c \tanh u, \quad \text{or} \quad u = \tanh^{-1}(q/c). \tag{7}$$

It is also of interest to note that

(7a) $$\cosh u = 1/\sqrt{1-(q/c)^2} \quad \text{and} \quad \sinh u = (q/c)/\sqrt{1-(q/c)^2}.$$

Since both q and c are expressed in the same units, u is a numeric which has been named by Robb the *rapidity* of a point, a body, or a system. According to equations (7), when q increases from zero to c, the corresponding rapidity increases from zero to infinity. It will be shown below that in the space under consideration motions are expressed in a much more natural way through rapidities than through velocities. Moreover, the above relationship between u and q shows why, in the space under consideration, the observed velocity can never be greater than that of light; the hyperbolic tangent of a real angle is always less than 1 and the hyperbolic tangent of infinity is 1.*

1. *Relativity Transformation of Velocities and of Rapidities*

A material point moves at a uniform velocity along the line of relative motion of the two observers; its velocities, as measured by S and S' respectively, are v and v'. It is required to find the relationship between v and v' in terms of the relative velocity q of the two observers, and also the relationship among the three rapidities corresponding to these velocities. By analogy with equation (7), we introduce rapidities w and w', as follows:

(8) $$v = c \tanh w \quad \text{and} \quad v' = c \tanh w'.$$

Dividing (4) by (5), we get

$$v = x/t = (x' \cosh u + ct' \sinh u)/[(t' \cosh u + (x'/c) \sinh u].$$

Hence

(9) $$v = [(x'/t') + c \tanh u]/[1 + (x'/ct') \tanh u].$$

Substituting the rapidities in place of $v = x/t$ and $v' = x'/t'$, we obtain

$$\tanh w = \tanh (w' + u)$$

and consequently

(10) $$w = w' + u.$$

Thus, *rapidities are added in this space directly*, just as velocities are added in our daily experience. On the other hand, the law of combination of velocities in a relativistic space is more complex. From (9) we have

(11) $$v = (v' + q)/[1 + (v'q/c^2)].$$

* In the author's article (Ref. 9) in which he uses oblique coordinates, an angle α is introduced which is related to the velocity q by the equation $q/c = \sin \alpha$, and is named the velocity angle. It will be seen that $\sin \alpha = \tanh u$. In hyperbolic trigonometry, when two angles are connected in this manner, α is known as the Gudermannian angle of u, and conversely, u is the anti-Gudermannian of α. Thus, it may be said that the present treatment is in terms of the anti-Gudermannian of the velocity angle previously used, and this relationship permits the two treatments to be connected.

This confirms the statement made above, namely that in the space under consideration motions are expressed more conveniently through rapidities than through velocities. If both w' and u are finite and positive, w is also finite and positive. In terms of velocities this means that two velocities each of which is less than that of light give a resultant velocity which is also less than that of light. Equation (10) also confirms the earlier proof that in the S-S' world the velocity of light is the same for all observers. Namely, if w' is infinitely great, the addition or subtraction of a finite u still leaves w infinitely great. An infinitely great rapidity corresponds to the velocity of light.

A different form of the formula for the addition of velocities is also of interest. From hyperbolic trigonometry we may write

$$u = \tanh^{-1}(q/c) = 0.5 \log \left\{(c+q)/(c-q)\right\}.$$

Hence,

$$e^{2u} = (c+q)/(c-q) = Q. \tag{12}$$

The quantity Q, introduced in (12), will be defined as the *quotient of* q. By analogy, we may write

$$e^{2w} = V \quad \text{and} \quad e^{2w'} = V', \tag{13}$$

V and V' being the quotients of v and v', defined as in equation (12). From (10) it follows directly that

$$V = V' \cdot Q. \tag{14}$$

When rapidities are added, and velocities combined according to equation (11), *the corresponding velocity quotients are multiplied.*

When q is very small compared to the velocity of light, the ratio q/c differs but little from u. This may be seen from the expansion

$$q/c = \tanh u = u - u^3/3 + 2u^5/15 - \cdots . \tag{15}$$

Thus, at small velocities, one may put $q/c = u$, and the general law of addition of rapidities becomes our usual law of addition of velocities. Therefore, the fact that an algebraic addition of velocities holds true in our world of experience is no proof that we live in an Euclidean space-time world. Our usual velocities are extremely small compared to that of light, and under such circumstances velocities are also additive in a hyperbolic space-time world. A crucial test comes when a velocity becomes comparable to that of light, and it is here that the experimental evidence seems to favor the theory of relativity and the space-time relationships expressed by (1) and (2). (See Ref. 4, index under Relativitätstheorie, Experimentelles.) Thus, it seems logical to conclude that perhaps we live in a hyperbolic space-time world which for practical everyday purposes may be assumed to be Euclidean.

2. *Slowing-down of Clocks and the Fitzgerald Contraction*

Let a clock be placed at the origin of $S'(x'=0)$ and let S' read it at some instant t'. Putting $x'=0$ in (5), we get

$$t = t' \cosh u. \tag{16}$$

Thus, an elapsed interval of time t', measured by S', means a greater interval of time, t, for the observer S (because the hyperbolic cosine is always greater than 1). In other words, to S a clock in the S' system seems to be running slow. Similarly, by putting $x=0$ in (2) it will be found that $t'=t \cosh u$, so that a clock in the S system also appears to the S' observer to be running slow. With small values of q/c (or u) the ratio t/t' differs but little from 1, and is always equal to unity in an Euclidean space-time world. When the relative velocity of the two observers is an appreciable fraction of the velocity of light, cosh u may be quite a large number, so that "the other observer's clock" may seem to run at a small fraction of its correct speed. This relationship is back of statements found in some popular works on relativity to the effect that one should stay eternally young when moving at the velocity of light. This means that in (16) t' must approach zero as u approaches infinity, if t is to remain finite.

Let now S' lay off a certain distance $x'=O'X'$ from his origin and mark the point X' with a hair-line visible to S. At the instant when the two origins are in coincidence, S measures the distance $O'X'$ with his own instruments and finds a certain length x. Substituting $t=0$ in (1), we find

$$x' = x \cosh u. \tag{17}$$

This means that the length x measured by S is smaller than that measured by S'. Since S considers his measurements to be correct, he reaches the conclusion that the yardstick of S' is contracted due to its motion. Yet, when the conditions of the experiment are reversed and S' measures a length x' laid off by S as a length x, he, S', also finds that the other observer's yardstick seems to be contracted. This reduction in the length of a moving body, in the direction of its motion, is known as the Fitzgerald contraction and is numerically equal to the ratio of 1 to cosh u. The Fitzgerald ratio is usually expressed in the form of the square root in the first equation (7a).

The important feature of the above experiment is that while one observer marks a length, it is the other who determines the "simultaneity" when applying his yardstick to the two ends of the moving distance. For example, when S' lays off a length x', it is S who acts in accordance with the condition $t=0$. Entirely different results are obtained when one and the same observer lays off a length and also gives an instantaneous signal for the other observer to measure the length. Thus, let S' lay off a distance $x'=O'X'$ and let him give two simultaneous (to him) flashes of light, at O' and X', when S is supposed to measure the length $O'X'$. Substituting $t'=0$ in (4), we find $x=x' \cosh u$, a condition opposite to equation (17), in the sense that the S observer finds a greater length than the original length x'. Should he analyze the reason he would find that for

him the two flashes of light were not simultaneous, the flash at X' occurring later. Consequently, the moving line covered some distance before he applied the yardstick to the distant end, and, as a result, he measured a greater apparent length. This difference in the conditions of the two experiments should be clearly kept in mind when discussing the Fitzgerald contraction.

3. *The Doppler Effect*

In addition to the systems S and S' imagine a third system, S'', moving along the same straight line XX' with the two others, at a uniform velocity. The motion of this third system with reference to the other two is determined by its velocities, v and v', as measured by the observers S and S'. Let the corresponding rapidities be w and w', the mathematical relationships being those discussed in Section 1 above. Let the system S'' carry a stick or a tape marked with equal divisions of length B, and let the axis of this tape or stick be in the direction of the motion of the three systems. In accordance with the Fitzgerald contraction, these divisions will appear to S to be of smaller length, b, where

$$b = B/\cosh w. \tag{18}$$

The number of divisions which S will count as passing by him per unit time, or the apparent frequency, will be

$$f = v/b = c \tanh w \cdot (\cosh w)/B = (c/B)(\sinh w). \tag{19}$$

Similarly, S' will measure divisions of length

$$b' = B/\cosh w' \tag{20}$$

and an apparent frequency

$$f' = (c/B)(\sinh w'). \tag{21}$$

The two rapidities, w and w', differ from each other by the amount of rapidity u of the relative motion of the two observers; see equation (10). The foregoing equations are sufficient for determining the so-called Doppler effect, that is, the change in the frequency or the wave length of a radiation, due to the motion of the observer himself. From (18) and (20) we have

$$b/b' = \cosh (w - u)/\cosh w = \cosh u - \sinh u \tanh w. \tag{22}$$

Equations (19) and (21) give

$$f'/f = \sinh (w - u)/\sinh w = \cosh u - \sinh u \coth w. \tag{23}$$

The latter relationship is more specifically known as the Doppler effect. From (23), knowing the rapidity of propagation of a radiation with respect to S and the rapidity of S with respect to S', the ratio of the frequencies of the radiation, as measured by the two observers, may be computed.

In the special case of an electromagnetic radiation in vacuum the rapidity w is infinitely great, so that $\tanh w = \coth w = 1$, and the preceding two equations give

$$b/b' = f'/f = \cosh u - \sinh u = e^u, \tag{24}$$

which means that the Doppler effect in this case is equal to the square root of the quotient Q of velocity q; see equation (12). It follows from the latter equation that

$$b/b' = f'/f = (c + q)/\sqrt{c^2 - q^2} \tag{25}$$

which is the usual expression for the Doppler effect.

4. *Reflection of a Perfectly Elastic Body from a Moving Rigid Target*

A plane rigid target is stationary in the S' system, its plane being normal to the direction XX' of the relative motion of the two observers. A perfectly elastic small sphere moves along XX' at a rapidity w_i' (the subscript i stands for incidence) as measured by the S' observer, and is reflected from the target at a rapidity w_r' (the subscript r stands for reflection). According to the laws of mechanics, the two rapidities are equal and of opposite sign; hence we have

$$w_i' + w_r' = 0. \tag{26}$$

The problem is to find the rapidities of incidence and reflection as measured by the observer S. In accordance with (10)

$$w_i = w_i' + u \tag{27}$$

$$w_r = w_r' + u. \tag{28}$$

Combining these equations with (26), we obtain

$$w_i + w_r = 2u \tag{29}$$

which answers the problem. Knowing u and w_i' we can determine w_i, and consequently can compute w_r. This example again illustrates the advantage of using, in relativity problems, rapidities rather than velocities. Since velocity quotients are multiplied when rapidities are added, the foregoing equation may also be written in a form similar to (14), namely

$$V_i \cdot V_r = Q^2. \tag{30}$$

The change in the frequency and wave length of a beam of light reflected from a moving mirror may be determined in a similar manner, the final result being

$$f_r/f_i = (c + q)/(c - q) = Q. \tag{31}$$

5. *Addition of Festinations*

Just as acceleration is defined in ordinary mechanics as the rate of change of velocity with time ($a = dq/dt$), so we shall define *festination*, du/dt, as the rate of change of rapidity with time.* Differentiating both sides of (7) with respect to t, we find

* The English word festination means haste, from Latin *festinus* meaning quick.

$$dq/dt = c \operatorname{sech}^2 u \cdot du/dt. \tag{32}$$

This is the relationship between an acceleration and the corresponding festination. Solving for the latter, gives

$$du/dt = (dq/dt)/\sqrt{(c^2 - q^2)}. \tag{33}$$

Differentiating (10) with respect to t, gives

$$dw/dt = dw'/dt + du/dt. \tag{34}$$

Using in this expression the accelerations instead of the festinations, in accordance with equation (32), gives

$$(dv/dt) \cosh^2 w = (dv'/dt) \cosh^2 w' + (dq/dt) \cosh^2 u. \tag{35}$$

A comparison of (34) and (35) shows that in the space under consideration a festination is a more convenient quantity to use than an acceleration, just as a rapidity is more convenient than a velocity.

In equation (34) it may be convenient in some applications to use dw'/dt' in place of dw'/dt, the former being a festination actually measured by the S' observer. We have

$$dw'/dt = (dw'/dt')(dt'/dt). \tag{36}$$

The ratio dt'/dt may be determined from (2). Since (34) is written from the point of view of the S observer, and he considers himself stationary, we must put $x = \text{const.}$, and consequently

$$dt'/dt = \cosh u. \tag{37}$$

Thus, (34) becomes

$$dw/dt = (dw'/dt') \cosh u + du/dt. \tag{38}$$

A similar transformation may be performed on (35). In applications, it is well to remember that the physical dimensions of festination are T^{-1}, whereas those of linear acceleration are LT^{-2}.

6. *Mass, Force, Energy, and Momentum*

The time-space relationships postulated in equations (1) and (2) are not sufficient for establishing the dynamics of material points in the space under consideration, and additional suppositions are needed in regard to mass, energy, force, and momentum. More than one starting point is possible in this respect (see Refs. 1–4). We shall adopt here the following postulates which seem reasonable in the light of our knowledge of atomic phenomena at extremely high velocities:

1. The mass of a material particle is a function of its velocity (or rapidity).
2. Mechanical force is defined as being proportional to the product of mass and festination (instead of acceleration), the mass being that corresponding to the instantaneous rapidity.

3. The work done upon a particle equals the force times the displacement.

4. Our usual laws of conservation of mass and energy are combined into one, the work done on a particle being a measure for the increase in its mass.

Let the mass, M, of a particle at any rapidity u be

$$M = m \cdot y \tag{39}$$

where m is its mass at rest and y an unknown function of u; at $u=0$, $y=1$. The problem is to determine y so as to be in accord with the remaining three postulates. We shall write the expression for the force, F, in the form

$$F = cM \cdot du/dt = cmy \cdot du/dt. \tag{40}$$

On account of the physical dimensions of festination, a coefficient of proportionality is needed whose dimensions are those of velocity. It is convenient to choose a unit of force such that this coefficient will be equal to c.

Let an elementary displacement of a particle be $q \cdot dt$, so that the corresponding work done on it is $F \cdot q\, dt$. The increase in the mass is $dM = m \cdot dy$, so that, in accordance with Postulate 4,

$$F \cdot q\, dt = km \cdot dy, \tag{41}$$

where k is a coefficient of proportionality which converts mass into energy and depends upon the chosen unit for energy. Substituting for F its value from (40) and for q from (7), the foregoing equation becomes

$$c^2 y\, du \cdot \tanh u = k\, dy.$$

It is evidently convenient to put $k = c^2$, so that

$$dy/y = \tanh u \cdot du. \tag{42}$$

This differential equation is satisfied by

$$y = \cosh u. \tag{43}$$

Thus, for the space under consideration we have the following fundamental expressions for dynamics of particles:

$$\text{Mass } M = m \cosh u \tag{44}$$

$$\text{Force } F = cm \cosh u \cdot du/dt \tag{45}$$

$$\begin{aligned} \text{Increase in energy} &= c^2 \times \text{Increase in mass} \\ &= c^2 m(\cosh u - 1). \end{aligned} \tag{46}$$

It is also convenient to define momentum as $M \cdot q$, because then the condition that force is the rate of increase in momentum is automatically satisfied. We have

$$F = d(Mq)/dt = m\, d(yq)/dt = cm\, d(\sinh u)/dt = cm \cosh u \cdot du/dt.$$

Thus, in addition to the foregoing three definitions we may also write

(47) $$\text{Momentum} = cm \sinh u.$$

Substituting in the foregoing expressions for $\cosh u$ and $\sinh u$ their equivalents in accordance with equations (7a), the usual formulas for relativistic mass, force, momentum, and energy are obtained. For applications and further development of relativity dynamics see Ref. 3, p. 73; and Ref. 4, vol. V, p. 586.

7. *Conservation of Momentum*

The foregoing principles of relativity dynamics are consistent with the principle of conservation of momentum for either the stationary or the moving observer, as may be seen from the following example. Let two identical inelastic spheres, of rest mass m each, move towards each other with equal and opposite rapidities w in the S system, in the XX' direction. After the collision they will remain stationary for the S observer, but the energy converted into heat should increase their rest mass in accordance with (46). Thus for the S observer the new rest mass of each sphere is $m \cosh w$, even though the velocity has been destroyed by the collision. For the S' observer the momenta before the collision are $cm \sinh (w-u)$ and $-cm \sinh (w+u)$, the sum being $-2m \cosh w \sinh u$. If the law of conservation of momenta be satisfied, this should be the total momentum measured by S' after the collision. Such is the case, the new total rest mass (for S) being $2m \cosh w$ and the relative rapidity of the two observers u. The minus sign indicates that to the S' observer the total mass appears to be moving in the negative direction, or to the reader's left.

8. *Pressure of a Beam of Particles or of Radiation*

Let a beam of particles or of radiation impinge at right angles upon a free target of mass M, uniformly and continuously, and be absorbed by it. Let the impinging mass be m per unit time, let the rapidity of the impinging particles be W and let the festination of M, as a result of bombardment, be w per unit time. The law of conservation of mass and energy gives

(48) $$M + m \cosh W = M' \cosh w$$

where M' is the increased mass of the target after having been exposed to bombardment per unit time. The total force acting upon the target, or the increase in momentum per unit time, is

(49) $$F = cm \sinh W = cM' \sinh w.$$

Equations (48) and (49) completely determine the phenomenon. Knowing M, m, and W, these equations may be solved for M' and w, and then the force F computed. In the special case of electro-magnetic radiation, W is infinitely great, but m is infinitely small, the product $p = c^2 m \cosh W$ being finite, since it represents the incident energy of radiation per unit time. Equation (48) may be re-written in the form

(50) $$c^2 M + p = c^2 M'.$$

In (49) sinh W may be replaced by cosh W, since the ratio between the two approaches unity when W approaches infinity. Hence

$$F = p/c. \tag{51}$$

If the cross-section of the beam equals unity, p/c is the energy per unit volume of the beam and is also equal to the pressure upon the target.

9. *Miscellaneous Formulas*

In the articles quoted under Refs. 5 to 10, oblique coordinates have been used and certain formulas derived in terms of circular trignometric functions. It may be of interest to re-state, without proof, some of the results in terms of the hyperbolic functions of rapidities.

(a) *Transformation of Electric and Magnetic Forces.* In Ref. 10, on top of page 240, equations (2) and (5) become respectively:

$$E_2' = E_2 \cosh u - H_3 \sinh u \tag{52}$$

$$H_2' = H_2 \cosh u + E_3 \sinh u. \tag{53}$$

These expressions are of the same general form as equations (1) and (2) of this article, thus confirming the general hyperbolic nature of the space-time under consideration.

(b) *Aberration of Light.* Let two observers, S and S', be moving relatively to each other at a velocity q, in a direction XX', as before, and let both study the same beam of light, by measuring the angle which this beam forms with the perpendicular to the line of relative motion. Let the measured angles be ϕ and ϕ' respectively. The usual relationship between these angles, or the law of aberration, is (Ref. 5, p. 233, equation 38)

$$\sin \phi' = [\sin \phi - (q/c)]/[1 - (q/c) \sin \phi]. \tag{54}$$

Let $\sin \phi = \tanh z$ and $\sin \phi' = \tanh z'$, where z and z' are the anti-Gudermannian angles of ϕ and ϕ'. Substituting these values and also the value of q/c from (7) in (54), we obtain

$$\tanh z' = \tanh (z - u) \tag{55}$$

so that the law of aberration is reduced to the simple form

$$z - z' = u \tag{56}$$

similar to the fundamental relationship (10). The difference between the anti-Gudermannians of the two measured angles of incidence is equal to the relative rapidity of the two observers. This result could perhaps be anticipated from the fact that $c \sin \phi$ is the component of the velocity of the ray in the direction XX', and the rapidities in that direction obey equation (10). Starting with (56), the relationship (54) could be deduced therefrom.

(c) *DeBroglie's Postulate.* The results given in Ref. 8 may be transformed as follows:

$$\text{Velocity of phase propagation} = c/\tanh u \tag{57}$$

$$\text{Ratio of frequencies} = f/f' = \cosh u \tag{58}$$

$$\text{Ratio of amplitudes} = A/A' = \operatorname{sech} u. \tag{59}$$

DeBroglie's quantum principle becomes

$$\lambda \cdot m \cdot c \cdot \sinh u = h. \tag{60}$$

A comparison with (47) will show that the momentum of the particle multiplied by the wave length is equal to Planck's action constant.

Some books on Relativity

1. Carmichael, R. D.; *The Theory of Relativity*, 1920 (John Wiley and Sons).

1a. Tolman, R. C.; *The Theory of Relativity of Motion*, 1917 (University of California Press).

1b. Born, M., translated by Brose; *Einstein's Theory of Relativity* (Methuen).

1c. Silberstein, L.; *The Theory of Relativity*, 1914 (Macmillan).

2. Eddington, A. S.; *The Mathematical Theory of Relativity*, 1924 (Cambridge University Press).

3. Birkhoff, G. D.; *Relativity and Modern Physics*, 1927 (Harvard University Press).

4. *Handbuch der Physik* (Springer); see index under *Relativität* and compound words beginning with it.

The author's previous articles on Relativity

5. *Aberration of light in terms of the theory of relativity, as illustrated on a cone and a pyramid.* Optical Society of America, Journal, vol. 9 (1924), p. 223.

6. *Einstein's restricted theory of relativity explained on a model.* Science and Invention, vol. 11 (1923), p. 442.

7. *Model to illustrate theory of relativity.* American Institute of Electrical Engineers , Journal, vol. 42 (1923), p. 712.

8. *Relativity transformation of an oscillation into a traveling wave, and deBroglie's postulate in terms of velocity angle.* Optical Society of America, Journal, vol. 19 (1929), p. 253. Same, Abstract, Physical Review, vol. 35 (1930), p. 127.

9. *Straight-line relativity in oblique coordinates; also illustrated by a mechanical model.* Optical Society of America, Journal, vol. 13 (1926), p. 155.

10. *Transformation of electric and magnetic forces in a plane wave, in a plane normal to the direction of relative motion of two observers.* The Physical Review, vol. 23 (1924), p. 239.

GEOMETRY AS A BASIS FOR CANONICAL FORMS

By T. E. RAIFORD, University of Michigan

1. *Introduction.* The object of this paper is the development of certain canonical forms of linear transformations through consideration of related geometrical properties. Some of the underlying ideas here presented have appeared in other publications but the different method of approach leads to some new theorems which prove valuable in further studies of transformations.

We consider transformations of the type

$$\alpha_i' = \sum_{j=1}^{n} a_{ij}\alpha_j, \qquad i = 1, 2, \cdots, n,$$

in which the α's may be considered as the coordinates of a point in a Cartesian system in a space of n dimensions. Such transformations, written for brevity $\alpha' = A(\alpha)$, always carry a straight line into a straight line, parallel lines into parallel lines, and leave unchanged the ratio of two directed distances on two parallel lines.* These properties are invaluable in studying certain relations existing between vector quantities and linear transformations.

The variables $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ collectively are called a vector and will be designated by the single letter x.

The n vectors $x_1, x_2, \cdots, x_n$, are said to be linearly dependent† if there exist n constants $c_1, c_2, \cdots, c_n$ not all zero such that $c_1x_1 + c_2x_2 + \cdots c_nx_n = 0$.

Given then n independent vectors C_i every vector x may be represented as a linear combination of them,

$$x = \sum_{i=1}^{n} \lambda_i C_i$$

and the numbers λ_i are called the components of x with respect to the vectors C_i.

A linear transformation may be considered as an operation assigning to a vector x the vector x' and is a function A which possesses the property $A(\nu x + \eta y) = \nu A(x) + \eta A(y)$. Conversely, if to every vector x a vector $x' = A(x)$ is assigned in such a way that $A(\nu x + \eta y) = \nu A(x) + \eta A(y)$, then the components of x' are obtained from those of x by a linear transformation of this same type.‡ In fact if $x = \sum_{i=1}^{n} \xi_i C_i$ we have $x' = A\sum_{i=1}^{n} \xi_i C_i = \sum_{i=1}^{n} \xi_i A(C_i)$. If now we have another set of linearly independent vectors D_i there is a non-singular transformation $P = (P_{ij})$ such that $D_i = \sum_{j=1}^{n} p_{ji} C_j$. If $y = \sum_{i=1}^{n} \xi_i D_i = \sum_{i=1}^{n} \eta_i C_i$ we have $\eta_i = \sum_{i=1}^{n} \xi_i p_{ij}$ hence $y = P(x)$. From $A(y) = y'$ and $A(x) = x'$ we have $PAP^{-1}y = y'$. Thus, a change in the coordinate system results in replacing A by a so-called "similar" matrix§ PAP^{-1}.

We shall need the following definitions:

A space is defined as the totality of vectors such that given any two vectors x and y of that totality every linear combination $\theta x + \phi y$, θ and ϕ being any two constants, is again a vector of that totality.

A zero vector of a transformation is defined as any vector x which satisfies the equation $A(x) = 0$.

The number of dimensions of a space is defined as the maximum number of independent vectors of that space.

---

* Graustein, *Introduction to Higher Geometry*, Chap. XI.

† Pincherle, *Le Operationi Distributive*, p. 5.

‡ Klein, *Vorlesungen über Höhere Geometrie*, p. 97.

§ Cf. Kowalewski, Berichte der Königlich-Sächsischen Gesellschaft der Wissenschaften, vol. 69 (1917), p. 325.

We shall denote $A[A(x)]$ by $A^2(x)$, and similarly refer to A^n as the nth power of the matrix A. Such a matrix is said to be singular if the determinant formed by its elements is zero.

2. *Sequences and related zero spaces.* From the foregoing definitions the totality of all vectors satisfying the equation $A(x)=0$ constitutes a zero space. There are zero spaces associated with powers of A other than those defined by this equation. Consider

THEOREM 1. *A necessary condition that a sequence of positive integers $a_1, a_2, \cdots, a_n$, represent the number of dimensions of the zero spaces S_k, of the successive powers of a singular matrix A of a transformation is that the differences $\Delta a_i = a_{i+1} - a_i$ should be non-negative and form a non-increasing sequence, and that $a_2 \leqq 2a_1$.*

Every solution of the equation $A(x)=0$ is also a solution of the equation $A^2(x)=0$ since $A^2(x)=A[A(x)]=A(0)=0$. Similarly every solution of $A^2(x)=0$ is a solution of $A^3(x)=0$ and so on for higher powers of A. This imposes the condition $a_1 \leqq a_2 \leqq \cdots \leqq a_n$ and consequently the first differences are either positive or zero.

The rest of the theorem can be shown by establishing the

LEMMA: *If p is the maximum number of new independent zero vectors of A^{k+1} no linear combination of which, say $y=\sum a_i x_i$, is a zero vector of A^k, then the p vectors obtained from them by applying A to each are zero vectors of A^k and no linear combination of them is a zero vector of A^{k-1}.*

The proof of this lemma follows readily from the definition of the terms used in its statement. The maximum number of independent zero vectors of A^{k+1} is a_{k+1} and the maximum number of independent zero vectors of A^k is a_k, and consequently the maximum number, p, of new zero vectors of A^{k+1} such that no linear combination of them is a zero vector of A^k is given by $p=a_{k+1}-a_k$. Similarly the maximum number of zero vectors, p', which belong to A^k such that no linear combination of them is a zero vector of A^{k-1} is $p'=a_k-a_{k-1}$. However if A is applied to any y the resulting vector is a zero vector of A^k since $A^k[A(y)]=A^{k+1}(y)=0$; but $A^{k-1}[A(y)]=A^k(y)\neq 0$, and consequently none of the zero vectors gotten in this way are zero vectors of A^{k-1}, establishing the truth of the lemma.

The fact that $a_2 \leqq 2a_1$ follows directly from this lemma since the number of zero vectors p gotten by applying A to each of the p zero vectors cannot exceed the maximum number of zero vectors p' and consequently p is at most equal to p', that is $a_{k+1}-a_k \leqq a_k - a_{k-1}$.

It will be shown later that the conditions of Theorem 1 are also sufficient.

3. *Choice of axes for $A^k=0$.* In studying the zero spaces associated with the singular matrix A^k, $k=1, 2, \cdots, n$, we wish to make the simplest choice of co-

ordinate axes that will exhibit its geometric properties. To do this we shall use the

DEFINITION: By a set of "coordinate vectors" is meant any set of n independent vectors, say $i^{(1)}, i^{(2)}, \cdots i^{(n)}$.

Consider the case in which for a sufficiently high power k of A the value of A^k is zero. Let the number of dimensions a_i of the zero spaces S_i corresponding to A^i, be given by the terms of the sequence

i	a_i	Δa_i	$\Delta^2 a_i$
1	2	2	1
2	4	1	
3	5		

Here there are five independent zero vectors of S_3 but only one (in this case) new independent zero vector. Choose then as $i^{(1)}$ the new zero vector of S_3. This means that $i^{(1)}$ is a vector to which A must be applied three times in order to produce a zero vector and consequently all lower powers of A when applied to $i^{(1)}$ produce vectors which are not zero vectors. We have $A^3(i^{(1)})=0$ and may set

$$A(i^{(1)}) = i^{(2)}, \qquad A^2(i^{(1)}) = i^{(3)}.$$

These relations also determine one of the zero vectors of S_2 and one of S_1 as is evident from writing

$$A(i^{(2)}) = i^{(3)} \qquad A^2(i^{(2)}) = 0 \qquad \text{and} \qquad A(i^{(3)}) = 0.$$

There remains one new zero vector of S_2. Choose $i^{(4)}$ as this vector and in a similar way

$$A(i^{(4)}) = i^{(5)} \qquad A^2(i^{(4)}) = 0 \qquad \text{and} \qquad A(i^{(5)}) = 0.$$

All of the five independent zero vectors of the 5-space are now accounted for and from them we can determine the form of A for which the number of dimensions of the zero spaces of the successive powers of A are the numbers of the given sequence. On choosing any vector

$$x = \alpha_1 i^{(1)} + \alpha_2 i^{(2)} + \alpha_3 i^{(3)} + \alpha_4 i^{(4)} + \alpha_5 i^{(5)} \tag{1}$$

and applying A to x the transformed vector becomes

$$A(x) = x' = \alpha_1 i^{(2)} + \alpha_2 i^{(3)} + \alpha_4 i^{(5)}. \tag{2}$$

Since by α_k' is meant the coefficient of $i^{(k)}$ in the expression for x' one has from the last two equations

$$\alpha_1' = 0, \ \alpha_2' = \alpha_1, \ \alpha_3' = \alpha_2, \ \alpha_4' = 0, \ \alpha_1' = \alpha_4. \tag{3}$$

The singular matrix of the transformation given by (3) is

$$A = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix} \tag{4}$$

and is indeed a very simple form for this special case. Moreover the method is easily generalized; and since it is always possible to choose the coordinate axes by the foregoing method with the resulting simple form for A the necessary conditions on the sequence as stated in Theorem 1 are also sufficient. Note that the above type of choice always yields an A whose every element a_{ij}, is zero except the elements $a_{i(i-1)}$ $(i=2, \cdots, n)$ which are either 0 or 1, and that the non-zero rows of A are the coordinate vectors of the space $A(x)$.

4. *Vectors of an n-space containing a zero space.* A vector v will be defined as a "value vector" of A if there exists a vector y such that $A(y)=v$. A value vector of A may be a zero vector of A; e.g., $v=(0,0,1,0,0)$ and $y=(0,1,0,0,0)$ for the example above.

Given any two vectors x and y with μ and ρ any two arbitrary constants then $A(\mu x+\rho y)=\mu A(x)+\rho A(y)$ and hence any linear combination of two or more value vectors is also a value vector. The totality of all such value vectors constitutes a space which will be called a "value space."

If a singular transformation A be applied to a vector x then the equation $A(x)=0$ defines a zero space of a_1 dimensions; if A^2 be applied to x the equation $A^2(x)=0$ defines a zero space of a_2 dimensions, and so on. Suppose the number of dimensions corresponding to the successive powers of A increases to a_k corresponding to the equation $A^k(x)=0$ but the number of dimensions corresponding to $A^{k+l}(x)=0$ for all $l>0$ remains a_k, then the zero space of a_k dimensions is defined as the "ultimate zero space" of A.

The characterization of a value space is facilitated by

THEOREM 2. *Any vector x may be expressed as the sum of a zero vector z of the ultimate zero space of A and a vector v of the value space.*

Proof: Given a vector x, form $A(x), A^2(x), \cdots, A^n(x)$. For $n+1$ vectors of an n-space there must be at least one linear combination that is zero,* and it is therefore possible to write

$$\alpha_0 x + \alpha_1 A(x) + \alpha_2 A^2(x) + \cdots \alpha_n A^n(x) = 0 \tag{5}$$

where not all of the α's are zero. If the constant α_0 is different from zero then it follows that x can be expressed as

$$x = A[\beta_0 x + \beta_1 A(x) + \beta_2 A^2(x) + \cdots \beta_{n-1} A^{n-1}(x)] \tag{6}$$

* Bôcher, *Higher Algebra*, p. 37.

which from the definition above says that x is a value vector. If the first term appearing in (5) different from zero is the term A^k then (5) can be written

$$A^k[x + \nu_1 A(x) + \nu_2 A^2(x) + \cdots \nu_{n-k} A^{n-k}(x)] = 0 \tag{7}$$

and consequently

$$x + A[\nu_1(x) + \nu_2 A(x) + \cdots \nu_{n-k} A^{n-k-1}(x)] \tag{8}$$

is a zero vector of A^k. Denote this vector by z. Then $x = z + v$ where v is a value vector given by the quantity, with the signs changed, in parenthesis. The theorem is thus proved.

5. *Non-singular transformations.* If there is no power k such that $A^k = 0$ then it becomes necessary to make a choice of coordinate axes for the value space.

Let A be the matrix of a non-singular transformation. We seek a number λ such that when subtracted from each element of the principal diagonal of A (usually described as subtracting the product, λ times the unit matrix I, from the matrix A) there results a matrix A_1 which is singular. Since a singular matrix is one whose determinant is zero such a value can be found by solving the nth degree equation in λ gotten by equating this determinant to zero, thus, $|A - \lambda I| = 0$. Let λ_1 be a root of this equation. The matrix $A_1 = A - \lambda_1 I$ is then a singular matrix. If it happens that there is some value k such that $A_1^k \equiv 0$ then the desired choice of coordinate axes for A_1 is known from the previous discussion. If there is no such value k then the question of the simplest choice of axes for the value space is still unanswered. We consider then the zero space of A_1 and the value space of A_1 separately as given by

THEOREM 3. *The zero space of A_1 and the value space of A_1 are invariant spaces of A.*

By applying A to each side of the equation $A_1 = A - \lambda I$ it is seen that A and A_1 are commutative since

$$AA_1 = AA - \lambda AI = (A - \lambda I)A = A_1 A. \tag{9}$$

If therefore v is a value vector of A_1, that is $v = A_1(y)$, the result of applying A to this relation gives

$$A(v) = A[A_1(y)] = A_1[A(y)] \tag{10}$$

showing the value space of A_1 to be invariant under the transformation A and similarly for the zero space of A_1.

6. *Vector coordinate system for any A.* Given then a non-singular transformation A, reduce it as indicated by equation $A - \lambda_1 I = A_1$, to a singular transformation having an ultimate zero space of dimensions s_1 and a value space of dimensions $n - s_1$. By Theorem 3 it is possible to treat the value space separately and in exactly the same way as was the matrix of the original transformation. Therefore let A_1 be written as

$$A_1 = \left\| \begin{array}{ccccc:cccc} 0 & 0 & 0 & \cdots & 0 & & & & \\ a & 0 & 0 & \cdots & 0 & & & & \\ 0 & a & 0 & \cdots & 0 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ 0 & 0 & 0 & \cdots & 0 & & & & \\ \hdashline & & & & & b_{s_1+1,s_1+1} & b_{s_1+1,s_1+2} & \cdots & b_{s_1+1,n} \\ & & & & & b_{s_1+2,s_1+1} & b_{s_1+2,s_1+2} & \cdots & b_{s_1+2,n} \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & b_{n,s_1+1} & b_{n,s_1+2} & \cdots & b_{n,n} \end{array} \right\| = \left\| \begin{array}{c:c} A_{01} & \\ \hdashline & A_{v1} \end{array} \right\|$$

where a is either zero or 1 as given by §3 and where A_{01} and A_{v1} denote the singular matrix of the ultimate zero zpace of A_1 and the non-singular matrix of the value space of A_1 respectively. Similarly in the $(n-s_1)$th degree equation $|A_{v1}-\lambda I|=0$ a value λ_2' can be found which on being substituted for λ will produce a singular matrix, say A_2. Every vector of this $(n-s_1)$-space can be expressed as the sum of a zero vector of the ultimate zero space of A_2 of dimensions s_2 and a value vector of the value space of A_2 of dimensions $n-s_1-s_2$. Again in a similar way we designate the ultimate zero space and that of the value space of A_2 by A_{02} and A_{v2} respectively. Since n is finite this process can be repeated until A has been completely expressed by a sequence of ultimate zero spaces and for each of these a very simple choice of coordinate axes is given by the method of §3. The transformation A can therefore be written

$$(11) \qquad A = \left\| \begin{array}{ccc:ccc:c:ccc} \lambda_1 & & & & & & & & & \\ a & \lambda_1 & & & & & & & & \\ & a & & & & & & & & \\ \hdashline & & & \lambda_2 & & & & & & \\ & & & a & \lambda_2 & & & & & \\ & & & & & & & & & \\ \hdashline & & & & & & \cdots & & & \\ \hdashline & & & & & & & \lambda_\omega & & \\ & & & & & & & a & & \\ & & & & & & & & a & \lambda_\omega \end{array} \right\|$$

where $\lambda_2=\lambda_1+\lambda_2'$, $\lambda_3=\lambda_1+\lambda_2'+\lambda_3'$, etc.

From (11) the numbers $\lambda_1, \lambda_2, \cdots, \lambda_\omega$ form the principal diagonal of A and are consequently the roots of $|A-\lambda I|=0$. This equation is called the characteristic equation* which we write

$$D(\lambda) = |A - \lambda I| = (\lambda - \lambda_1)^{s_1}(\lambda - \lambda_2)^{s_2}(\lambda - \lambda_3)^{s_3} \cdots (\lambda - \lambda_\omega)^{s_\omega}. \tag{12}$$

Moreover the roots of such an equation of a transformation are independent of the form of the transformation† and therefore the values of λ thus obtained are also all roots of the characteristic equation $|A-\lambda I|=0$, of the original matrix A.

8. *Conclusion.* The very simple form given by (11) in which any transformation can be written by choosing a coordinate system as here developed is readily recognized as the well known "classical canonical form" and has resulted from a search for the simplest form of coordinate system possible in terms of which certain geometric properties of the transformation may be displayed.

ON THE CEVIAN TETRAHEDRON

By N. A. COURT, University of Oklahoma

1. Given the tetrahedron $(T)=DABC$ and a point M, not in a face of (T), the traces D', A', B', C' of the lines DM, AM, BM, CM in the respective faces of (T) determine a tetrahedron $(T')=D'A'B'C'$ which may be referred to as the "cevian tetrahedron" of the point M with respect to (T), or, more briefly, for (T).

The tetrahedron (T) may be called the "anticevian tetrahedron" of the point M for (T').

2. The two tetrahedrons (T), (T') are obviously homological, the point M being the center of homology. The plane of homology σ is the tetrahedral polar plane of M with respect to both (T) and (T').‡

If the line is drawn through M to meet the two opposite edges DA, BC of (T), say, in U', X', the harmonic conjugate U of U' with respect to D, A and the harmonic conjugate M' of M with respect to X', U' both lie in the plane σ. Let S, L be the traces of the lines UX', UM' on the line DD'.

Cutting the two harmonic pencils $U(U'X', MM')$, $X'(DA, UU')$ by the transversal DD' we have

$$(DSML) = -1, \qquad (DD'SM) = -1,$$

hence, eliminating S from these two harmonic ratios, we have

$$(MLDD') = -3.$$

---

* Dickson, *Modern Algebraic Theories*, p. 47.

† Bôcher, *Higher Algebra*, p. 269.

‡ Nathan Altshiller-Court, *Modern pure solid geometry*, The Macmillan Company, 1935, p. 234.

Now the points U, M' of the line $UM'L$ lie in the plane σ, hence L is the trace of the line DD' in σ. Consequently: *A tetrahedron and the cevian tetrahedron, for this tetrahedron, of any point in space correspond to each other in an homology the anharmonic ratio of which is equal to* -3.

3. If the point M coincides with the centroid G of the tetrahedron (T), the cevian tetrahedron (T') coincides with the medial tetrahedron (i.e., with the tetrahedron having for its vertices the centroids of the faces of the given tetrahedron). The tetrahedral polar plane σ of G with respect to (T) coincides with the plane at infinity, the two tetrahedrons thus becoming homothetic, and from the formula in the preceding section we obtain $GD:GD' = -3$. We have thus a new proof of Commandino's theorem.

Conversely, assuming Commandino's theorem, the proposition of section (2) may be derived from it by projection.

The reader may find it interesting to make M coincide with remarkable points of either the general or some special tetrahedron and formulate the corresponding propositions. The point M may, for instance, be made to coincide with the circumcenter of the general tetrahedron, or with the orthocenter of an orthocentric tetrahedron, etc.

4. Consider four mutually tangent spheres (A), (B), (C), (D), all contact being external. If X, U; Y, V; Z, W are the pairs of points of contact lying on the pairs of opposite edges BC, DA; CA, DB; AB, DC of the tetrahedron $DABC$ formed by the centers of the spheres, the lines XU, YV, ZW meet in a point M.*

The points X, Y, Z are the points of contact of the sides BC, CA, AB of the triangle ABC with the circle inscribed in this triangle, hence the line DM meets the plane ABC in the Gergonne point of the triangle ABC, and similarly for the other faces of the tetrahedron $DABC$. Hence:

Given four mutually tangent spheres, the contacts being external, the four Gergonne points of the four faces of the tetrahedron formed by the centers of the given spheres determine a tetrahedron homological to the tetrahedron of the centers, the ratio of homology being equal to -3.

A similar proposition may be stated about the Nagel points of the faces of the tetrahedron $DABC$ just considered.

5. Problem. *Construct the anticevian tetrahedron of a given point for a given tetrahedron.*

If σ is the tetrahedral polar plane of the given point M for the given tetrahedron (T'), the required tetrahedron (T) corresponds to (T') in the homology having M and σ for center and plane of homology, the anharmonic ratio of the homology being equal to $-1/3$.

The proof follows immediately from section (2), since the given point M has one and only one tetrahedral polar plane with respect to the given tetrahedron (T').

---

* Nathan Altshiller-Court, *Sur quatre sphères tangentes deux-à-deux;* Mathesis, vol. 47 (1933), p. 227 and p. 350.

6. If (T'') is the cevian tetrahedron of the point M for (T'), we have $(MLD'D'')=-3$. Eliminating the point L between this anharmonic ratio and the anharmonic ratio $(MLDD')=-3$ found before, we have

$$(MDD'D'') = -2 \tag{a}$$

which becomes, if M is supposed to be at infinity,

$$DD':DD'' = -1/2,$$

a special result which was noticed before.*

7. Returning to the general case, if (T''') is the cevian tetrahedron of M for (T''), we have, by virtue of (a),

$$(MD'D''D''') = -2. \tag{b}$$

Eliminating M between (a) and (b) we have

$$(DD'D''D''') = 4/7,$$

and similarly for the other groups of corresponding vertices of the four tetrahedrons considered.

The process may be continued by constructing the cevian tetrahedron of M for (T'''), and so on, indefinitely. It may also be extended indefinitely in the opposite direction by constructing the anticevian tetrahedron of M for (T), etc.

The anharmonic ratio of corresponding vertices of any four consecutive tetrahedrons so constructed is constant and equal to 4/7.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE McCAY CUBIC

By J. R. MUSSELMAN, Western Reserve University

In a recent note J. H. Weaver† has shown that the McCay cubic of a triangle cuts the circumcircle at the three vertices T_i and in three other points Q_i which form an equilateral triangle. This set of points, entirely independent of this new property, has been studied by Goormaghtigh‡ in the problem of locating those points M_i whose Simson lines are perpendicular to OM_i. Produce T_1 diametrically through O till it cuts the circumcircle at B_1, let T_1' be the midpoint of the arc T_2T_3 which does not include T_1. Then locate M_1 so that $2\angle T_1'OM_1$

* Educational Times, Reprints, vol. 44 (1886), p. 28, Q. 8047.

† This MONTHLY, vol. 42 (1935), p. 496.

‡ Mathesis, vol. 31 (1911), p. 163.

$= \angle M_1OB_1$ where M_1 lies on the arc $T_1'B_1$ to the right of T_1'. Similarly M_2 and M_3 can be constructed. Now these points M_i are Weaver's points Q_i since their coordinates when calculated turn out to be the complex numbers $-\sigma_3^{1/3}$, $-\omega\sigma_3^{1/3}$, $-\omega^2\sigma_3^{1/3}$. Hence we have the theorem that *the McCay cubic of a triangle intersects the circumcircle at the vertices and at three other points, forming an equilateral triangle, whose Simson lines are perpendicular to the lines joining the circumcenter to these points.*

The set of points M_i' diametrically opposite to M_i, together with M_i form a regular hexagon. Calling the Simson lines of these points S_i' and S_i respectively, Goormaghtigh has shown that the lines S_i' meet at the center of the Nine-point circle of $T_1T_2T_3$, and are the bisectors of the angles of the equilateral triangle formed by the lines S_i. The circumradius of this latter triangle is equal to that of the circle on $T_1T_2T_3$; the inscribed circle is the Nine-point circle of $T_1T_2T_3$. Boutin* has stated that the lines T_1M_1, T_2M_2, T_3M_3 meet at a point V and the lines T_1M_1', T_2M_2', T_3M_3' meet at a point W, where V and W are the Hessian or isodynamic points of the triangle $T_1T_2T_3$.

On McCay's Cubic

By R. Goormaghtigh, Bruges, Belgium

1. The contact properties believed to be new in J. H. Weaver's paper *On isogonal points* (this Monthly, 1935, p. 496) have already been given and even in a more general form.

When the straight line joining a point P to its isogonal conjugate P' passes through a fixed point Q, having as normal coordinates $\alpha_1, \alpha_2, \alpha_3$, the locus of P and P' is a cubic, anallagmatic in the normal coordinates inversion, and having

$$\sum \alpha_1 x_1(x_2^2 - x_3^2) = 0 \tag{1}$$

for equation.

The curve is tangent at Q to the straight line joining Q to its isogonal conjugate Q'; the cubic passes also through the incenter and excenters and the tangents at these points are concurrent at Q; the tangents at the vertices pass through Q'.†

When Q is the circumcenter, Q' is the orthocenter, QQ' is the Euler line and AQ', BQ', CQ' are the altitudes.

2. Similarly, when the straight line joining a point P to its isotomic conjugate P'‡ passes through a fixed point Q, having as barycentric coordinates

---

* Mathesis, vol. 32 (1912), p. 264.

† Van Aubel, Nouvelle Correspondance Mathématique, 1878, p. 261; Neuberg, *Mathesis*, 1923, p. 97.

‡ The points P and P' are isotomic conjugates when the joins from the vertices to P cut the opposite sides at the images, through the mid-points of these sides, of the points where the sides meet the joins from the vertices to P'. If x_1, x_2, x_3 are the barycentric coordinates of P, those of P' are $x_1^{-1}, x_2^{-1}, x_3^{-1}$.

A Tetrahedron, a Point, and a Triangle

By Maud Willey, Siloam Springs, Arkansas

Let A_1, A_2, A_3, A_4 be vertices of a tetrahedron; C any point not in the plane of any face of the tetrahedron; l_1, l_2, l_3 lines through C, l_1 meeting A_1A_2 in point B_{12} and A_3A_4 in point B_{34}, l_2 meeting A_1A_3 in point B_{13} and A_2A_4 in point B_{24}, l_3 meeting A_1A_4 in point B_{14} and A_2A_3 in point B_{23}. Let $B_{12}CB_{34}C_1$, $B_{13}CB_{24}C_2$, and $B_{14}CB_{34}C_3$ be harmonic ranges.

There is a one-to-one correspondence between the points represented by C and the planes of the triangles represented by $C_1C_2C_3$. Given any plane not containing a vertex of the tetrahedron, the corresponding point may be found by use of the principle of duality. The triangle $C_1C_2C_3$ is the diagonal triangle of the quadrilateral whose vertices are the traces of the edges of the tetrahedron in the given plane.

The two tetrahedrons $A_1A_2A_3A_4$ and $CC_1C_2C_3$ are perspective from four points, D_1, D_2, D_3, D_4. Every edge of each of the three tetrahedrons $A_1A_2A_3A_4$, $CC_1C_2C_3$ and $D_1D_2D_3D_4$ meets two opposite edges of each of the others in points which separate harmonically the two vertices on that edge. The three tetrahedrons $B_{12}B_{34}E_{12}E_{34}$, $B_{13}B_{24}E_{13}E_{24}$, $B_{14}B_{23}E_{14}E_{23}$, where E_{ij} is the trace of line A_iA_j in the plane of $C_1C_2C_3$, are related in the same way as tetrahedrons $A_1A_2A_3A_4$, $CC_1C_2C_3$ and $D_1D_2D_3D_4$.

Let Q_1, Q_2 and Q_3 be the ruled quadric surfaces determined by l_2, A_1A_4, A_2A_3; l_3, A_1A_2, A_3A_4 and l_1, A_1A_3, A_2A_4 respectively. Each side of triangle $C_1C_2C_3$ is a ruling of two of these quadric surfaces. There is a group of 192 collineations interchanging the points $A_1A_2A_3A_4CC_1C_2C_3$ and transforming the set of rulings of Q_1, Q_2 and Q_3 into itself. This group may be generated by the permutations $(A_1A_2)(A_3A_4)$, $(A_1A_3)(A_2A_4)$, $(CC_1)(C_2C_3)$, $(CC_2)(C_1C_3)$, $(A_1A_2)(CC_1)$, $(A_1A_3)(CC_2)$ and $(A_1C)(A_2C_1)(A_3C_2)(A_4C_3)$.

Let R_1, R_2, R_3 and S_1, S_2, S_3 be ruled quadric surfaces related to pairs of tetrahedrons $A_1A_2A_3A_4$, $D_1D_2D_3D_4$ and $CC_1C_2C_3$, $D_1D_2D_3D_4$ in the same way as Q_1, Q_2, Q_3 to the pair $A_1A_2A_3A_4$, $CC_1C_2C_3$. The group of collineations interchanging points $A_1A_2A_3A_4CC_1C_2C_3D_1D_2D_3D_4$ and transforming the set of rulings of $Q_1Q_2Q_3R_1R_2R_3S_1S_2S_3$ into itself is of order 576.

RECENT PUBLICATIONS

Edited by R. A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Challenging Problems in American Schools of Education. The 1934–1935 Sachs Lectures. By David Eugene Smith. New York, Teachers College, 1935. 48 pages. 55 cents.

Mathematics and the Question of Cosmic Mind, with other Essays. By C. J. Keyser. New York, Scripta Mathematica, 1935. vi+122 pages. 75 cents.

Statistical Method in Education. By C. W. Odell. New York, Appleton-Century Company, 1935. xx+458 pages. $3.50.

A First Course in Calculus. By H. L. Slobin and M. R. Solt. New York, Farrar and Rinehart, 1935. xii+426 pages. $3.00.

Financial Mathematics. By A. W. Richeson. New York, Prentice-Hall, 1935. xvi+361 pages. $2.50.

A Text Book of Algebra for Colleges and Engineering Schools. By W. H. H. Cowles and J. E. Thompson. New York, D. Van Nostrand Company, 1935. xii+402 pages. $2.25.

Introductory College Mathematics. By W. E. Milne and D. R. Davis. Boston, Ginn and Company, 1935. xiv+384+64 pages. $3.00.

Plane Trigonometry. By H. L. Rietz, J. F. Reilly, and Roscoe Woods. New York, The Macmillan Company, 1935. xii+142+72+x pages. $2.20.

Elementary College Algebra. By H. W. Kuhn and J. H. Weaver. New York, The Macmillan Company, 1935. xviii+360 pages. $2.00.

Elementary Differential Equations. By L. M. Kells. Second Edition. New York, McGraw-Hill Book Company, 1935. xii+248 pages. $2.00.

A First Course in Differential Equations. By Norman Miller. Oxford University Press, 1935. 148 pages. $2.50.

Intermediate Algebra. By R. W. Brink. New York, Appleton-Century Company, 1935. xii+268 pages. $1.35.

The Mathematical Theory of Finance. By K. P. Williams. New York, The Macmillan Company, 1935. xiv+280 pages. $2.75.

REVIEWS

A First Course in Calculus. By H. L. Slobin and M. R. Solt. New York, Farrar and Rinehart, 1935. xi+426 pages. $3.00.

The authors have presented us with a traditional type of calculus, the first 135 pages devoted entirely to differential calculus, the next 199 pages to integral calculus, and the last chapter of 42 pages to ordinary differential equations. The remaining pages contain 137 calculus formulas, answers to practically all the problems and an index.

The book is written more from a mathematical standpoint than from a technical or an engineering one. This is seen in the amount of algebra and trigonometry supposed familiar to the student who reads the chapter on "The Integral as a Sum." Here the authors determine the areas under a straight line, under a parabola, under an exponential curve and under a sine curve by summing a series of n rectangles and passing to the limit. We must remember that in 1932 Slobin and Wilbur published *Freshman Mathematics*, a book of 438 pages, dividing the subjects of college algebra, trigonometry, and analytic geometry in the ratio 2:1:2. While this calculus text is a completely independent book (save in some half dozen references to the other) the authors imply a greater familiar-

ity with algebra than seems to the reviewer to be the case unless the student has previously studied *Freshman Mathematics* or its equivalent.

A second indication of the mathematical viewpoint underlying the calculus is the 21 page discussion of variables, functions and limits which is the opening chapter. The authors include "as much rigor in the demonstrations as is necessary and possible in a first course"; hence we have precise and accurate definitions of terms and ideas including "removable and non-removable discontinuities." Of course it is a debatable question how much to tell the initiate about limits. The ordinary student feels we are making a lot of fuss about nothing when we try to be rigorous, but this may be somewhat due to poor teaching on our part. The reviewer feels that the authors have included too much material here unless there is a mathematical maturity greater than ordinary on the part of the student, or else time and labor is spent which could be used to better advantage elsewhere.

The text has been well planned and carefully written. One typographical error occurs on page 327 where .0008 should read .00008. The applications are the ordinary ones to mechanics. The chapters on "*Infinite Series*" and "*Functions and Their Power Series*" are especially commendable. The greater majority of the 185 diagrams are carefully drawn and in particular the authors should be complimented on those in the chapter on "*Multiple Integration*." It is unfortunate that figures 16, 38, 61 and 65 are below the standard set up by the others. To acquire facility about two thousand exercises are scattered throughout the text. It will be interesting to try this book on a class which can absorb and appreciate some theoretical discussion of the calculus.

J. R. MUSSELMAN

Introductory College Mathematics. By W. E. Milne and D. R. Davis. Boston, Ginn and Company, 1935. xiv+383+64 pages. $3.00.

The present book was written to offer a unified treatment of elementary mathematics to all groups of interested students. These, as the authors suggest, may be classified as those studying the subject for but one year, those interested in it as a tool, and those who may become teachers and mathematicians. In preparing this text, which is new in a significant sense, the authors have anticipated only two years of secondary school mathematics, on the part of the student. In the opinion of the reviewer, *Introductory College Mathematics* is an excellent work which merits careful consideration by teachers.

A mere perusal of the table of contents will prove profitable, as well as indicate the objective of the authors, who have presented their material under the following chapter divisions: I. Graphs and Functions; II. Differentiation; III. Integration; IV. Functions of the First Degree; V. Functions of the Second Degree; VI. Rational Integral Functions; VII. Rational Fractions; VIII. Irrational Functions; IX. Common Logarithms; X. Trigonometric Functions; XI. Circular Functions; XII. Trigonometric Formulas; XIII. Solution of Oblique Triangles; XIV. Further Properties of Circular Functions; XV. Natural Logarithms;

XVI. Exponential and Hyperbolic Functions; XVII. Simultaneous Equations and Determinants; XVIII. Analytic Geometry, Lines and Points; XIX. Loci, The Circle; XX. Parabola, Ellipse, and Hyperbola; XXI. Polar Coordinates; XXII. Tangents and Normals, Poles and Polars; XXIII. Transformation of Coordinates; XXIV. Parametric Equations and Loci; XXV. Permutations, Combinations, Probability; XXVI. Methods of Approximation.

The authors have built up their structure largely as an elementary theory of functions, whereby their unity of treatment is attained. This is not to suggest that the treatment is advanced in method, for the material has been presented with simplicity and clarity. The notions of the calculus are introduced naturally by means of the "average rate of change of a function," and the "exact rate of change of a function." Without going too deeply into differentiation, integration is sufficiently developed so that thereafter the algebra, geometry, and calculus of the functions considered can be studied together. The theory developed is always aptly applied to real problems seeming to bring out the role of the functions involved rather than that of one branch of mathematics as opposed to another. Hence to the analytic geometry appearing in the early portions of the book, notions from the calculus are applied. In many ways this results in a more elegant treatment than might have been given otherwise.

The material covered in Chapters IX to XIII is essentially a course in trigonometry. Thereafter the function concept again assumes a central role, appearing in the chapters devoted primarily to analytic geometry in the guise of a locus. In their treatment of loci authors do not consider the question of sufficiency. Theorems from a brief treatment of determinants are applied to the straight line and to the area of a triangle. The book closes with an excellent chapter on methods of approximation. This material, apart from introducing the tools he will use to the student in fields remote from, but applying mathematics, should prove to be stimulating to any one who comes to grips with it. It can well serve as a starting point for outside reading. The treatment of Approximation by Maclaurin's Series is an excellent introduction to infinite series, in an understandable and intelligently motivated manner.

With the exception of their treatment of the differential* the authors have achieved a beautiful simplicity, clarity, and unity of exposition. There is an abundance of illustrative material and of exercises. Certainly those students who come to mathematics not by choice will be well served by this text, which is as "lively and exciting" as an elementary mathematics text can be. After comparison with many other books written in keeping with the modern trend, *Introductory College Mathematics* is likely to be judged as well worth the writing.

S. B. LITTAUER

* The notation D_xy is not used, as is now customary, nor is dy defined as $(dy/dx)dx$. The authors attempt to show that $dy=(dy/dx)dx$, and that dy/dx can be treated "legitimately" as a fraction, and "dx, dy, etc., as ordinary quantities." The procedure and the implications therefrom might lead to confusion.

New Numbers. By F. E. Andrews. New York, Harcourt, Brace, & Co., 1935. 161 pages. $2.00.

Mathematicians have long known that for many purposes twelve would be a better base than ten for our number system, but few or none of them have worked out the details and tried to publicize their knowledge. This is the purpose of this most interesting little book, *New Numbers*.

An introductory chapter on the origins of counting and number systems is followed by a discussion of the practical disadvantages of counting by tens in ordinary life. These are, of course, closely linked to our common measures, but the thesis advanced is that the system of counting is the thing to be changed, and the arguments are surprisingly convincing. The author then systematically investigates the merits of various bases from the standpoint of arithmetic alone. The verdict in favor of twelve is based on both its superior factorability and other considerations less generally known, introduced here and in the later chapters on Logarithms and Curiosa. For the benefit of the amateur five chapters are given to instruction in arithmetic on a duodecimal basis. The final chapter is a temperate plea for reforming our number system, and also contains some suggestions for rationalizing certain common measures.

As a piece of propaganda it is unlikely that this book will accomplish much beyond stirring up discussion, for obvious reasons recognized by the author. It is, however, a valuable addition to the growing list of popular books on mathematics, because the author has not only done his work well, but has been fortunate in choosing a subject so simple that any reader can really understand it. The reader who already knows the mathematics will find his interest sustained by the new aspect which the author's novel proposals put upon more or less familiar material. This book should be added to the list published in this MONTHLY, vol. 35 (1928), p. 221.

W. A. WILSON

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

SPECIAL ANNOUNCEMENT

Since many clubs are already following the plan of rewarding the undergraduate presenting the best paper at a club meeting, this department suggests that each club choose the best paper presented at any of its meetings during the academic year 1935–36; that a copy of this paper be sent to this department before July 1, 1936 together with a list of books and articles consulted. These papers will be carefully considered, notice will be given of all received, and selections will be made for publication in this department.

Club Topics

Volume XXV of this MONTHLY contains a series of carefully compiled lists of references on subjects especially suited for club programs. There are fourteen topics in the series, more than enough for a year's work in any club.

This volume also contains a compilation of thirty-seven clubs active during 1917–18. It is worthy of note that nineteen of these clubs, or their logical successors, have reported programs for 1934–35. Only two of the remaining ones are known to have gone out of existence. We hope to have reports from all of those still active.

Over seventy clubs have reported to this department since a questionnaire was sent out in the autumn. The programs will appear as rapidly as space permits. It is hoped that all mathematics clubs in America may make some report before June 1, 1936, the dead line for entry in the summer compilation of active organizations.

CALCULATING MACHINES

Although we are apt to consider calculating machines as a product of the nineteenth century, Pascal built an adding machine in 1642 and a few years later issued a warning against imitators. He wrote:

"The possibility for distrust, dear reader, might be the imperfect reproductions of this machine which have been produced by the presumption of certain artisans. In these cases I beg of you to carefully consider, to guard yourself from surprise, to distinguish between 'la lepre et la lepre,' and not to judge the true original by the imperfect productions of the ignorance and the temerity of the mechanics."

Leibniz, about 1671, built a machine primarily for multiplication.

Napier's Rods and Galileo's Geometric and Military Compasses preceded these, as did the invention of the slide rule with its heated debate with regard to priority.

For discussions of the history and development of modern calculating machines the following references are suggested:

D. E. Smith, *Source Book of Mathematics*, pp. 156, 160, 164, 173, 186; F. Cajori, *A History of Mathematics*, p. 485; Vera Sanford, *A Short History of Mathematics*, p. 350; Encyclopaedia Britannica, Nineteenth Edition, vol. 16, p. 712; F. A. Willers, *Methoden der Praktischen Analysis*, p. 35; The Comptometer Company, *Applied Mechanical Arithmetic as Practiced on the Comptometer*; J. Turck, *Origin of the Modern Calculating Machine*, pp. 11, 132; R. Mehmke, *Numerisches Rechnen*, Encyclopaedia der Mathematischen Wissenschaften, Band 1, seite 960; Oeuvres de Blaise Pascal, Paris, 1908, vol. 1, pp. 303–314; University of California Publications in Mathematics, vol. 1, No. 9, pp. 187–209.

DEVELOPMENT OF PRESENT DAY NUMERALS

From the day when men called any group containing more than two objects "many," through the processes of making individual marks for each object to the present simple decimal notation is a long story. For some of its more interesting chapters the reader is referred to the following:

L. C. Karpinski, *History of Arithmetic*, Chap. 2; and *Rara Arithmetica*, pp. 4–6, 26, 37, 57, 88; F. Cajori, *History of Mathematical Notations*, Chapters II, III; and *History of Elementary Mathematics*, pp. 1–19; D. E. Smith, *History of Mathematics*, vol. 2, p. 36 et seq.; T. L. Heath, *Manual of Greek Mathematics*, Chap. II. L. L. Conant, *Primitive Number Systems*, Smithsonian Reports, 1892, pp. 583–594; and *The Number Concept*.

Club Activities

Intercollegiate Mathematics Association, Milwaukee, Wisconsin

Milwaukee has five institutions of higher learning: Mount Mary College, Downer College, Milwaukee State Teachers College, University of Wisconsin Extension Division and Marquette University. All of these Colleges have had Mathematics Clubs for many years: At the instigation of Mrs. Ethelwynn R. Beckwith of Downer College and Professor H. P. Pettit of Marquette University, the Clubs banded together in 1934 and formed the Intercollegiate Association. The objects of the Association are to stimulate research in the field of mathematics and to serve as a forum for the discussion of problems which arise in the separate clubs.

Meetings are held once every two months, the schools serving as hosts in turn. Papers are given by the members and discussions follow. Officers for 1935–36 are: Miss Brady, Milwaukee State Teachers College, President; Mr. Christof, University of Wisconsin Extension Division, Vice President; Miss D. Beringer, Mount Mary College, Secretary; Miss Lange, Downer College, Treasurer; E. H. Stock, Marquette University, Corresponding Secretary.

The Association publishes a yearly bulletin "The Circle."

At the first annual banquet Professor G. A. Parkinson of the University of Wisconsin Extension Division spoke on "Mathematics and civilization."

Pi Mu Epsilon of Marquette University

During the year 1934–35 this chapter issued a one page monthly bulletin containing "a record of business transacted at the previous meeting, an abstract of the paper presented and information concerning the next meeting."

The Bulletin contained abstracts of the following papers, "Plane linkages" by Professor H. P. Pettit; "Fifth postulate of Euclid" by E. E. Boren; "Brocard points of a triangle" by Geraldine Stein; "Certain physical laws" by Sherman Baker; "Mathematics of illumination" by Dean F. A. Kartak of the College of Engineering.

A banquet and an evening given to the solution of mathematical puzzles were features of the social program.

This chapter annually conducts the Frumveller Competition. This is a competitive examination, open to properly prepared high school seniors in Milwaukee County. It consists of three parts, 1. Elementary algebra, 2. Plane geometry, 3. Choice of advanced algebra, solid geometry or trigonometry. First prize is a semester's scholarship at Marquette University donated by the University; second prize is a gold medal donated by the chapter; while Honorable Mention certificates are awarded the next eight. In the spring of 1935 there were 52 contestants. Robert Schultz of Washington High School won first place with Laura Gilbert of Riverside High School second. They were guests at the chapter's annual banquet.

Officers for 1935–36 are, J. Sevenich, Director; S. Baker, Vice Director; F. Adler, Secretary; W. Nelson, Treasurer; Eleanor Bartozak, Librarian.

Junior Mathematics Club, University of Wisconsin, Extension Division

The club sponsored a Mathematical Exhibit to which high school mathematics clubs were invited. Two hundred guests saw the exhibits, looked through the telescope and heard a short program. Dolores Weaver spoke on "The history of mathematics."

A prize, known as the Euler Prize, is offered by the club. It was won in 1934–35 by Eugene Hensel with a paper on "Linkages": Other papers offered in this competition were, "The fourth dimension" by Mr. Feinstein, "Pythagorean theorem" by H. Jacobi. Professor W. E. Roth spoke to the Club on "Integers" and Florence Axen spoke on "Determinants." A Christmas party with mathematical puzzles was largely attended. Officers for 1934–35 were, H. Jacobi, President; Carolyn Bauer, Secretary-Treasurer; Florence Axen, Club Adviser.

The Mathematics Club of Connecticut College

Eight meetings were held during the year 1934–35, including a picnic and a Christmas party with mathematical games. Two members, Ruth Grolotzhe and Geraldine Coon, were sent to speak

at the Student Science Conference at Massachusetts State College. The club purchased a set of models to be used in mathematics classes of the College.

Papers and speakers for the year were as follows:

"The summer mathematics meeting at Williamstown" by Dr. Julia Bower; "The Influence of mathematics on contemporary thought," Professor D. D. Leib; "Mathematical signposts," Audrey LaCourse; "Stars and angles" by Ruth Howell; "The beginning of algebra" by Harrette Webster; "Euler's partion numerorum" by Professor H. Rademacher of the University of Pennsylvania; "The curve of quickest descent" by Ruth Grodotzhe; "Determination of orbits" by Gertrude Coon; "Famous women in mathematics" by Dr. Ruth Mason of Wellesley College.

The moving picture, "Einstein's theory of relativity" from the Kodascope Library was shown at one meeting.

The Mathematics Club of Rutgers University

Meetings were held monthly during 1934–35 with the following programs; "Some properties of the medians of a triangle" by B. Julius; "Indirect methods in geometry" by H. Selby; "Isolation of the roots of a quartic equation" by L. Chando; "Interpolation methods" by B. M. Walter; "Quaternions" by R. H. Elliott; "The fourth dimension" by R. Leviton; "This simple universe" by Professor B. H. Brown of Dartmouth College; Mathematics used in statistics" by E. A. Darby; "Mathematics and science" by Professor C. N. Moore of the University of Cincinnati; "Mathematics and Economics" by Professor A. E. Meder.

Officers for 1935–36 are: John B. McIlroy, Jr., President; Edward A. Darby, Vice President; Paul F. Stryker, Secretary-Treasurer; Professor E. P. Starke, Faculty Adviser.

Pi Mu Epsilon of Washington University

With fifty active members the chapter had a very successful year in 1934–35. The 14 new initiates were drawn from various schools as follows: Liberal Arts, 7; Engineering and Architecture, 5; Graduate Studies, 1; Faculty, 1.

Eight regular meetings were held. Papers and speakers were as follows: "Convergence criteria for infinite series with positive signs" by Dr. C. E. Buell; "The solution of Diophantine equations by congruences" by R. R. Middlemiss; "*P*-adic numbers" by Mr. Gottleib; "Pell equations" by Ona Defoe; "Photo elasticity" by Professor E. Siroky; "Some geometrical applications of complex numbers" by Viola Muench; "Gibbs-Fourier series phenomenon" by J. A. Joseph; "Some elementary transformations of the plane with respect to the parabola" by Professor O. Dunkel; "One hundred million years of astronomical history" by Professor Jessica Y. Stephens.

In addition the chapter sponsored a lecture, attended by about 70 people. Professor G. Szegö spoke on "Some recent developments of the theory of numbers."

Officers for 1935–36 are Professor Jessica Y. Stephens, Director; A. E. Martin, Vice Director; Goldine Cohnberg, Secretary; H. Clark, Assistant Secretary and Librarian; H. R. Grummann, Treasurer; Elizabeth Ham, B. Black, J. A. Joseph, Helen Mardorf, Student Members Executive Committee.

Echols Mathematics Club of University of Virginia

During 1934–35 the club held nine regular meetings besides enjoying a special address by Professor J. Williamson of Johns Hopkins University on "Matrices whose elements are matrices." The list of papers and speakers follows: "The functions defined by the expression $(e^x-e^a)/(x-a)=f_1(a)+xf_2(a)+\cdots+x^{n-1}f_n(a)+\cdots$." by A. D. Wallace; "Residue-indices and the number of values of b for which the congruence $x^n\equiv b$, mod m is solvable" by D. W. Hall; "Certain ordered sets" by Professor G. T. Whyburn; "A classification of bicircular quartics" by J. W. Blincoe; "Gergonne's solution of the problem of Apollonius" by Professor B. Z. Linfield; "The purposes and methods of the American Mathematical Society" by Professor M. H. Ingraham of the University of Wisconsin; "Some familiar loci in areal coordinates" by G. C. Watson; "Orthogonal hyperspheres in Euclidian n-space" by M. W. Aylor; "The Hessian points of a one-parameter family of triangles" by E. K. Ritter; "Matric differential equations" by A. D. Wallace; " Systems of

coordinates and mathematical spaces" by Professor J. J. Luck; "A matric identity in algebraic geometry" by Professor B. Z. Linfield; "A degenerate bicircular quartic" by J. W. Blincoe; "Residue-indices" by D. W. Hall.

The officers for 1935–36 are: J. W. Blincoe, President; A. D. Wallace, Secretary-Treasurer:

Kappa Mu Epsilon of Northeastern Teachers College, Oklahoma

This chapter, the first organized of this national fraternity, had a very active year, its fourth. Besides initiation socials, Founder's Day banquet, and a drive in the Cherokee Hills and entertainment given for the Wagoner High School mathematics classes, the chapter held ten regular meetings. The speakers and their subjects are as follows:

"Changing numbers to different bases" by Dorothy Bixby; "Theory of the divisability of numbers" by L. P. Woods; "Developing trigonometric formulas" by L. Smith; "Tricky formulas" by Lorene Williams; "Solid geometry" by Phillip Bohart; "Analytic formulas" by Evelyn Clemens; "Physics formulas" by W. King; "Advantages of mathematics" by L. Towry; "The theory of equations" by L. P. Woods; "The position of the high school teacher" by Wilma Fulbright; "Partial fractions" by Mary Sullivan; "Pascal's triangle" by S. Capehart; "Magic squares" by Leota Lindsey; "A citizen of the universe" by E. Durham; "Radio" by J. Cowherd; "Electrolysis of Water" by T. Bedwell; "Mathematical Recreations" by W. King. Evelyn Clemens, J. Williams and Iva Holbrook told of the history of Kappa Mu Epsilon and discussed its constitution.

The annual issue of *The Exponent* is a four page paper filled with reports of the year, alumni notes and amusing skits.

The William S. Hall Mathematics Club of Lafayette College

The club held six regular meetings at which the following papers were presented, chiefly by students, occasionally by faculty members: "Transfinite numbers," "The view point of infinity applicable to descriptive geometry," "Geometry of the Michelson-Morley experiment," "The geometry of growth," "Path of a charged particle under the influence of an electrostatic and an electromagnetic field," "The life and work of Newton," "Discussion of the cylinder, sphere and pseudo-sphere," "The mathematical theory of ruled surfaces."

At an open meeting, Professor J. S. Taylor, University of Pittsburgh, spoke on "Some practical aspects of the theory of probability."

Pi Mu Epsilon of Oregon State College

Regular monthly meetings were held with papers by students and faculty members. The most outstanding papers were "Photo-elasticity" by E. H. McAllister and H. Eberhart, and "Curve fitting and graphical solution of equations" by W. Miller.

The chapter joined with the other scientific honorary societies on the campus in supporting the Willamette Valley Students' Scientific Conference, at which papers were presented by science students of universities and colleges in western Oregon.

The annual initiation and banquet were held in May. For 1935–36 M. M. Day is Director and G. A. Williams is Secretary-Treasurer.

Van Vleck Club of Wesleyan University

The club held four regular meetings besides a steak roast in the autumn and an outing on Long Island Sound in the spring.

The papers presented were as follows: "History of mathematics" by Professor M. C. Foster; "The mechanics of three monkeys and a rope" by A. G. Bagg; "The monkey's age" by B. Reynolds; "The nine-point circle" by E. Hudson; "Simpson's line" by A. R. Parente; "Projective geometry of conic sections with special reference to Pascal's line" by K. Yoneda; "Determination of coefficients for a special type of differential equations" by S. Wardwell; "Solution of equations," by J. L. Melnick; "Scales of notation" by E. C. Hammarstrom.

Officers for 1934–35 were; A. G. Bagg, President; A. R. Parente, Vice President; E. C. Hammarstrom, Secretary; Professor B. H. Camp, M. C. Foster and H. E. Arnold, Faculty Advisers.

The Mathematics Club of St. Xavier College

The club holds bi-monthly meetings. In 1934–35 the officers were Catherine Butler, President; Margaret Predergast, Vice President; Bernice Peterman, Secretary. Following is a list of speakers and their subjects:

"Mathematical fallacies" by Geraldine Buchner; "Cell of the honey bee" by Margaret Prendergast; "History of Arabic numerals" by Imelda Davidson; "Slide rule" by Kathleen Cooney; "Finding cube roots" by Beatrice Brenken; "Mathematical notes" by Margret Engler; "Japanese computing machines" by Mary Matsumoto; "A letter concerning the improvement of statistics written by Walter Willcox" by Bernice Peterman; "Strange case of the theorem of Pythagoras" by Margaret Engler; "Curious case of mathematical induction applied to geometry" by Imelda Davidson; "Adding machines" by Bernice Brenken; "Monroe calculating machines" by Dorothy Normoyle; "The tree of knowledge" by Catherine Butler. A short play was presented by a dozen of the members.

The club has twenty-five active members. An annual open meeting is held to which all students and faculty are welcomed.

Officers for 1935–36 are: Margaret Engler, President; Catherine Regan, Vice President; Ellen Leonard, Secretary.

The Junior Mathematical Club of the University of Chicago

The Club sponsors the social affairs of the department besides its regular meetings, each preceded by a social half hour in the Common Room. In 1934–35 there were 45 members and the club held two bridge parties, a bridge tea and three bridge dances and 14 regular programs. The speakers and their subjects follow: "Famous contributors to the theory of the calculus of variations," Professor G. A. Bliss; "A property of harmonic functions" by M. Coral; "The theory of demand for completing and competing goods" by Professor H. Schulte of the Department of Economics; "Some famous problems in the theory of numbers" by D. M. Dribin; "History of algebra" by Professor L. M. Graves; "Transfinite numbers" by M. F. Smiley; "Some famous problems in the calculus of variations" by H. H. Goldstine; "Some applications of matrix theory to psychology" by Professor L. L. Thurstone, Department of Psychology; "The development of the multiplier rule in the calculus of variations" by N. A. Moscovitch; "The scientific works of Eliakim Hastings Moore" by Professor G. A. Bliss; "Descriptive geometry" by F. A. Valentine; "Fundamentals of continued fractions" by I. E. Perlin; "The evolution of the number concept" by Miss M. G. Humphreys; "Projective and metric geometries" L. R. Wilcox. Officers for 1935–36 are: D. M. Dribin, President; C. H. Denbow, Treasurer; H. H. Goldstine, Program Chairman; M. F. Smiley and Kitty Cartwell, Social Chairmen.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 192. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A merchant marked his merchandise in code, replacing each real digit by a

preassigned code digit. His new bookkeeper made out the following bill, *every* digit in which is thus coded.

4 Gimcracks	at	81¢	\$7.85
2 Baubles	at	70¢	.37
6 Gee Gaws	at	9¢	.16
			\$8.43

What were the correct figures in this bill?

E 193. *Proposed by A. Gloden, Luxembourg.*

Find a perfect square of six digits, such that if we consider the numbers formed by the successive pairs of its digits, the second of these three two-place numbers exceeds by unity the sum of the other two.

E 194. *Proposed by Abe Gelbart, Paterson, New Jersey.*

Of all the lines through vertex A of triangle ABC, there is just one which meets BC in the point D such that the incircles of triangles ABD and ACD are equal. Show that this line AD may be constructed by ruler and compasses.

E 195. *Proposed by V. Thébault, Le Mans, France.*

Show that there is just one number whose square, expressed in the scale of six, contains each of the six different digits just once.

E 196. *Proposed by J. Rosenbaum, Hartford Federal College.*

The construction by ruler and compasses of the edge of a cube, whose volume equals the sum of the volumes of two cubes of arbitrary edges, is known to be impossible. However, the construction is possible in particular cases, as for instance when the two given cubes have edges proportional to $9+\sqrt{5}$ and $9-\sqrt{5}$. Derive a general formula for determining such constructible cases.

E 197. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Construct a parallelogram, given both its altitudes and the acute angle between its diagonals.

E 198. *Proposed by J. E. Trevor, Cornell University.*

A multiplication of a three-place number by a two-place number has the form

$$\begin{array}{r} p\,p\,p \\ p\,p \\ \hline p\,p\,p\,p \\ p\,p\,p\,p \\ \hline p\,p\,p\,p\,p \end{array}$$

The p's are all prime digits, different from unity. Determine their values and show that the solution is unique.

Also solved by Frank Ayres, J. E. Burnam, H. E. H. Greenleaf, C. A. Murray, D. B. Perry, E. P. Starke, C. W. Trigg, Ruth Woerner and the proposer.

E 166 [1935, 444]. *Proposed by W. E. Buker, Leetsdale High School, Pa.*

In the triangle PQR, M and N are points in PQ and PR respectively, such that $PM:MQ=a:b$, and $PN:NR=c:d$. In what ratio, in terms of a, b, c and d, is MN cut by the median of PQR drawn from P?

Solution by J. M. Feld, New York City

A more general problem is arrived at if, instead of letting MN be cut by the median drawn from P, MN is cut by a line PS which divides QR in the ratio $QS:SR=e:f$.

Let PS intersect MN at T. Let $MT=x$ and $NT=y$. Draw QQ' and RR' parallel to MN, intersecting PS at Q' and R' respectively. Let $QQ'=x'$ and $RR'=y'$. From pairs of similar triangles the following three proportions are obtainable:

$$x/x' = a/(a+b)$$
$$y'/y = (c+d)/c$$
$$x'/y' = e/f.$$

Consequently

$$x:y = ae(c+d):cf(a+b)$$

In the given problem, $e=f$, and the required ratio is then

$$x:y = a(c+d):c(a+b).$$

Also solved by J. E. Burnam, K. W. Crain, Wm. Douglas, H. E. H. Greenleaf, S. S. Henry, L. M. Kelly, D. L. MacKay, W. G. McGavock, Leon Recht, J. T. Rule, E. P. Starke, C. W. Trigg, T. G. Thurston and Simon Vatriquant.

E 167 [1935, 444]. *Proposed by J. M. Feld, New York City.*

If $A+B+C=180°$, and if $2S=\sin A+\sin B+\sin C$, prove that

$$\sin^2 A \sin^2 B \sin^2 C = 4S(S-\sin A)(S-\sin B)(S-\sin C).$$

Solution by L. M. Kelly, Lawrence, Massachusetts

The similarity between this expression and Heron's formula for the area of a triangle of sides a, b and c, is at once apparent. We also use $2R=a/\sin A=b/\sin B=c/\sin C$, where R is the circumradius, and the elementary formula, $Area=\frac{1}{2}ab\sin C$. Combining gives us

$$\sqrt{s(s-a)(s-b)(s-c)} = 2R^2 \sin A \sin B \sin C.$$

Squaring and substituting for s, a, b and c gives us

$$16R^4S(S-\sin A)(S-\sin B)(S-\sin C) = 4R^4 \sin^2 A \sin^2 B \sin^2 C$$

which immediately reduces to the desired formula.

Also solved by J. E. Burnam, Arthur Danzl, O. E. Eggert, L. S. Johnston, M. W. Keller, C. W. Trigg and Simon Vatriquant.

E 168 [1935, 445]. *Proposed by V. Thébault, Le Mans, France.*

a, b and c are three consecutive digits in some order, and in some scale of notation the number $aabb=(cc)^2$. It is required to determine the base of the system of enumeration and the values of a, b and c.

Solution by Simon Vatriquant, Brussels, Belgium

Denoting the base of the system by B, we may write

$$(1) \qquad aabb = (cc)^2 \quad \text{or} \quad (aB^2 + b)(B + 1) = (B + 1)^2c^2$$

from which we note that $a<c$, and that aB^2+b is divisible by $B+1$. Since $aB^2+1=a(B^2-1)+(a+b)$, it follows that $(a+b)=k(B+1)$. But $a<B$ and $b<B$, so $k=1$. Consequently, $B=a+b-1$. If we use this equation to eliminate a from the second part of (1), we get

$$(2) \qquad B^2 - Bb + b = c^2 \qquad [c < B].$$

We must now consider the three possible orders for a, b and c. First, if $a<b<c$, $a=c-2$, $b=c-1$, and $B=2c-4$. Then (2) reduces to $c^2-9c+11=0$, which has no integer root. Second, if $a<c<b$, $a=c-1$, $b=c+1$, and $B=2c-1$. Then (2) reduces to $c^2-4c+3=0$, which has the roots $c=1$ and $c=3$. The former is ruled out, since $a\neq 0$, but the latter gives a solution, namely

$$2244 = (33)^2 \qquad [\text{base } 5].$$

Finally, if $b<a<c$, $b=c-2$, $a=c-1$, and $B=2c-4$. Then (2) reduces to $c^2-7c+6=0$, with roots $c=1$ and $c=6$. The former is ruled out as before, but when $c=6$, the following solution emerges:

$$5544 = (66)^2 \qquad [\text{base } 8].$$

There are no other solutions.

Also solved by W. E. Buker, O. E. Eggert, E. P. Starke, C. W. Trigg, G. W. Wishard, B. C. Zimmerman and the proposer.

E 169 [1935, 445]. *Proposed by A. A. Bennett, Brown University.*

Prove that n, the number of years in which a sum of money will double itself at the interest rate r, compounded annually, is given approximately by $n=1/3+9/13r$, and determine the approximate error.

Solution by W. R. Ransom, Tufts College

From $(1+r)^n=2$, we get $n=\ln 2/\ln(1+r)$. Using the approximations $\ln(1+r)=r(1-r/2)$ and $1/(1-r/2)=1+r/2$, this becomes

$$(1) \qquad n = \left(\frac{1}{2} + \frac{1}{r}\right)\ln 2.$$

If ln 2 is converted into a continued fraction, the successive convergents are found to be $2/3, 7/10, 9/13, 61/88, \cdots$. If we use $2/3$ for ln 2 when we multiply it by $\frac{1}{2}$ in (1), but use $9/13$ for ln 2 when we multiply it by the (ordinarily) much larger $1/r$, we get $n=1/3+9/13r$, as was proposed.

To estimate the error, we shall assume that the rate is between 1% and 10%. Since $1/\ln(1+r)=1/r+1/2-r/12+r^2/24- \cdots$, the error introduced by replacing $\ln(2)/\ln(1+r)$ by $(\frac{1}{2}+1/r)\ln(2)$ is of the order of $r/12$, and hence ranges from $+.0006$ to $+.006$ as r ranges from 1% to 10%. The error introduced by using $1/3$ for $\frac{1}{2}\ln(2)$ is roughly $-.013$, and that introduced by using $9/13r$ for $(\ln 2)/r$ is about $-.0008/r$, which ranges from $-.08$ to $-.008$. Hence the combined error ranges from about $-.093$ to $-.015$ as r ranges from 1% to 10%. Consequently the given formula will generally underestimate the time by a month or less.

Also solved by E. P. Starke.

E 170 [1935, 445]. *Proposed by T. C. Fry, Bell Telephone Laboratories, New York City.*

It is a simple matter to erect equilateral triangles on the sides of any triangle ABC, thereby determining three points, P, Q and R, which constitute the new vertices of these equilateral triangles. It is here proposed to invert this process and construct the triangle ABC when given only the points P, Q and R.

Solution by J. Rosenbaum, Hartford Federal College

Let the equilateral triangles on the sides of the original triangle ABC be ABR, BCP and CAQ. Now if the pairs of points A and P, B and Q, C and R are on opposite sides of BC, CA and AB respectively, then it is known that the line segments AP, BQ and CR are equal and also concurrent. Denote this point of concurrency by O. It can be proved (see Altshiller-Court's *College Geometry*, Theorem 196) that if none of the angles of triangle ABC exceed 120°, then this point O is the isogonal point of both triangles ABC and PQR. Accordingly, if a triangle DEF is obtained from triangle PQR by drawing the equilateral triangles PQF, QRD and RPE outside triangle PQR, the lines PD, QE and RF will be concurrent at O. From this it follows that the points A, B and C lie on PD, QE and RF respectively. Furthermore, in the triangles DRA and QRB, DR and QR are sides of the equilateral triangle DRQ; also RA and RB are sides of the equilateral triangle RAB, and the angles DRA and QRB are each equal to 60° plus (or minus, as the case may be) angle ARQ. Hence triangles DRA and QRB are congruent, and DA equals QB. But, as mentioned above, QB equals AP, so that DA equals AP. Similarly, EB equals BQ, and FC equals CR. It thus follows that A, B and C are the midpoints respectively of PD, QE and RF. This gives the construction of the triangle ABC of the kind mentioned, when PQR is such that a solution of this kind exists.

In a similar manner it is shown that if the points D, E and F are obtained from triangle PQR by erecting the equilateral triangles so that the pairs of

points P and D, Q and E, R and F, lie on the same sides of QR, RP and PQ respectively, then the midpoints of PD, QE and RF will give a solution ABC where the pairs of points A and P, B and Q, C and R, lie on the same sides of BC, CA and AB respectively.

In order to admit a solution of the first kind, PQR has to satisfy certain restrictions. When these restrictions are not satisfied, then both the constructions outlined above will yield solutions of the second kind, so that in this case there will be two solutions of the second kind, but none of the first kind.

Next, consider the triangle PQR obtained by taking P and A on the same side of BC, and the remaining two pairs of points on opposite sides of their respective corresponding sides. Here it is seen by drawing the figure that the triangles RBP and ABC are congruent by two sides and the included angle. Hence AC equals PR, and AC is a chord of length PR lying on the circle centered at Q and with radius equal to PR. In the same way, the triangle QPC is congruent to ABC, and hence AB is a chord of length QP lying on a circle centered at R with radius equal to QP. The point A is therefore one of the intersections of these two circles. (It seems that for the case under discussion, that intersection must be used which is separated from P by QR). This, together with the conditions of the problem, enables us to locate the remaining points B and C. In the same way the other two solutions of this third kind (in which one of the original equilateral triangles overlaps triangle ABC) may be obtained.

Finally, consider the case in which two of the original three equilateral triangles overlapped triangle ABC. Suppose PBC does not overlap ABC, but QCA and RAB do. Then the triangles QPC and RBP are each congruent to ABC, so that AB equals PQ and AC equals PR. As before, A is at an intersection of the circles centered at Q and R with respective radii PR and PQ, using this time the intersection which was not used in the third kind of solution. Here again there are three solutions, according as PBC, QCA or RAB is taken as the non-overlapping equilateral triangle.

Hence for every possible set of non-collinear points P, Q and R, we may determine eight different triangles ABC having the equilateral triangles PBC, QCA and RAB constructed on their sides.

J. R. Musselman calls attention to the fact that this problem was proposed by Lemoine and one solution printed in the *Nouvelles Annales de Mathématique* (1869) page 40. A more recent statement of the problem with an indicated solution appears in the *Mathematical Gazette* of May 1935, p. 91, by V. Thébault.

D. L. Mackay further cites L'Intermédiaire des Mathématiciens, vol. 9 (1902), p. 332; Mathesis, vol. 9 (1889), p. 173 and 2nd ser., vol. 9 (1899), p. 131.

Also solved by W. B. Clarke, L. M. Kelly, S. A. Schelkunoff, E. P. Starke, Norman Stein, Simon Vatriquant and the proposer.

Editorial Note. The "certain restrictions" mentioned in the above solution which must be satisfied if a "solution of the first kind" exists may be stated in

an interesting geometrical form. Consider any side of the given triangle PQR, say the side QR, and the corresponding *outside* equilateral triangle QRD, and let the sides DQ and DR be extended beyond Q and R. Construct a circle tangent to the rays DQ and DR and with a radius $2\overline{QR}/\sqrt{3}$. If the vertex P lies inside this circle, there will be one solution of the first kind and one of the second kind; if P is outside of the circle, there will be no solutions of the first kind but two of the second kind; but if P is on this circle there will be one solution of the second kind and a set of three points A, B, C *on a line.* This last case furnishes a slight exception to the statement that there will always be eight different triangles ABC.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3769. *Proposed by Don Wallace, Charlottesville, Va.*

If the medians of a triangle be extended to meet the circumcircle in points forming a second triangle, then the centroid of the given triangle is that point for which the product of the distances to the vertices of the second triangle is an extreme.

3770. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres, the spheres orthogonal to one of them and belonging to the coaxial net determined by the remaining three form a coaxial pencil. The four lines of centers of the four pencils thus obtained form a hyperbolic group of lines.

Note. The corresponding property of three circles was discussed in Nouvelles Annales de Mathématique, 1852, p. 198, and 1858, p. 79, Q. 195.

In regard to coaxial pencils see, for instance, Altshiller-Court, *Modern Pure Solid Geometry* (Macmillan, 1935), p. 200, ex. 4.

3771. *Proposed by Hansraj Gupta, Hoshiarpur, India.*

Prove that

$$61! + 1 \equiv 0 \bmod 71,$$
$$63! + 1 \equiv 0 \bmod 71.$$

Prove the general result of which these are particular cases.

3772. *Proposed by H. D. Ruderman, Brooklyn, N. Y.*

Given two perspective triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$, let d_{ij} denote the

perpendicular distance from A_i' to A_iA_k, where d_{ij} is positive or negative according as A_i' and A_j are on the same or opposite sides of A_iA_k. Prove that $d_{12}d_{23}d_{31}=d_{13}d_{32}d_{21}$.

3773. *Proposed by William Hoover, Columbus, Ohio.*

Two planes are tangent to a central quadric surface; they are also at right angles to each other and their line of intersection has a given direction. Find the form of the cylinder generated by their line of intersection.

SOLUTIONS

3691 [1934, 395]. *Proposed by E. P. Starke, Rutgers University.*

Show that

$$\sum_{j=k}^{[n/2]}\binom{j}{k}\binom{n}{2j} = 2^{n-2k-1}\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right],$$

where n is a positive integer, k is a positive integer or zero, $[n/2]$ is the greatest integer in $n/2$, and $k \leqq [n/2]$.

Solution by the Proposer

From trigonometry we have

$$\text{(1)}\qquad \cos n\theta = \frac{1}{2}\sum_{k=0}^{[n/2]}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](2\cos\theta)^{n-2k}.$$

Since the left side of (1) is also the real part of $(\cos\theta+i\sin\theta)^n$ we have

$$\begin{aligned}\cos n\theta &= \sum_{j=0}^{[n/2]}(-1)^j\binom{n}{2j}\cos^{n-2j}\theta\,\sin^{2j}\theta\\ \text{(2)}\qquad &= \sum_{j=0}^{[n/2]}(-1)^j\binom{n}{2j}\cos^{n-2j}\theta\sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}\cos^{2j-2k}\theta\\ &= \sum_{k=0}^{[n/2]}(-1)^k\sum_{j=k}^{[n/2]}\binom{j}{k}\binom{n}{2j}\cos^{n-2k}\theta.\end{aligned}$$

The desired result follows immediately upon equating coefficients of $\cos^{n-2k}\theta$ in (1) and (2).

It is easily verified that the right member of the equation of the problem is

$$\text{(3)}\qquad 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k},$$

and that the left member is $\phi^{(k)}(1)/k!$, where

$$\text{(4)}\qquad \phi(x) = \tfrac{1}{2}\left[(1+x^{1/2})^n+(1-x^{1/2})^n\right],$$

and the index (k) means the kth derivative.

Solved also by Frank Ayres, Jr., and Harry Langman.

Editorial Note. The formula (1) has been developed in a slightly different form as equation (7) on page 575 in the note following the solution of 3677 [1935, 572]. In the formula (7) of that note, if $a=\cos\theta$, $y_n=\cos n\theta$; whereas, if $a=\cosh\theta$, $y_n=\cosh n\theta$. By use of Starke's (3) the two expressions for $\cos n\theta$ may be identified. It will be observed that there is no necessity for introducing the trigonometric function; for y_n may be developed by the binomial formula in a different way by setting

$$2y_n = (a+\sqrt{a^2-1})^n + (a-\sqrt{a^2-1})^n;$$

and a comparison of the two results for y_n gives the equation of the problem.

The solution of Ayres starts with the right member. The equation of the problem is easily verified to be true for $n=2k$ and $2k+1$, and for the remaining cases we develop

$$2^{n-2k-1} = (1+1)^{n-2k-1}$$

by the binomial theorem. The right side then becomes

$$[\quad]+\sum_{s=0}^{n-2k-2}\left\{\binom{n-k}{k}\binom{n-2k-1}{s}+\binom{n-k-1}{k-1}\binom{n-2k-1}{s+1}\right\}$$

$$=[\quad]+\sum_{s=0}^{n-2k-2}\binom{k+s+1}{k}\binom{n-k-1}{k+s}=\sum_{i=0}^{n-2k}\binom{k+i}{k}\binom{n-k-1}{k-1+i},$$

where the contents of the brackets are given in the problem. Ayres then remarks that this last sum may be shown to be equal to the left member of the problem by equating coefficients of x^{2k} in the developments of the two members of

$$(1-x^{-2})^{-k-1}(1+x)^n = x^{2k+2}(x-1)^{-k-1}(1+x)^{n-k-1}.$$

Langman used the identity

$$v = (1-x^2)^{-k-1}(1+x)^n = (1-x)^{-k-1}(1+x)^{n-k-1}$$

by equating coefficients of x^{n-2k} in the developments of the two sides. The coefficient from the right member is the same as Ayres' last summation above after setting $i=j-k$. Several reductions are then made in order to put it in the form of the right member of the problem.

He also obtained another form for the left member of the problem

$$\sum_{j=0}^{k}(-1)^j 2^{n-k-1-j}\binom{n-k-1}{j}\binom{n-k-j}{n-2k},$$

by setting

$$v = \frac{[2-(1-x)]^{n-k-1}}{(1-x)^{k+1}}.$$

3692 [1934, 395]. *Proposed by E. P. Starke, Rutgers University.*

Show that there are four distinct sets of integers which satisfy the equations

$$x_1 + x_2 + x_3 = 54, \qquad x_1^2 + x_2^2 + x_3^2 = 1406.$$

Develop a general method of attack for similar problems in which 54 and 1406 are replaced by a and b.

I. *Solution by W. C. Janes, Kansas State College*

The general case requires the distinct sets of integers which satisfy the equations

$$x_1 + x_2 + x_3 = a \tag{1}$$

$$x_1^2 + x_2^2 + x_3^2 = b. \tag{2}$$

Elimination of x_1 between (1) and (2) gives

$$2(x_2^2 + x_2x_3 + x_3^2 - ax_2 - ax_3) = b - a^2. \tag{3}$$

Hence $b-a^2$ is an even integer, and consequently b and a must be either both even or both odd.

Let

$$x_1 = \frac{a}{3} - x \tag{4}$$

$$x_2 = \frac{x-y}{2} + \frac{a}{3} \tag{5}$$

$$x_3 = \frac{x+y}{2} + \frac{a}{3}. \tag{6}$$

The difference of (6) and (5) shows y to be always an integer (possibly zero). Furthermore, y can always be considered as a positive integer, for a change in the sign of y merely interchanges x_2 and x_3 and does not lead to a distinct set of integers x_1, x_2, and x_3. In case y is zero x_2 and x_3 are equal.

Substitution of (4), (5) and (6) in (2) gives

$$3x^2 + y^2 = 2\left(b - \frac{a^2}{3}\right); \tag{7}$$

(7) shows $b \geqq a^2/3$.

Case 1: When a is a multiple of 3.

(4) shows that in this case x is an integer. Since x_2 and x_3 are integers, x and y must be either both even or both odd. Any pair of integers which satisfy (7) will give a set of integers x_1, x_2, and x_3 which satisfy (1) and (2) simultaneously. If x is positive the required set of integers will usually be distinct from

the set obtained when x is negative. If x and y are numerically equal only one set of integers is obtained regardless of the sign of x. Not all distinct pairs of integers which satisfy (7) will necessarily lead to distinct sets x_1, x_2, x_3; for the symmetry of (1) and (2) shows that if a value of x not previously used leads to a value of x_1 which is identical with a previous value of x_2 or x_3 no new set of required integers is obtained. In fact, except for special cases, one should expect to get three distinct pairs of values of x and y for every distinct set of values of x_1, x_2, x_3.

Case 2: When a is not a multiple of 3.

In this case x is of the form $n/3$, where n is an integer such that $(a-n)/3$ is an integer. Pairs of integers n and y used to obtain x_1, x_2, x_3 must satisfy (7) and the sign of n must be such as to satisfy (4). y and n must be either both even or both odd.

Given specific values of a and b, the problem can be completely solved. The following examples illustrate the method.

For the given case $a=54$, $b=1406$, the right member of (7) is 868. The distinct solutions are given by

x	y	x_1	x_2	x_3
3	29	15	5	34
-3	29	21	2	31
8	26	10	9	35
-8	26	26	1	27

The remaining sets

$$(x, y) = (\pm 9, 25), (\pm 13, 19), (\pm 16, 10), (\pm 17, 1)$$

lead to no other distinct sets of x_1, x_2, x_3.

For $a=5$, $b=41$, the right member of (7) is $196/3$; and we have

x	y	x_1	x_2	x_3
$\frac{14}{3}$	0	-3	4	4
$-\frac{13}{3}$	3	6	-2	1

The remaining sets

$$(x, y) = (11/3, 5), (-7/3, 7), (2/3, 8)$$

give no other distinct solutions.

II. *Solution by M. A. Heaslet, Stanford Univ.*

To get the essential solutions of the equations we assume $x_1 \leqq x_2 \leqq x_3$. If we set $x_1=x$, $x_2=x+p$, $x_3=x+q$; $0 \leqq p \leqq q$, the given equations become

$$(1)\qquad \begin{aligned} 3x + p + q &= a \\ 3x^2 + 2px + 2qx + p^2 + q^2 &= b. \end{aligned}$$

For integral solutions to exist a, b must necessarily be of the same parity. Squaring the first equation and subtracting it from three times the second we have, after dividing by 2,

$$q^2 - pq + p^2 = \frac{3b - a^2}{2}$$

hence

$$(2)\qquad q - p = \sqrt{\frac{3b - a^2}{2} - pq}.$$

If p_1, q_1 is a solution of (2) then $q_1 - p_1, q_1$ is also; hence we need merely investigate possible integers for which $\sqrt{(3b-a^2)/2} \geqq p \geqq 0$ and $(3b-a^2)/2 - pq$ is a perfect square.

After determining the integral solutions of (2), we find x from (1) and x_1, x_2, x_3 follow immediately.

When $a=54$ and $b=1406$, $(3b-a^2)/2=651$, and the solutions of (2) are (1, 26), (25, 26), (10, 29), (19, 29). It follows that the solutions of the original equations are (9, 10, 35), (1, 26, 27), (5, 15, 34), (2, 21, 31).

Solved also by J. W. Clawson, E. B. Escott, S. W. Klanfer, Harry Langman, J. Rosenbaum, C. W. Trigg, F. Underwood, S. Vatriquant, and the proposer.

3693 [1934, 452]. *Proposed by H. K. Fulmer, Georgia School of Technology.*

Solution of the differential equations, $dx/dt = k(a-x)^3$ and $dx/dt = k(a-x)(b-x)(c-x)$, $x=0$ when $t=0$, leads to

$$(1)\qquad k = \frac{2ax - x^2}{2a^2t(a - x)^2}$$

$$(2)\qquad k = \frac{(b - c)\log\dfrac{a - x}{a} + (c - a)\log\dfrac{b - x}{b} + (a - b)\log\dfrac{c - x}{c}}{t(b - c)(c - a)(a - b)}.$$

In certain applications it is physically necessary that the second expression for k should approach the first expression when b and c approach a simultaneously in any manner. Show mathematically that this is true.

I. *Solution by H. E. Bray, Rice Institute*

The problem amounts to showing that

$$\lim_{b,\,c=a} k(a, b, c) = k(a)$$

with $(a-x)(b-c)(c-a)(a-b)\neq 0$, where

$$k(a) = \frac{2ax - x^2}{2a^2(a-x)^2},$$

$$k(a, b, c) = \frac{(b-c)\log\dfrac{a-x}{a} + (c-a)\log\dfrac{b-x}{b} + (a-b)\log\dfrac{c-x}{c}}{(b-c)(c-a)(a-b)}.$$

If we let

$$f(z) = \log\left(\frac{z-x}{z}\right),$$

it is readily verified, after writing $(b-c)=(b-a)-(c-a)$, that

$$k(a, b, c) = -\frac{\dfrac{f(c)-f(a)}{c-a} - \dfrac{f(b)-f(a)}{b-a}}{c-b}.$$

$$= -\left[\frac{d}{dz}\left\{\frac{f(z)-f(a)}{z-a}\right\}\right]_{z=z'},$$

where z' lies between b and c. But since $f(z)$ is analytic at a we have, from the Taylor's expansion about a,

$$[f(z) - f(a)]/(z-a) = f'(a) + \frac{1}{2!}f''(a)(z-a) + \cdots.$$

On differentiating, and setting $z=z'$, we find

$$k(a, b, c) = -\tfrac{1}{2}f''(a) - \tfrac{1}{3}f'''(a)(z'-a) - \cdots$$

and since, when b, c approach a, z' does so also, we have finally

$$\begin{aligned}\lim_{b,c \doteq a} k(a, b, c) &= -\frac{1}{2}f''(a) \\ &= \frac{1}{2}\left[\frac{1}{(z-x)^2} - \frac{1}{z^2}\right]_{z=a} \\ &= \frac{2ax - x^2}{2a^2(a-x)^2} \\ &= k(a).\end{aligned}$$

II. *Solution by J. P. Dalton, University of the Witwatersrand, Johannesburg, South Africa*

In equation (2) of the problem let $a<b<c$. Write $b=n_1a$, $c=n_2a$, $x=\lambda a$, and

$$\psi(n) = \frac{1}{n-1} \log \frac{1-\lambda/n}{1-\lambda}. \tag{1}$$

The equation then becomes

$$kta^2 = \psi(n_1)/(n_2 - n_1) + \psi(n_2)/(n_1 - n_2). \tag{2}$$

The degeneration as $n_2 \to n_1 \to 1$ depends upon the properties of ψ.

Successive derivatives of $\psi(n)$ are given by

$$(n-1)\psi^r(n) + r\psi^{r-1}(n) = (-1)^{r-1}(r-1)![(n-\lambda)^{-r} - n^{-r}]. \tag{3}$$

$\psi(n)$ and its derivatives are indeterminate at $n=1$.

The limiting values as $n \to 1$ are easily seen to be

$$\psi(1) \to [(1-\lambda)^{-1} - 1] \tag{4}$$

$$\psi^{r-1}(1) \to (-1)^{r-1}r^{-1}(r-1)![(1-\lambda)^{-r} - 1]. \tag{5}$$

Now make $n_2 \to n_1$ in (2). We obtain

$$kta^2 = -\psi'(n_1) \tag{6}$$

Let $n_1 \to 1$, and use (5), so that

$$kta^2 = \tfrac{1}{2}[(1-\lambda)^{-2} - 1]. \tag{7}$$

Returning from λ to x we obtain the first equation of the problem.

The function ψ was introduced in a paper entitled: "*The integrated equations of chemical reactions*" published in Trans. R.S.S.Af. vol. ix, 1921. Short tables of ψ, ψ' and ψ'' were there given, and it was shown how the integrated equation of a reaction of any order might be written down in terms of ψ and its derivatives. In a later number of the same Transactions (vol. xi, 1923) the use of the function was extended to the homogeneous balanced action.

Solved also by Harry Langman and S. Vatriquant.

Editorial Note. In solution II a special manner of the approach of b and c to a is used. The method of I may be used to obtain the general manner of approach as stated in the problem. Thus we may restrict n_1 and n_2 to an interval about unity which does not contain λ. Then the right side of (2) is equal to $\psi'(\theta)$, where θ lies in the n interval between n_1 and n_2; and then the desired result follows from the properties of ψ deduced in II.

Langman set $b=a+\beta$, $c=a+\gamma$, in the expression in (2) for kt, and then developed this result in a series whose first term is the right side of kt in (1) and whose remaining terms contain integral powers of β and γ. Hence as β and γ approach zero simultaneously in any manner the limit is the first term. This proof has the advantage that it is valid for complex variables.

It is simpler to use the facts given in the problem as follows: Let b and c lie in a small interval about a which contains no point of the interval $0 \leqq z \leqq x$; then for (2)

$$kt = \int_0^x \frac{dz}{P(z)}, \qquad P(z) = (a - z)(b - z)(c - z).$$

Thus kt is defined as a continuous function of b and c, and the theorem is obviously true in this case. For real variables, however, this restricts the relative positions of x and a. But we easily side-step this difficulty by taking in the complex plane a path of integration from the origin to x which lies entirely outside the small region containing a, b, c. Here also the given variables may be complex. If, for example, it is desirable to write the series for kt in powers of α and β, it is simpler to expand the integrand $1/P(z)$ and then integrate the resulting series term by term.

3694 [1934, 453]. *Proposed by J. D. Leith, University of North Dakota.*

A given ellipse, $b^2x^2+a^2y^2=a^2b^2$, $a<b$, is the envelope of a family of circles of radii r and centers at points $(0, c)$ on the y-axis, where r is a function of c. Determine this function, and discuss the variation of the circles as r and c vary.

Solution by E. P. Starke, Rutgers University

Evidently $x^2+(y-c)^2=r^2$ and $b^2x^2+a^2y^2=a^2b^2$ are tangent for a certain value of r. Eliminating x we have

$$(a^2 - b^2)y^2 + 2b^2cy + b^2(r^2 - a^2 - c^2) = 0.$$

Let the discriminant of this equation in y vanish, and solve for r. We obtain $r^2=a^2+a^2c^2/(a^2-b^2)$. With this value of r^2, we find the common points of the ellipse and the circle to be given by $x^2=a^2-a^2b^2c^2/(a^2-b^2)^2$ and $y=-b^2c/(a^2-b^2)$. In order that the points shall have real abscissas we must have $|c| \leqq (a^2-b^2)/b$.

The variation of the circles is now easy to trace. For $c=(a^2-b^2)/b$, the circle with radius a^2/b and center $(0, c)$ touches the ellipse at $(0, -b)$. As the center $(0, c)$ of the circle moves down the y-axis, the points of contact move up the ellipse until, for center $(0, 0)$, they are the ends of the major axis. Results symmetric in the x-axis to these are obtained when c is negative.

Solved also by W. B. Campbell, Hansraj Gupta, H. Halperin, J. E. LaFon, Harry Longman and S. Vatriquant.

Editorial Note. Calculus methods were used in most of the other solutions; and Langman and Vatriquant discussed the extraneous cases arising from their methods where the circles were not tangent to the ellipse. Gupta noted that, if S is a focus of the ellipse of eccentricity e and if O is the center of a circle of the family, $er=OS$.

It is obvious that each circle of the family, with two exceptions, is tangent to the ellipse at two distinct points symmetric to the minor axis, and that there can be no other points in common. The only case in which a circle of the family can be an osculating circle is when the point of contact is at an end of the minor axis. When $c=0$ the circle must be the major auxiliary circle, and it follows that each circle of the family contains the ellipse in its interior.

3696 [1934, 453]. *Proposed by J. B. Reynolds, Lehigh University.*

A dog directly opposite his master on the banks of a stream, flowing with uniform speed, swims at a still-water speed of two miles per hour heading directly towards his master at all times. The man notes that the dog does not stop drifting down stream until he is two-thirds across measured perpendicularly to the banks, and that it takes five minutes longer to make the trip than if the water had been still. How wide is the stream?

Solution by J. E. Trevor, Cornell University

Taking the origin at the dog's initial position, draw an x axis containing the straight line from the dog to his master, and let distances along the y axis be measured down stream. Write w and c for the width and speed of the stream, and let θ be the positive acute angle between the straight line from the representative point to the point $(w, 0)$ and the x axis. Then we have the equations

$$c - \frac{dy}{dt} = 2 \sin \theta, \qquad \frac{dx}{dt} = 2 \cos \theta. \tag{1}$$

Putting $\tan \theta = z$, we find

$$\frac{dy}{dx} + z = \frac{c}{2}\sqrt{z^2 + 1}. \tag{2}$$

Now, obviously,

$$z = \frac{y(x)}{w - x}.$$

Differentiating this with regard to x, eliminating dy/dx from (2), and equating the reciprocals of the members,

$$\frac{d}{dz} \log (w - x) = -\frac{2}{c} \frac{1}{\sqrt{z^2 + 1}}.$$

Integrating,

$$\log (w - x) = -\frac{2}{c} \log (z + \sqrt{z^2 + 1}) + \log k. \tag{3}$$

To determine k we note that x and z vanish together, wherefore, from (3), $k = w$. Substituting this value in (2), and solving for z, we find

$$2z = \left(\frac{w}{w - x}\right)^{c/2} - \left(\frac{w}{w - x}\right)^{-c/2} \tag{4}$$

If we replace z by its value $y/(w-x)$, the equation of the dog's path is found to be

$$2y = (w - x)\left[\left(\frac{w}{w - x}\right)^{c/2} - \left(\frac{w}{w - x}\right)^{-c/2}\right]. \tag{5}$$

On differentiating (5), the equation $dy/dx=0$ reduces to

$$\left(\frac{w}{w - x_0}\right)^c = \frac{2 + c}{2 - c},$$

where x_0 is the abscissa of the maximum point. Since x_0 is two-thirds of w, the equation becomes

$$3^c = \frac{2 + c}{2 - c},$$

whence $c=1$.

In seeking the time t required for the swim, we return to the second of the equations (1), and find that

$$\frac{dt}{dx} = \frac{\sqrt{z^2 + 1}}{2}. \tag{6}$$

On eliminating z between (6) and (4), and integrating from $x=0$ to $x=w$, we find

$$t = \frac{2w}{4 - c^2}. \tag{7}$$

Since the dog could cross the stream in t_0 hours in still water, and does require 1/12 hours more,

$$t_0 = \frac{w}{2}, \qquad t - t_0 = \frac{1}{12}.$$

Eliminating t_0 and t between these equations and (6), and solving for w,

$$w = \frac{1}{6} \frac{4 - c^2}{c^2}.$$

But $c=1$. So the width w of the stream is one-half mile. The dog swims it in twenty minutes, and would swim it in fifteen minutes in still water.

Solved also by J. E. LaFon, Harry Langman, A. S. Merrill, E. P. Starke and the proposer.

Editorial Note. The equation for c has its positive roots in the interval $0 \leqq c < 2$, and we may write

$$f(c) = c \log_e 3 - \log_e (2 + c) + \log_e (2 - c),$$

$$f'(c) = \log_e 3 - \frac{4}{4 - c^2}.$$

Hence $f'(c)$ vanishes for only one positive value of c; and, since $f(0)=f(1)=0$, 0 and 1 are the only roots not negative. Incidentally this shows that $1 < \log_e 3 < 4/3$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

On November 9, 1935, the Association of Mathematics Teachers of New Jersey held their annual meeting at Atlantic City. The theme of the meeting was: "The New Mathematics Requirement of the College Entrance Examination Board." The point of view of the commission which framed the requirements was given by Professor Arnold Dresden of Swarthmore College. The point of view of the college was given by Professor Robert M. Walter of the New Jersey College for Women. The point of view of the secondary school was given by Mr. Rolland R. Smith of the Classical High School of Springfield, Massachusetts. The discussion was led by Professor J. R. Kline of the University of Pennsylvania.

Simon Newcomb (1835–1909) was elected to the Hall of Fame of New York University in the eighth quinquennial election. The election has been ratified by the senate of the University and a commemorative tablet and bust will be placed in the Colonade at University Heights at ceremonies to take place, probably, next spring.

The following volume of the Colloquium Publications of the American Mathematical Society has recently appeared: volume 20, J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain.*

The National Academy of Sciences held its autumn meeting on November 18–29, 1935, at the University of Virginia under the presidency of Dr. F. R. Lillie. At this meeting Professor Edward Kasner of Columbia University and Professor G. T. Whyburn of the University of Virginia, presented papers entitled *Conformal geometry*, and *Continuous transformations of certain manifolds*, respectively.

Professor E. P. Lane has been appointed the representative of the American Mathematical Society on the editorial committee of the Duke Mathematical Journal.

Dr. S. F. Barber has been appointed an associate in mathematics at the University of Iowa.

Dr. L. M. Blumenthal, formerly a National Research Fellow, has been appointed an assistant at the Institute for Advanced Study.

Professor J. W. Drew, of Storer College, Harper's Ferry, West Virginia, has been appointed dean of men and associate professor of mathematics at Virginia Union University, Richmond.

Dr. E. G. Harrell has been appointed head of the department of mathematics at Ouachita College, Arkadelphia, Arkansas.

Assistant Professor J. M. Harrington, of the Michigan College of Mining and Technology, has been promoted to an associate professorship in mathematics.

Dr. Nathan Jacobson, of Princeton University, has been appointed a lecturer at Bryn Mawr College.

Professor A. J. Maria is visiting assistant professor at Duke University this academic year.

Dr. J. R. Mayor has been appointed professor of mathematics and head of the department at the Southern Illinois State Teachers College.

Professor Boris Podolsky, formerly at the Institute for Advanced Study, has been appointed assistant professor of mathematical physics at the University of Cincinnati.

Assistant Professor H. P. Robertson has been promoted to an associate professorship in mathematics at Princeton University. Professor Robertson is on leave of absence during this academic year and is at the California Institute of Technology and the University of California.

Dr. C. W. Strom has resigned his professorship of mathematics at Luther College to accept an appointment as U. S. Foreign Service Officer and has been assigned as United States Vice Consul of Career at Vancouver, B.C.

Dr. W. R. Hutchings has been appointed assistant instructor in mathematics and physics at Rollins College.

Dr. Phyllis H. Hutchings has been appointed instructor in astronomy at Rollins College.

G. W. Petrie III has been appointed instructor at the South Dakota State School of Mines.

Among the members of the American Mathematical Society who have recently entered the field of secondary teaching are the following: Dr. E. P. Northrop, at the Hotchkiss School, Lakeville, Connecticut; Dr. R. S. Pieters, at the George School, Bucks County, Pennsylvania.

The following are in residence at the Institute for Advanced Study for the whole or part of the academic year 1935–36: Professor Gregory Breit, of the University of Wisconsin; Professor Leonard Carlitz, of Duke University; Professor E. W. Chittenden, of the University of Iowa; Dr. E. H. Cutler, of Lehigh University; Professor Arnold Dresden, of Swarthmore College; Professor Phillip Franklin, of the Massachusetts Institute of Technology; Professor B. P. Gill, of the College of the City of New York; Professor L. M. Graves, of the University of Chicago; Professor S. H. Kimball, of the University of Rochester.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N. Y.

BOOKS FOR REVIEW should be addressed to R. JOHNSON, Brooklyn College, 66 Court Street Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Summer Meeting, Harvard University, Aug. 31, 1936.

Twenty-first Annual Meeting, Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, May 8–9.
INDIANA, North Manchester, May.
IOWA, April.
KANSAS, March.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov. 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 4.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S. C., April 17–18.
SOUTHERN CALIFORNIA, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, May 2.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 3, MARCH

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE TWENTIETH ANNUAL MEETING OF THE ASSOCIATION

The twentieth annual meeting of the Mathematical Association of America was held at St. Louis, Missouri, on Monday and Tuesday, December 30–31, 1935, in affiliation with the American Association for the Advancement of Science, the American Mathematical Society and the National Council of Teachers of Mathematics. Three hundred thirty-two were in attendance at the meetings, including the following two hundred twenty-nine members of the Association:

V. M. Adkisson, University of Arkansas
N. B. Allison, Kentucky Wesleyan College
Nola L. Anderson, Sophie Newcomb College
W. E. Anderson, Miami University
R. C. Archibald, Brown University
Mary M. Arnoldy, Marymount College
C. S. Atchison, Washington and Jefferson College
Edith I. Atkin, Illinois State Normal University
H. T. R. Aude, Colgate University

R. W. Babcock, Kansas State College
Wealthy Babcock, University of Kansas
Clara L. Bacon, Goucher College
H. W. Bailey, University of Illinois
G. A. Baker, Mississippi Woman's College
R. W. Barnard, University of Chicago
I. A. Barnett, University of Cincinnati
W. D. Baten, University of Michigan
A. A. Bennett, Brown University
William Betz, University of Rochester
M. T. Bird, Southwestern College
Florence L. Black, University of Kansas
Archie Blake, U. S. Coast and Geodetic Survey
Henry Blumberg, Ohio State University
L. M. Blumenthal, Institute for Advanced Study
O. K. Bower, University of Illinois
Jessie W. Boyce, State Teachers College, Wayne, Nebr.
J. W. Bradshaw, University of Michigan
W. C. Brenke, University of Nebraska
R. W. Brink, University of Minnesota
R. S. Burington, Case School of Applied Science
Jewell Hughes Bushey, Hunter College

W. D. Cairns, Oberlin College
M. Patricia Callaghan, Fontbonne College
B. H. Camp, Wesleyan University
C. C. Camp, University of Nebraska
C. S. Carlson, St. Olaf College
R. D. Carmichael, University of Illinois
C. C. Carter, States Attorney, Scott County, Illinois
E. W. Chittenden, University of Iowa
M. Borgia Clarke, Webster College
A. B. Coble, University of Illinois
L. M. Coffin, Coe College
L. W. Cohen, University of Kentucky
J. B. Coleman, University of South Carolina
Julia T. Colpitts, Iowa State College
E. E. Colyer, State Teachers College, Hays, Kansas
Rose Margaret Cook, Loretta Heights College
J. J. Corliss, De Paul University
Byron Cosby, State Teachers College, Kirksville, Mo.
N. A. Court, University of Oklahoma
A. T. Craig, University of Iowa
C. C. Craig, University of Michigan
A. R. Crathorne, University of Illinois
D. R. Curtiss, Northwestern University
J. H. Curtiss, Johns Hopkins University
E. H. Cutler, Institute for Advanced Study

R. D. Daugherty, Kansas State College
H. T. Davis, Indiana University
L. L. Dines, Carnegie Institute of Technology
Mariola Dobbin, Rosary College
Lucy T. Dougherty, Junior College, Kansas City, Kans.
Otto Dunkel, Washington University
O. L. Dustheimer, Baldwin-Wallace College

P. D. Edwards, Ball State Teachers College
H. J. Ettlinger, University of Texas
H. S. Everett, University of Chicago
J. P. Everett, Kalamazoo State Teachers College

B. F. FINKEL, Drury College
C. H. FISCHER, Wayne University
C. W. FOARD, Youngstown College
T. C. FRY, Bell Telephone Laboratories
GORDON FULLER, Sioux Falls College

F. J. GERST, Loyola University, Chicago, Ill.
J. L. GIBSON, University of Utah
R. M. GINNINGS, Western Illinois State Teachers College
W. C. GRAUSTEIN, Harvard University
F. L. GRIFFIN, Reed College

LAURENCE HAMPTON, Auburn, Alabama
E. H. HANSON, North Texas State Teachers College
ELIZABETH L. HARRIS, Southwestern Bell Telephone Co.
W. L. HART, University of Minnesota
W. W. HART, University of Wisconsin
M. C. HARTLEY, University of Illinois High School
M. L. HARTUNG, University of Wisconsin High School
J. O. HASSLER, University of Oklahoma
T. W. HATCHER, Virginia Polytechnic Institute
I. L. HEBEL, Colorado School of Mines
E. R. HEDRICK, University of California at Los Angeles
GERTRUDE HENDRIX, Eastern Illinois State Teachers College
CORA B. HENNEL, Indiana University
DEBORAH M. HICKEY, Delta State Teachers College
T. H. HILDEBRANDT, University of Michigan
H. M. HOSFORD, University of Arkansas
E. MARIE HOVE, State Teachers College, Wayne, Nebr.
M. GWENETH HUMPHREYS, Mount St. Scholastica College
MILDRED HUNT, Illinois Wesleyan University
W. A. HURWITZ, Cornell University
C. A. HUTCHINSON, University of Colorado
EMMA HYDE, Kansas State College

M. H. INGRAHAM, University of Wisconsin

JOSEPH JABLONOWER, Feildston School
DUNHAM JACKSON, University of Minnesota
R. D. JAMES, University of California
G. H. JAMISON, State Teachers College, Kirksville, Mo.
W. C. JANES, Kansas State College
R. L. JEFFERY, Acadia University
L. W. JOHNSON, State Teachers College, Edmond, Okla.
MARIE M. JOHNSON, Oberlin College
F. C. JONAH, Western Reserve University

L. C. KARPINSKI, University of Michigan
DORA E. KEARNEY, State Teachers College, Cedar Falls, Iowa
CLARIBEL KENDALL, University of Colorado
EULA WEEKS KING, St. Louis, Mo.
ELIZABETH E. KNIGHT, State Teachers College, Milwaukee, Wis.
J. H. KUSNER, University of Florida

RUTH O. LANE, University of Iowa High School
C. G. LATIMER, University of Kentucky
SOLOMON LEFSCHETZ, Princeton University
HARRY LEVY, University of Illinois
C. F. LEWIS, Kansas State College
MAYME I. LOGSDON, University of Chicago
S. A. LYNCH, North Texas Agric. College
W. H. LYONS, Kansas State College

C. C. MACDUFFEE, University of Wisconsin
DOROTHY MCCOY, Belhaven College
JANET MCDONALD, Hinds Junior College, Raymond, Miss.
F. M. MCGAW, Cornell College
J. V. MCKELVEY, Iowa State College
A. S. MCMASTER, University of Colorado
V. S. MALLORY, State Teachers College, Montclair, N. J.
MORRIS MARDEN, University of Wisconsin, Extension Division
ANNA MARM, Bethany College, Lindsborg, Kans.
MARY FELICE, Mount Mary College
J. R. MAYOR, Southern Illinois State Normal School
H. A. MEYER, Hanover College
H. J. MILES, University of Illinois
E. B. MILLER, Illinois College
C. N. MILLS, Illinois State Normal University
W. E. MILNE, Oregon State Agric. College
W. L. MISER, Vanderbilt University
U. G. MITCHELL, University of Kansas
E. C. MOLINA, Bell Telephone Laboratories
C. N. MOORE, University of Cincinnati
G. E. MOORE, University of Illinois
E. J. MOULTON, Northwestern University
ARRIA MURTO, High School, Carthage, Mo.

MARIE M. NESS, University of Minnesota
C. V. NEWSOM, University of New Mexico
G. D. NICHOLS, University of Arkansas
MABEL I. NOWLAN, Bethel Woman's College

C. O. OAKLEY, Haverford College
G. A. O'DONNELL, Boston College
RUFUS OLDENBURGER, Armour Institute of Technology
E. G. OLDS, Carnegie Institute of Technology
ARTHUR OLLIVIER, Mississippi State College
JESSE OSBORN, Harris Teachers College

W. V. PARKER, Georgia School of Technology
W. O. PENNELL, Southwestern Bell Telephone Co.
O. J. PETERSON, State Teachers College, Emporia, Kans.
D. T. PETTY, F. W. Parker School, Chicago
A. E. PITCHER, Institute for Advanced Study
H. S. POLLARD, Miami University

J. W. QUERRY, State Teachers College, Huntsville, Texas

J. F. RANDOLPH, Cornell University
B. L. REMICK, Kansas State College
R. G. D. RICHARDSON, Brown University
H. L. RIETZ, University of Iowa
R. F. RINEHART, Ashland College
W. C. RISSELMAN, Northern Arizona Teachers College
ROBIN ROBINSON, Dartmouth College
S. L. ROBINSON, College of the City of New York
W. H. ROEVER, Washington University
C. F. ROOS, Colorado College
M. F. ROSSKOPF, John Burroughs School, Clayton, Mo.
W. E. ROTH, Univeristy of Wisconsin, Extension Division
J. T. RULE, Taylor School for Boys, Clayton, Mo.
LULU RUNGE, University of Nebraska

R. G. SANGER, University of Chicago
M. G. SCHERBERG, University of Minnesota
I. J. SCHOENBERG, Swarthmore College
E. W. SCHREIBER, Western Illinois State Teachers College
E. E. SCOTT, Greenville College
JOSEPH SEIDLIN, Alfred University
J. A. G. SHIRK, State Teachers College, Pittsburg, Kans.
L. S. SHIVELY, Ball State Teachers College
EDMOND SIROKY, Washington University
C. H. SISAM, Colorado College
C. D. SMITH, Mississippi State College
EDWIN R. SMITH, Iowa State College
G. W. SMITH, University of Kansas
H. L. SMITH, Louisiana State University
J. P. SMITH, St. Peters College
R. G. SMITH, State Teachers College, Pittsburg, Kans.
F. W. SOHON, Georgetown University
I. S. SOKOLNIKOFF, University of Wisconsin
ANNA A. STAFFORD, University of Nebraska
NORMA K. STELFORD, Northern Illinois State Teachers College
EUGENE STEPHENS, Washington University
MRS. EUGENE STEPHENS, Washington University
GUY STEVENSON, University of Louisville
RUTH W. STOKES, Duke University
E. B. STOUFFER, University of Kansas
C. J. STOWELL, McKendree College
W. T. STRATTON, Kansas State College
G. B. SWEAZEY, Westminster College
J. L. SYNGE, University of Tornoto
GABRIEL SZEGÖ, Washington University

J. D. TAMARKIN, Brown University
E. H. TAYLOR, Eastern Illlinois State Teachers College
MILDRED E. TAYLOR, Mary Baldwin College
H. W. TYLER, Library of Congress

A. L. UNDERHILL, University of Minnesota

J. I. VASS, University of Wisconsin, Extension Division

G. E. WAHLIN, University of Missouri
R. J. WALKER, Cornell University
L. E. WARD, University of Iowa
J. H. WEAVER, Ohio State University
WARREN WEAVER, Rockefeller Foundation
MARIE J. WEISS, Sophie Newcomb College
W. D. A. WESTFALL, University of Missouri
J. J. WHEELER, University of Kansas
A. E. WHITE, Kansas State College
W. M. WHYBURN, University of California at Los Angeles
K. P. WILLIAMS, Indiana University
F. L. WREN, George Peabody College
ALICE KELSEY WRIGHT, Southern Illinois Normal University
E. KATHRYN WYANT, Athens College
C. C. WYLIE, University of Iowa
MABEL M. YOUNG, Wellesley College

The opening session of the A.A.A.S. occurred on Monday evening. The address of the retiring president, Professor E. L. Thorndike, on "Science and values" was followed by a reception tendered to the Association and guests by the St. Louis local committee at the Municipal Auditorium. The Mathematical Association was represented on the Council of the A.A.A.S. each morning by Professor C. S. Atchison and Secretary Cairns. The Council elected Professor G. C. Evans vice-president and chairman of Section A for the year 1936 and Professor L. M. Graves member of the Section committee. The Council adopted a resolution endorsing the twelve-month, equal-quarter plan for the simplification of the calendar, known as the World Calendar.

The headquarters for the mathematicians was the Coronado Hotel, which was within convenient distance of the lecture halls in the Commerce and Finance Building of St. Louis University where the meetings were held. Registration headquarters for mathematicians were provided in the lobby of the hotel, including a table for registration for the American Association where validation of tickets, etc. could be made. At the conclusion of the Tuesday afternoon program several double-deck busses carried the visiting mathematicians and guests on a sight-seeing tour about St. Louis, and then to the Women's Building on the campus of Washington University where the ladies of the faculty and the members of the Washington University chapter of Pi Mu Epsilon served tea in these very pleasant surroundings. A resolution was offered by Professor E. R. Hedrick at the joint dinner, and heartily adopted by the guests, expressing our appreciation of the excellent provisions that had been made for the meetings, of the facilities afforded by St. Louis University and Washington University, and of the courtesy of those who arranged the tea and the sight-seeing trip.

About two hundred thirty attended the annual dinner in the Club Caprice of the Coronado Hotel. Professor R. D. Carmichael acted as toastmaster and introduced the two speakers. Professor Roever gave a hearty welcome to the mathematicians and spoke of the pleasure and honor which this gave to the group at St. Louis. Professor Graustein spoke very feelingly of the passing of Professor W. E. Byerly, for thirty-seven years assistant professor and professor of mathematics at Harvard University until the time of his retirement in 1913. Professor Byerly received in 1873 the first degree of Ph.D. in mathematics conferred by Harvard. He contributed not a little to the shaping of mathematics in this country. He was not a productive mathematician, as we understand the term today, but he was an erudite scholar and an outstanding teacher, as indicated in his published works. These had a wide influence, his two books on the calculus being the forerunners of the current texts. In all his relations with students he showed a deep understanding. The keynote of the man was simplicity alike in dress and manner.

Professor Graustein continued by saying that we should strengthen and solidify the position of mathematics in this country, not its financial support or its position in the colleges and universities, but we should be interested rather

in the problem of establishing mathematics on a firm, solid basis with the rank and file of the public. Certain classes of people attract our attention and we handle each in our own way; it might be advisable and profitable to handle the situation in an organized way whereby people who have specific needs may have advice, encouragement and direction. We can thus spread the gospel of mathematics more widely and win an appreciation of what we are doing.

Professor Carmichael reported on the large supply of papers for the *Transactions*, larger than at any time in the last five years, and expressed the editors' appreciation of the considerateness of the writers of these papers. He said that he has not known many of the younger group in recent years, but found that he had met a larger number of older acquaintances at St. Louis than at any meeting for a long term of years. He congratulated the younger group who are utilizing the opportunity of meeting one another in this wise.

At the conclusion of the dinner and of the annual banquet of the National Council of Teachers of Mathematics, held in adjoining rooms, the Society and Association members were the guests of the National Council in listening to an address by Professor R. C. Archibald on "Babylonian mathematics, with special reference to recent discoveries." Professor Archibald made a very able presentation of the newly discovered facts as given by Professor Neugebauer in *Quellen und Studien der Mathematik* for June 1935, wherein it appears that algebra involving the quadratic and cubic, etc. were known to the Babylonians one thousand to fifteen hundred years earlier than has been previously supposed.

The American Mathematical Society held sessions Tuesday morning, Wednesday morning and afternoon, and Thursday morning and afternoon for the reading of short papers. Thursday morning a joint session was held with the Econometric Society and the Institute of Mathematical Statistics. Following the business meeting on Wednesday afternoon, Professor Gabriel Szegö addressed the Society, by invitation, on "Some recent investigations concerning sections of trigonometric and related series." Thursday afternoon Professor Vannevar Bush, of the Massachusetts Institute of Technology, gave the Twelfth Josiah Willard Gibbs lecture on "Mechanical analysis" in the Municipal Auditorium under the auspices of the Society with the cooperation of the American Association.

The National Council of Teachers of Mathematics held separate sessions on Tuesday afternoon and evening and on Wednesday morning. Papers were given by Professor W. H. Roever, J. T. Rule, R. R. Smith, Edith Woolsey, E. W. Schreiber, Ruth Lane, H. C. Christofferson and W. D. Reeve. The full report will appear in the Mathematics Teacher. Miss Martha Hildebrandt of the Proviso Township High School, Maywood, Illinois, was elected president for the coming two years. The meetings at St. Louis gave another valuable opportunity for numerous consultations between officers and committee members of the National Council and the Mathematical Association concerning common interests.

The Mathematical Association held two sessions on Monday morning and

afternoon, a joint session with Section A and the National Council on Tuesday morning, and with Section A and the Society on Tuesday afternoon. Professor Curtiss presided at the Monday sessions, Professors Hildebrandt and Hassler Tuesday morning, and Professor Hildebrandt Tuesday afternoon. The indebtedness of the Association to the Program Committee, under the chairmanship of Professor Roever, was expressed in the resolution adopted at the joint dinner. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

First Session of the Association

1. "Modern pure solid geometry" by Professor N. A. Court, University of Oklahoma.

2. "Improving our calendar" by Professor C. C. Wylie, University of Iowa.

3. "Some mathematical aspects of motion and causality" by Professor H. J. Ettlinger, University of Texas.

1. Modern pure solid geometry may be conveniently thought of as being an extension to space of the properties of the triangle and the circle. Here are some examples:

a. F. Commandino (1509–1575) pointed out that the four medians of a tetrahedron (i.e., the lines joining the vertices to the centroids of the opposite faces) have a point, G, in common, and are divided by this point in the ratio 3:1. The point G is also the mid-point of each of the three bimedians (i.e., the lines joining the mid-points of the pairs of opposite edges) of the tetrahedron.

b. Four given spheres (A), (B), (C), (D), with non-coplanar centers, taken two by two have twelve centers of similitude. It is at once apparent that these twelve points lie by sixes in the faces of the tetrahedron $ABCD$ determined by the centers of the spheres. It may also be shown that the six external centers of similitude lie in a plane, the "external plane of similitude" of the spheres. Again, the three internal centers of similitude of the sphere (A) with each of the spheres (B), (C), (D) determine a plane which passes through the external centers of similitude of the latter three spheres taken in pairs. We obtain in this manner the four "mixed planes of similitude" of the given spheres. Finally, if we divide the four spheres into two couples, say (A), (B), (C), (D), the four internal centers of similitude obtained by pairing off each sphere of one couple with each sphere of the other couple, are coplanar, and this plane contains the two external centers of similitude of the two couples of spheres considered. These considerations yield the three "internal planes of similitude" of the spheres. The eight planes of similitude were considered by Gaspar Monge (1746–1818) in his Géometrie descriptive (Paris, 1795).

c. Given the tetrahedron $DABC$, the three external bisecting planes of three dihedral angles whose edges are coplanar meet in a point which also belongs to the internal bisecting planes of the three remaining dihedral angles. This point is the center of a sphere touching three faces of the tetrahedron, produced, and the fourth face not produced. The four "escribed" spheres thus obtained were

noticed by J. L. Lagrange (1736–1813) in 1773. Half a century later J. Steiner (1796–1863) added three more escribed spheres obtained in the following way. If the two pairs of edges AC, AD; BC, BD are produced beyond A, B respectively, the space thus formed may contain a sphere tangent to the four faces of $DABC$, produced, and if it does, the analogous space obtained by producing the edges through the vertices C, D cannot contain a similar sphere. Likewise for the other pairs of opposite edges.

d. The three figures considered are special cases of a configuration to which C. Stephanos called attention in 1879 (Bulletin des sciences mathématiques et astronomiques). Given the tetrahedron $(T)\equiv DABC$ and the point M, not in a face of (T), the line is drawn through M meeting the opposite edges DA, BC, say, in U', P'; let V', Q'; W', R' be the analogous points on the edges DB, CA; DC, AB. If M', M'', M''' are the harmonic conjugates of M with respect to the pairs of points U', P'; V', Q'; W', R', the plane $\mu\equiv M'M''M'''$ cuts the edges of (T) in the harmonic conjugates P, Q, R, U, V, W of $P', Q', R' \cdots$ with respect to the corresponding pairs of vertices of (T). The line PQR lies in the plane $U'V'W'$, and similarly for the traces of μ in the other faces of (T). Moreover, the line PQR is the trilinear polar, for the triangle ABC, of the trace D' of the line DM in the plane ABC; similarly for the other faces of (T). The plane μ is the "tetrahedral polar plane" of M for (T). It may be shown that each face of the tetrahedron $(M)\equiv MM'M''M'''$ is the tetrahedral polar plane of the opposite vertex for the tetrahedron, and reciprocally, each face of (T) is the tetrahedral polar plane of the opposite vertex with respect to (M). The two tetrahedrons are mutually self-polar.

If D', A', B', C' are the traces of the lines MD, MA, MB, MC in the faces of (T), and M_d, M_a, M_b, M_c the harmonic conjugates of M with respect to the pairs of points D, D'; A, A'; B, B'; C, C', the tetrahedron $(M_d)=(M_dM_aM_bM_c)$ is self-polar with respect to both (M) and (T). Moreover, each two of the three tetrahedrons (T), (M), (M_d) correspond to each other in four harmonic homologies, the centers and planes of homology being the vertices and the respectively opposite faces of the third tetrahedron of the group. The three tetrahedrons $PP'UU'$, $QQ'VV'$, $RR'WW'$ bear to each other the same relations as the three tetrahedrons just considered.*

However, no selection of propositions, no matter how felicitously chosen, can convey an adequate idea on a body of doctrine in pure geometry. The proofs of pure geometry are as much a part of this doctrine as are the propositions. In fact, the subject matter of pure geometry is best thought of as a continuous whole. The propositions are singled out not on account of their greater importance, but in view of more ready reference in what is to follow. This procedure may be unavoidable, but it lends to the subject an aspect of heterogeneity that is fundamentally foreign to it. Besides, it is not always possible to judge correctly beforehand what part of the argument may be needed in later

* Editor's note. The interested reader may find further details in the author's *Modern Pure Solid Geometry*. The Macmillan Company, 1935.

considerations. It happens quite frequently that the part that is wanted for reference at a given moment is inconspicuously imbedded in a previous proof.

Proofs in pure geometry are usually short. To prove a proposition means to show that it is a consequence of one or more propositions already known. If the chain that connects the new to the old is unduly long, this may mean that too much has been left unexplored in between, or that the proper contact has not been made. This "proper contact" is often very difficult to establish, but when it is finally achieved, it yields ample reward in the form of delightful esthetic satisfaction. "The 'indeed' of such a proof has sometimes the mellowness of a condescending smile, sometimes the swiftness of epigrammatic sarcasm, and sometimes the surprise of a pointed, witty anecdote."*

2. Professor Wylie gave an extensive description of the complications which astronomical situations create in the calendar, of the various modifications which have been made in the past in order to meet these, and of the main reforms which have been more recently proposed. He indicated further how, until a definite calendar reform has been adopted, a fixed schedule of appointments for chambers of commerce, business engagements, academic appointments, etc. could be shifted from year to year with a minimum of confusion.

3. Proceeding from the thesis that there has been mutual benefit and convenience between mathematics and natural science by interchange of ideas, Professor Ettlinger exhibited mathematical models illustrating the two following concepts which are of fundamental importance. The first example is that of a type of highly irregular motion pointing in the direction of the Brownian dance of a molecule. This type of motion is continuous but admits no finite velocity over an everywhere dense set of points. The example assumes the initial position of a point and the prescription of a velocity function. The method follows that of Carathéodory, generalizing the earlier method of the Cauchy polygon. Not only the final limiting result but the intermediate stages are of interest as bearing a definite resemblance in a simple fashion to quantum situations, in this case, with respect to time and velocity.

The second type of mathematical model illustrates a situation in which the presence of exact information concerning the initial state and the prescription of an exact law of causality (i.e., a definite one-valued velocity at each point) nevertheless permits a type of indeterminacy suggestive of the situations postulated in quantum mechanics. The principal example exhibited is $y'=2\sqrt{y}$, $y(0)=0$ on $t=0$ to $t=1$. The maximal solution is the parabola, $y=t^2$ and the minimal solution is singular, $y(t)=0$. Between these, there is an infinite set of solutions emanating from the origin, which may be written in the form, $y(t)=0$, when $t\leqq t_0$, and $y(t)=(t-t_0)^2$ when $t\geqq t_0$, where t_0 is any value on $t=0$ to $t=1$. The following situation is of particular interest, namely, that the motion is not determinate, but if the time and position are given, the velocity is determined, and if the velocity is given, the position is determined. Other types of examples were given having these same general properties.

* "*Art and Mathematics*," Scripta Mathematica, vol. 3 (1935), p. 103.

SECOND SESSION OF THE ASSOCIATION

1. Annual business meeting and election of officers.

2. "An elementary theory of regular convex and star-shaped polygons" by Professor W. D. CAIRNS, Oberlin College.

3. Report of the Commission on the Training and Utilization of Advanced Students in Mathematics, by Professor E. J. MOULTON, Northwestern University, *Chairman*.

4. "Application of matrix theory to linear differential equations with constant coefficients" by Professor R. W. BARNARD, University of Chicago.

2. If, beginning with the left end of a horizontal diameter of a unit circle, the circumference be divided into n equal parts, a line drawn from this point to the kth point of division will be the side of a regular polygon, convex or star-shaped. If k is prime to n, this will give a polygon of n sides; if k is not prime to n, the polygon will have fewer sides. For a given n there will be as many regular polygons as there are integers $<n/2$ which are prime to n. The purpose of Professor Cairns's paper was to find the equation of which these definite sides for a given n are the roots. The field of all possible polygons was covered by three different methods, with minor exceptions in the last two methods, the resulting equations having relatively simple coefficients. The results led also to the well-known theorems for the constructibility of regular polygons.

3. Professor Moulton reported that the activities of the Commission on the Training and Utilization of Advanced Students of Mathematics have led to (1) a formulation of a report on the Training of Teachers of Mathematics which was printed in the May 1935 issue of the MONTHLY; (2) the development of a placement bureau under his direction to assist persons holding the doctorate in mathematics to secure positions; and (3) a report on The Ph.D. Degree and Mathematical Research in America, which will be offered for publication in the MONTHLY this year. He reported that about sixty persons had registered with the placement bureau in 1935. A check-up on the employment of persons who have recently received their doctorate in mathematics showed that, of 95 persons reported, 57 were employed as instructors or professors in colleges and universities, 2 in teachers colleges and junior colleges, 9 in secondary schools, 14 were assistants in universities, 4 were fellows and 2 were otherwise registered as students, 2 were actuaries, one was a clerk and 4 were unemployed. Most of Professor Moulton's report was devoted to a presentation of excerpts from the report on The Ph.D. Degree and Mathematical Research in America, which was prepared under the direction of Dean Richardson; several tables from the report were shown by means of slides.

JOINT SESSION OF THE ASSOCIATION WITH SECTION A OF THE AMERICAN ASSOCIATION AND THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1. Report of the joint Commission on the Place of Mathematics in the

Secondary Schools, by Professor K. P. WILLIAMS, Indiana University, *Chairman.*

2. "The main purposes and objectives in teaching high school mathematics" by Mr. WILLIAM BETZ, Rochester, N. Y., representing the National Council, and Professor W. W. HART, University of Wisconsin, representing the Mathematical Association.

1. The report of the Chairman of the joint commission was entirely of a preliminary character, being in the nature of an outline of the various subjects which it is investigating and which it proposes to discuss in its final report. Because the joint commission of the Association and the National Council has been organized only recently, the occasion of the St. Louis meetings afforded the first opportunity for a majority of the commission members actually to meet and to organize the commission's activities. It is expected that reports of progress will be made from time to time as the work develops.

2. Mr. Betz disclaimed any intention of discussing so broad a theme within the brief period assigned to him. Instead, he preferred to consider some aspects of the *present crisis in the teaching of elementary mathematics*. That there is such a crisis must be apparent to any one who has followed recent developments in the field of education. Reference was made to Professor E. T. Bell's corresponding comments in the November, 1935 number of this MONTHLY, page 559, and also to the investigation summarized in the National Mathematics Magazine for October and November, 1934, and February, 1935. It appears that in seventeen states mathematics is no longer a required subject in the high schools. Forty-seven state superintendents of instruction replied to a circular letter, requesting a personal explanation of their attitude toward this situation. The breakdown of secondary mathematics is attributed by these men primarily to (1) poor teaching, (2) poor textbooks, and (3) an unconvincing and ineffective formulation of the objectives of elementary mathematical instruction.

Mr. Betz asserted that these criticisms, while entirely true not only of mathematics, but of every other subject-matter field, do not go to the heart of the matter. They ignore the even more fundamental crisis in education as a whole, from the kindergarten to the university, of which our present troubles are the inevitable symptoms. Accordingly, the National Council will devote a large section of its next Yearbook, to be issued in the near future, to a detailed discussion of the really basic issues in education. Only in the light of such a broad setting can one appraise intelligently the present mathematical dilemma. It will be shown that among the real factors that underlie the obvious chaos in education are the following: (1) a confusion of objectives due to aimlessness, caused largely by Dewey's pragmatic philosophy; (2) the doctrines of "progressive" education, with their emphasis on immediate experience, individual interests and "felt needs," and their disregard of race experience and sequential learning; (3) a policy of continuous and incoherent curriculum-revision based on momentary interests, "social reconstruction," superficial "orientation," to the exclusion of continuity and foundational training in essential lines of work;

(4) the problem of mass education, with the resulting attempts at "adaptation" to individual needs and interests, all of which attempts have been unsuccessful because they have ignored basic causes and problems, have rejected standards, and have preferred an inconsequential tinkering with opportunist expedients or surface adjustments; (5) a wrong psychology of learning based on a mechanistic conception of the mind and avoiding real understanding, thus making the ruination of mathematical instruction almost inevitable; (6) a narrowly specific and totally inadequate training of secondary teachers.

According to Mr. Betz, the crucial question which teachers of mathematics are called upon to answer may be stated as follows:

"Does secondary mathematics, when taught adequately, represent such a vitally essential educational element that it should be made mandatory in the curriculum of all normal pupils, even though many of them may never have occasion to use it for immediate, vocational purposes?"

The deplorable fact is that influential groups of educators and administrators have already answered this question negatively, and hence reject in advance any "defense" of mathematics as a belated and futile attempt at "special pleading." They refuse to investigate the fundamental issues, and to endorse vital improvements such as effective teaching in the primary grades and a continuous, six-year curriculum in secondary mathematics, of the type we find in all other leading countries. A characteristic example of the prevailing attitude toward prescribed, sequential training in secondary mathematics is that of Butler Laughlin, President of the Chicago Normal College, as stated in the Chicago Schools Journal of December, 1935. He says that "mathematics instruction should be strictly adapted to individual needs, and the program should be limited to the barest essentials. The major portion of the core curriculum in mathematics should be concerned with quantitative thinking in genuine social situations." The usual formal courses, he states, should be made elective after the ninth year, being limited to superior or special students.

The time has come for a more vigorous exposition of the affirmative side of the story. It has already been given so many times, throughout the ages, that it would seem to be a mark of ignorance to have escaped the impact of all these impressive rehearsals. Each succeeding age has only confirmed and extended this positive verdict rendered by many of the world's leading thinkers. In recent years we have been made increasingly conscious, by a long series of syllabus reports, yearbooks, treatises and studies, as well as newspaper and magazine articles, of the fact that mathematics is indeed the "Queen of the Sciences." Reference may be made, for example, to the beautiful essays of Professor Cassius J. Keyser and of Professor David Eugene Smith, issued by Scripta Mathematica, and to scores of papers in the Mathematics Teacher, in School Science and Mathematics, and in the American Mathematical Monthly.

Hence those who are informed know that mathematics, the oldest of all the sciences, needs no "defense." It is the product of human civilization. Without

it, our entire technical, industrial and economic structure would collapse and primitive conditions would return. That mathematics is the most basic and hence the most indispensable of all the sciences, was recognized in a spectacular way at the recent Chicago Exposition by John Norton's famous mural entitled "The Tree of Knowledge." In this allegorical painting, mathematics was shown to be the main root of all "the basic and applied sciences." On the wall opposite this "Tree," the story of electricity was told in a striking manner. Its culminating sentence read as follows: "Mathematics is the Key and Applied Mathematics is the Tool wherewith Man Conquers the Universe."

But the "educator" who insists on ignoring all this overwhelming evidence has a convenient excuse for his negative attitude. Strangely enough it was given currency by so competent a scholar as the late Professor Inglis. It is to the effect that no one denies the direct "service-values" of mathematics, but that most of them are of such a technical character that they have no place in a scheme of general education. Instead they should be assigned exclusively to "vocational" courses. Hence our "defense" of secondary mathematics is seen to rest almost entirely on its "indirect" or "cultural" values.

Here again recent psychological investigations, such as those of Lashley and Orata, have given us ample reason for complete confidence in the essential and permanent place of mathematics in the education of our young people. It is now held by competent judges that the old question of the "transfer of training" is, in essence, a problem of "training for transfer." When thus understood, the general or "cultural" value of adequate mathematical instruction is seen to be unexcelled. In particular, three aspects of mathematical education are of outstanding importance. They may be described as symbolic thinking, postulational thinking, and relational thinking. Each is absolutely fundamental and each has unlimited "transfer" possibilities.

Unfortunately the "educators" as a class are still ignorant of these facts. They continue to repeat the outworn and misleading assertion that "mental discipline is a myth" and dismiss "formal" mathematics as "obsolete." How long will it take before this state of regrettable and uninformed backwardness becomes obsolete?

What will be the effect of the present indifference or hostility to mathematics, not found in other countries? A breakdown of all dependent types of training, whether direct or indirect, will be the inevitable result. Already, colleges and universities are obliged to arrange for elementary courses in algebra and geometry, to correct the sad consequences of a defective or pauperized type of secondary education. Adults are attending evening schools to obtain the mathematical education they need, postponed through unwise guidance. Eventually we may expect a wholesale rebellion against our present educational leadership, such as occurred in Russia a year ago.

Much publicity should be given to a recent decision of the University of Wisconsin. It now refuses admission to a vast number of courses (including agriculture, commerce, economics, pharmacy, pre-medicine, philosophy, psy-

chology, sociology, the natural sciences) in the case of all students who do not offer at least a minimum preparation in secondary mathematics.

The immediate future will be a critical one. Of course, mathematics will return to its deserved place of importance, in spite of all uninformed propaganda against it. Nothing can prevent it. The world is "incurably mathematical." We shall, however, hasten the return to sanity by better teaching, based on a clearer understanding of the true functions of mathematical instruction.

In his address which followed, Professor Hart submitted some modifications of some premises of educational philosophy and psychology which are popular at present, applied these modified premises to the formulation of aims and objectives of the study of high school mathematics, and made a suggestion about means of giving publicity to pro-mathematics doctrine.

While education is indeed "preparation for living," it is, for youth, a desirable form of present living. The present makes a greater appeal to youth than the future. Preparation, interpreted narrowly as vocational, is futile at best, and is, for the most part, unnecessary for those pupils who do not go into the higher professions. Many leaders in the field of vocational education are proposing that a broad general education should precede special vocational training. Preparation limited by study of that "which is worth while" or that "which will function in life" is handicapped by failure to define "worth while"; it is often concerned with superficial activities of living which are mastered easily just by living, with indefinite sociological, economic, and political problems and questionable solutions of them far beyond the experience of youth, with bits of unorganized knowledge, such as "general science," and with apparently useful materials from the business world, such as so-called "junior business practice." In contrast to such experiences, contacts with the fields of organized knowledge have much greater educational significance and possibilities. Such contacts cannot be made effectively or economically when they are only incidental in connection with some so-called "integration program."

Education conceived as the process of forming bonds between specific stimuli and responses appears less desirable than a process which looks to development of general ideas,—a conception which is based on the apparently established fact that the mind "functions as a whole." This latter conception of education discredits the tendency so popular recently to emphasize learning of facts and acquiring skills "which will be used" and only such; it emphasizes as the chief objectives of education general ideas, attitudes, ideals, and appreciations.

The dictum that educational experiences are most effective when the learning is accompanied by satisfaction is not accepted by all psychologists; it is usually accompanied by the unsound implication that the chief source of interest is the subject matter itself, whereas the personality, attitude, and resourcefulness of the teacher are more likely to produce interest; and, when the learning is directed toward interesting things, the resultant education may indeed

be quite inferior to that which focuses attention on fundamental things in an interesting way.

The pedagogical deductions from real and apparent differences among pupils are subject to criticism. The differences are as much in respect to will and preparation as in respect to native intelligence. As a matter of fact, the differences are no more marked than are the likenesses of pupils because all normal youths have potential capacity of some sort. This latter is a much more optimistic view and is much more stimulating to the teacher. The deduction commonly made that some students should be excused from or denied the opportunity to secure contacts with some subjects because "they cannot get them" displays a thoroughly erroneous and fatalistic attitude toward education. The real question should be, "What fields of knowledge are fundamentally important?" When that question is answered, the proper attitude of the school should be to assist each child to secure such mastery as is possible through diligent use of the ability the child possesses.

On the basis of the foregoing modifications of current philosophy and psychology of education, it is submitted that high school mathematics should be studied by all normal pupils for three reasons:

(a) For the pleasure derived while studying it.

(b) For the direct and indirect usefulness of the information and training secured from study of it.

(c) For the purpose of knowing mathematics and its contributions to past and present material progress and to intellectual life.

The first of these aims suggests a desirable new view point for all concerned. Pupils who come to the study of mathematics unbiased are easily led to pleasurable experiences with it,—an assertion which can be developed. Whether or not the pupils do secure such pleasurable experiences depends very largely upon the personality, attitude, and technique of the teacher, rather than upon the subject itself, although the subject does make its contributions. This is an inspiring view point for the teacher; it places on the teacher, however, a great present responsibility. When teachers of mathematics have become impressed with this aim as a primary one and succeed in meeting it, much of the current criticism of mathematics as a subject will disappear, and with it the current opinion among many parents and administrators that mathematics is at best only a necessary evil.

The second general aim is the one usually emphasized, but even here, only two of the three possible classes of "uses of mathematics" are discussed commonly. Psychologists classify possible objectives of a course as: (1) knowledge (2) habits and skills, and (3) general patterns of conduct, such as attitudes, appreciations, and ideals. In respect to the first two classes of objectives, lists of specific items of useful knowledge, habits and skills which can be secured from study of high school mathematics are available. Of course, such contributions of the study should be emphasized. They must be regarded only as minima in the courses. There is a tendency, however, for lists of minima auto-

matically to become maxima, particularly among those who teach only that which will be used. Pupils who can do more should be encouraged to use all the ability they possess in accordance with the premise that real education is a process of stimulating learning to capacity. Also, this tendency to limit instruction to that which will be used overlooks the necessity of a degree of overtraining in order to instill confidence and to develop facility in the learner. Instead of striving to build up only specific bonds in the narrow range of those which are useful, it is in all respects more desirable to develop understanding of broad general ideas, and to build up gradually a degree of mathematical maturity.

Interest is growing in analysis of the objectives called general patterns of conduct. Just what habits, appreciations, ideals, and attitudes can be secured through study of mathematics is being asked. In respect to appreciations, there must be included appreciation of the use of mathematical formulas in industry and in science, of signed numbers and literal numbers, of geometry in various fields of design, of the importance of exactness of definition and phraseology, and of logical reasoning in non-mathematical situations. These suggest additions to traditional subject matter of secondary mathematics, or at any rate they suggest by-products of the instruction which must be taught if the training is to transfer. That there is hope of such transfer has been asserted by Orata, provided there is training for transfer. Mathematics is rich in possible contributions of such kinds. It would be unfortunate, however, if the resultant courses become largely descriptive, if the books become filled with pretty pictures, and if there is an excessive use of examples from non-mathematical fields to impress the meaning and use of definitions, assumptions, and logical processes in such fields. The statement was once made that geometry is only logic. It was meant to be a criticism of instruction in geometry. The present tendency would seem to be in the direction of making it only logic, and stressing that as its important contribution to secondary education. In other words, there must be a balance between the new type of materials and the traditional material, else we shall have courses in mathematics with much of the mathematics left out.

The third general aim listed above has least reputation among educationists. Just why is study of a subject for itself not worthwhile? No apology appears to be necessary for interest in music, art, or literature, both poetry and prose, and yet the sole criterion for judging the acceptability of subject matter in mathematics and in some of the other sciences appears to be its possible usefulness. It is submitted that mathematics occupies in the cultural world of the past and present at least as significant a place as any of the more popular fields mentioned above. When to this there is added the fact that mathematics has made a much more significant contribution to the material progress of the world than any of the fields mentioned above, one may almost be pardoned for insisting that it is narrow prejudice or personal bias which dictates this anti-mathematics attitude.

Some educationists, on the other hand, very sanely propose that a principal

function of the secondary school is to secure to pupils intellectual interests. This was formerly known as the exploratory function of the school. In a recent article, Bagley has said that the organized knowledge of the world must always be the core of educational experiences. It does not appear to be an unfair criticism either of the youth in the schools of all grades or of the adults who have graduated from them to say that intellectual interests as such are not conspicuous. Is it any wonder that this is true when the whole emphasis in educational philosophy and the force of opinion and attitude of many administrators and educationists is directed only toward that which they call "worthwhile"? To be specific, can pupils be expected to develop interest in mathematics as such, when the chief means of motivation is to make mathematics appear largely as the hand-maid to the pursuit of its applications? *Instead of teaching mathematics as the hand-maid of science, it should be taught as the queen of sciences.* Was pursuit of knowledge for its own sake as disreputable a generation ago as it appears to be among some people today? Is the present attitude a result of a generation of elective education in high school and college? Is the elective system resulting in a lack of that sympathy for and appreciation of the intellectual interests of others which by many has been considered a characteristic of educated people? Will this elective system not have, and is it not having, an undesirable effect upon the students? It is submitted that the third general aim of study of high school mathematics is a worthy one; that it can be attained; that the chief requisite for attaining it is a group of teachers who know their subject and who themselves have real interest in and respect for it.

What mathematics shall be taught to accomplish the general aims listed? The answer should be teach broad general ideas, rather than specific facts; develop understanding first and such skill only as may be necessary; strive not to do less and less mathematics, but rather more and more mathematics; teach so that each child will master as much as possible, instead of being content with merely getting by. Any pupil and any class can do less; the able pupils and the ambitious teachers will strive to do more.

Groups interested in the teaching of mathematics can readily agree upon most of the foregoing. Our problem is to get over such ideas to the public, the school administrators, and the educationists. A vigorous campaign of education directed toward the people who do not attend these meetings is the only remedy. Meetings such as this and reports such as will be published by the joint commission of our two organizations will serve their purpose. In addition, there is need of a continuing activity directed by a national committee, which will in turn stimulate local committees to various forms of proper pro-mathematics publicity. These may include publication and distribution of articles such as that of Hedrick recently in the Scientific American, of Langer in School and Society, stimulation of talks by friends of mathematics before parent-teacher associations and school administrators, and in general, a degree of fair minded, but nevertheless, vigorous opposition to all who are exhibiting a narrow unsympathetic attitude toward the subject. As an example of pro-mathematics doc-

trine which may be distributed, let me in closing quote a statement made not long ago by Professor Judd of Chicago to the effect that mathematics has established its place in civilization; that it is no trivial assault on the educational institutions of the race that is made by those who would curtail instruction in mathematics or reduce it to a bare minimum.

Joint Session of the Association with Section A of the American Association and the American Mathematical Society

1. "Linear differential equations of infinite order" by Professor R. D. Carmichael, University of Illinois, retiring chairman of Section A.

2. "Tensorial methods in dynamics" by Professor J. L. Synge, University of Toronto.

1. This address will appear in an early issue of the *Bulletin*.

2. The idea of regarding the motion of a general dynamical system as that of a particle in Riemannian N-space goes back over sixty years, but the systematic application of tensor calculus to this idea is only about ten years old. The manifold employed is either the manifold of configurations or that of configurations and time, the former being suited to scleronomic systems and the latter to rheonomic. In the manifold of configurations the most interesting line-element is

$$ds^2 = 2Tdt^2 = a_{ij}dx^i dx^j.$$

Using generalized definitions which immediately suggest themselves, a number of known results of particle dynamics are found to hold for general systems. Thus acceleration = force; acceleration may be resolved into components vdv/ds and κv^2 along the tangent and first normal to the trajectory; a constrained trajectory satisfies a condition of least curvature; and so on. The theories of brachistochrones, hodographs and apsides may also be extended to general systems. In all this work the tensor notation enables us to express results very compactly and is a powerful source of suggestion.

Systems with non-holonomic constraints are most conveniently discussed by means of tensors, as is also the case for quasi-coordinates. These dynamical ideas have led to the development of "non-holonomic geometry" (Vranceanu, Schouten).

Wundheiler has developed a "rheonomic geometry" for systems with moving constraints. Since we are dealing with Newtonian dynamics, t is privileged, and we may be satisfied with invariance under transformations identical in t in the manifold of configurations and time. We have in fact a privileged family of surfaces t = constant.

The tendency of the application of tensorial methods is to draw dynamics into close association with Riemannian and associated geometries. The objective is a compressed but easily interpretable expression of the general properties of dynamical systems.

Meetings of the Board of Trustees

Twelve members of the out-going and of the incoming Board were present at the St. Louis meetings.

The following twenty-seven persons were elected to membership on applications duly certified:

Eleanor G. Anderman, A.M.(New Mexico) Teacher, High School, Albuquerque, N.M.
Mae R. Andersen, A.M.(Chicago) Asst. Prof., Math. and French, Concordia Coll., Moorhead, Minn.
H. G. Ayre, M.S.(Michigan) Teacher, High School, Waukegan, Ill.
Elizabeth J. Breuer, A.B.(U.C.L.A.) Teaching Asst., Univ. of California at Los Angeles, Los Angeles, Calif.
N. K. Chatterjee, M.S.(Caclutta) Grad. research fellow, Calcutta Univ., Calcutta, India.
A. E. Currier, Ph.D.(Harvard) Instr., U. S. Naval Acad., Annapolis, Md.
N. A. Gilbert, A.B.(Oberlin) Instr., Muskingum Coll., New Concord, Ohio
E. H. Hanson, Ph.D.(Ohio State) Prof., Dir. of Dept., N. Texas State Teachers Coll., Denton, Texas
A. O. Hizon, B.S.C.E.(Philippines) Civil Engr., P. I. Scholar of Philippine Gov't, Washington, D.C.
M. Gweneth Humphreys, Ph.D.(Chicago) Instr., Mt. St. Scholastica Coll., Atchison, Kans.
Ruth O. Lane, M.S.(Iowa) Instr., Univ. of Iowa; Head of Dept., University High School, Iowa City, Iowa
G. B. Lang, A.M.(Georgia) Math. and Physical Sci., West Georgia Coll., Genola, Ga.
S. A. Lynch, M.S., E.M.(Missouri School of Mines) Prof., N. Texas Agric. Coll., Arlington, Tex.
J. R. Mayor, Ph.D.(Wisconsin) Chm. of Dept., S. Illinois State Normal Univ., Carbondale, Ill.
Mary McKenna, A.M.(Columbia) Teacher, Washington Irving High School, New York, N. Y.
Z. I. Mosesson, A.M.(Harvard) Instr., Harvard Univ., Cambridge, Mass.
E. A. Nordhaus, M.S.(Chicago) Instr., Univ. of Wisconsin, Extension Div., Milwaukee, Wis.
J. M. H. Olmsted, A.B.(Minnesota) Grad. student, Univ. of Minnesota, Minneapolis, Minn.
Theresa L. Podmele, A.M.(Buffalo) Teacher, East High School, Buffalo, N.Y.
R. F. Rinehart, Ph.D.(Ohio State) Prof., Ashland Coll., Ashland, Ohio
S. L. Robinson, Ph.D.(Iowa) Instr., Coll. of the City of New York, New York, N.Y.
P. J. Rulon, Ph.D.(Minnesota) Asst. Prof., Harvard Grad. School of Educ., Cambridge, Mass.
Mrs. Alta H. Samuels, A.M.(Louisiana) Instr. Louisiana State Univ., Baton Rouge, La.
J. L. Stearn, M.S.(C.C.N.Y.) Jr. Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C.
W. R. Talbot, Ph.D.(Pittsburgh) Asst. Prof., Lincoln Univ., Jefferson City, Mo.
C. B. Tompkins, II, Ph.D. (Michigan) Instr., Univ. of Maryland, College Park, Md.
J. F. Wyckoff, A.M.(Yale) Instr., South Dakota State Coll., Brookings, S.D.

The financial report of the Secretary-Treasurer for the year 1935 was presented, approved by Professor Slaught for the Finance Committee; and this was accepted, subject to inspection by a sub-committee. Professors Brink and Rietz later examined the report and found it to be satisfactory; Professors Sinclair and Yeaton certified to the evidences of assets. The Finance Committee was authorized to transfer approximately \$4000 to the General Endowment Fund.

The Trustees approved the recommendation of the committee on the award of the Chauvenet Prize, which was announced Monday afternoon at the business meeting. They also unanimously recommended to the Association a resolu-

tion on the World Calendar. Professor Curtiss announced the addition of Professor Dunham Jackson to the editorial committee on the Carus Monographs. He also reappointed Professor C. S. Atchison and Secretary Cairns as representatives of the Association on the Council of the American Association for the year 1936.

Invitations for future meetings of the Association were considered and referred to an appropriate committee for consideration. The following were appointed associate editors of the MONTHLY for the year 1936, as nominated by Professor Carver:

W. F. Cheney	R. E. Gilman	F. W. Owens
N. A. Court	R. A. Johnson	Helen B. Owens
Otto Dunkel	B. W. Jones	R. G. Sanger
B. F. Finkel	J. R. Musselman	D. E. Smith
T. C. Fry	H. L. Olson	J. H. Weaver

ANNUAL BUSINESS MEETING

The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He also reported the deaths of the following members:

G. G. CHAMBERS, Professor of mathematics, University of Pennsylvania. (October 24, 1935)
B. H. CRENSHAW, Professor of mathematics, Alabama Polytechnic Institute. (Novermber 25, 1935)
A. B. DINWIDDIE, Professor of mathematics and President, Tulane University. (November 21, 1935)
RAYMOND GARVER, Associate Professor of mathematics, University of California at Los Angeles. (November 7, 1935)
D. C. GILLESPIE, Professor of mathematics, Cornell University. (October 31, 1935)
E. D. GRANT, Professor of mathematics, Earlham College. (September 2, 1935)
LOUIS INGOLD, Professor of mathematics, University of Missouri. (January 25, 1935)
E. H. JONES, Professor of mathematics, Southern Methodist University. (August 7, 1935)
E. A. PARTRIDGE, Head of science department, West Philadelphia High School. (March 22, 1934)
E. B. SKINNER, Professor of mathematics emeritus, University of Wisconsin. (April 3, 1935)
W. P. WEBBER, Professor of mathematics, Louisiana State University. (June 26, 1935)

President Curtiss announced the fourth award of the Chauvenet Prize of $100 to Professor Dunham Jackson for the following series of related papers:

1. *The convergence of Fourier series.* American Mathematical Monthly, volume 41 (1934), pp. 67–84.
2. *Series of orthogonal polynomials.* Annals of Mathematics, volume 34 (1933), pp. 527–545.
3. *Orthogonal trigonometric sums.* Annals of Mathematics, volume 34 (1933), pp. 799–814.

This award covers the years 1932–34 and is given "for a noteworthy expository paper published in English by a member of the Association."

On the behalf of Professor E. V. Huntington, the Secretary presented a resolution on the adoption of the World Calendar. After an extended discussion the resolution formulated as follows was adopted with one dissenting vote.

Whereas, in the opinion of this Association a simplification of the calendar would be of great advantage to the world at large and to educational institutions in particular, and

Whereas, of the two proposals mentioned in the League of Nations Report of 1931 the 12-month equal-quarters plan known as the World Calendar appears to be the more desirable, be it therefore

Resolved, that the Mathematical Association of America hereby endorses the 12-month equal-quarters plan for the simplification of the calendar and would welcome its world-wide adoption.

Professors Edwin R. Smith and L. S. Shively acted as tellers for the election of officers for 1936, the result being as follows:

Vice-Presidents: N. A. COURT, University of Oklahoma; T. C. FRY, Bell Telephone Laboratories.

Additional members of the Board of Trustees, to serve until January 1939: G. C. EVANS, University of California; W. R. LONGLEY, Yale University; E. J. MOULTON, Northwestern University; MARY EMILY SINCLAIR, Oberlin College.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 12, 1935

RECEIPTS

Balance Dec. 12, 1934		$8,488.90
1934 indiv. dues	560.60	
1934 instit. dues	7.00	
1934 subscriptions	13.00	
1935 indiv. dues	6,444.33	
1935 instit. dues	645.30	
1935 subscriptions	840.02	
Initiation fees	214.00	
Advertising	427.00	
Authors' reprints	239.46	
Sales copies of MONTHLY	61.85	
Sale First Carus Mon.	13.75	
Sale Second Carus Mon.	13.75	
Sale Third Carus Mon.	12.50	
Sale Fourth Carus Mon.	8.75	
Sale Fifth Carus Mon.	35.00	
Sale Archibald's Outline of Hist. of Math	151.87	
Annals subscription	2.50	
Life membership fees	109.11	
Placement bureau fees	59.00	
Sale Rhind Papyrus	61.25	
Sale Moulton reports	47.10	
Drury Coll. int. Hardy Fund	120.00	
Sale C. T. Securities & Invest. Co. Bonds	2,050.00	
Sale Pacif. Power & Light Bonds, less commission	820.69	

EXPENDITURES

Publisher's bills (Nov. '34–Oct. '35)		$4,703.01
Reprints		243.68
Manager's office		4.66
Editor-in-chief's office		459.93
Committee on membership		61.00
Expense Com. on Training & Utilization		118.28
Reprints Moulton's Report		120.57
Expense placement bureau		55.36
Expense Com. on Place of Math.		32.63
Expense Com. on Testing		32.84
Secretary-Treasurers' office:		
Postage	403.50	
Bond	11.26	
Safety Deposit	4.40	
Office supplies	58.59	
Express, tel., etc	90.50	
Clerical work	1,971.66	
Printing	230.21	
Bank tax	10.50	2,780.82
Annals subvention		225.00
Duke Journal subvention		100.00
Expense sections from init. fees		192.79
Pittsburgh meeting		87.36
Ann Arbor meeting		61.39
St. Louis meeting		7.00
Forwarded *Annals* subscription		2.50
Paid *Annals* subscriptions		12.50

Int. Oberlin Savgs. Bk.	29.48	
Int. Peoples Bkg. Co.	38.59	
Int. Cleveland Trust Co.	77.04	
Int. certif. of deposit	.46	
Int. Genl. End. Fund	397.46	
Int. Carus Fund	118.06	
Int. Chace Fund	222.82	
Int. Chauvenet Fund	15.00	
Int. current funds	39.07	
Payment from restricted Carus Fund	99.40	
Payment from restricted Chace Fund	4.40	
Profit from sale of Iowa Elec. Lgt. & Power Co. Bonds	347.03	14,346.64
Total 1935 receipts to date		22,835.54
Total expenditures		15,490.93
Balance to end of 1935 business		7,344.61
Received on 1936 business		717.75
Book balance Dec. 12, 1935		8,062.36

Sust. memb. in Amer. Math. Soc.	100.00
Refund subscriptions	7.65
Expense mailing Outline	10.11
Storage & inventory back copies of MONTHLY	54.83
Paid back copies MONTHLY	20.40
Paid B. F. Finkel int. Hardy Fund	120.00
Expense *Register*	96.46
Ins. back copies MONTHLY	17.70
Loss exchange HOLC bonds for N. W. Elec. bonds	29.05
Cost above par U. S. Treas. bonds	9.80
Cost above par for HOLC bonds	14.31
Paid Union Pacific Debentures	3,199.51
Transfer to Chace Fund	509.79
Transfer to Genl. End. Fund	2,000.00
Total expenditures	15,490.93
Checking account	146.39
Oberlin Savgs. Bk. acct restricted	1,166.20
Peoples Banking Co. acct	1,098.52
Cleveland Trust Co. savgs. acct	4,401.25
U. S. Savings Bonds	750.00
U. S. Treasury Bond. 1946–49	500.00
Bank balance Dec. 12, 1935	8,062.36

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

CARUS MONOGRAPH FUND

Balance Dec. 12, 1934		$6,084.78
Receipts: Sales	$ 83.75	
Interest	175.19	
Profit from sale C. T. Securities Co. Bond	25.00	
Profit from sale Pacific Power and Light Bond	50.69	334.63
		$6,419.41
Cost above par Union Pacific Debentures		133.00
		$6,286.41
Certificate of deposit	$1,685.58	
Union Pacific Debentures $4\frac{1}{2}\%$	2,000.00	
3 1/8% U. S. Treasury Bond of 1946–49	1,000.00	
3% HOLC Bond 1944–52	1,000.00	
Cash in bank, restricted, certificate of participation	695.80	6,381.38
Due general treasury		94.97
Balance Dec. 12, 1935		$6,286.41

ARNOLD BUFFUM CHACE FUND

Balance Dec. 12, 1934		$5,987.03
Receipts: Sale Papyrus	$ 61.25	
Interest	227.75	
Profit from sale Iowa Elec. Light and Power Co. Bond	77.12	366.12
		$6,353.15

Expense purchase U. S. Treasury Bond		9.79
		$6,343.36
3 1/8% U. S. Treasury Bonds 1946–49	$1,500.00	
3% HOLC Bond 1944–52	1,000.00	
Western United Gas and Elec. Co. Bonds	2,370.00	
Certificate of deposit	126.67	
Certificate of deposit, Northern Trust Co., Chicago	884.96	
Cash in bank, restricted, certificate of participation	30.80	
Cash in bank, unrestricted	430.93	
Balance Dec. 12, 1935		$6,343.36

Chauvenet Prize Fund

Balance Dec. 12, 1934		$589.38
Receipts: Interest	$15.00	
Profit from sale Iowa Elec. Light and Power Co. Bond	38.56	53.56
		$642.94
3% HOLC Bond 1944–52	$500.00	
Cash in bank, unrestricted	142.94	
Balance Dec. 12, 1935		$642.94

Life Membership Fund

Liability on life memberships as of Jan. 1, 1935	$661.71
Life membership payments	109.11
	770.82
To be transferred to current funds, surplus	17.85
Liability on life memberships as of Jan. 1, 1936	$752.97

General Endowment Fund

Balance Dec. 12, 1934		$12,585.00
Transferred from current funds		2,000.00
		$14,585.00
Less depreciation Northwestern Electric Co. Bonds		500.00
		$14,085.00
3¼% U. S. Treasury Bonds 1944–46	$1,000.00	
3¼% U. S. Treasury Bond 1943–45	1,000.00	
3% HOLC Bonds 1944–52	5,500.00	
Land Trust Certificate	700.00	
5% Idaho Power Co. Gold Bonds 1947	2,000.00	
5% Texas Power and Light Co. Gold Bonds 1956, market value	885.00	
4½% Union Pacific Debenture 1967	1,000.00	
Oberlin Savings Bank savings account	2,000.00	
Balance Dec. 12, 1935		$14,085.00

Of the funds on hand, indicated in the first division of this financial report, $430.93 belongs to the Arnold Buffum Chace Fund, $142.94 belongs to the Chauvenet Prize Fund, $752.97 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1936, while $94.97 is due the general treasury from the Carus Monograph Fund.

When the accounts were closed Dec. 12, 1935, there remained on the total business for 1935 the following items:

BILLS RECEIVABLE

1935 individual dues	$200.00
Advertising	50.00
Due from Carus Mon. Fund	94.97
	$344.97

BILLS PAYABLE

Publisher's bills (Nov.–Dec. 1935)	$1,000.00
Printing *Register*	425.00
President's office	20.00
Manager's office	20.00
Editor-in-chief's office	120.00
Secretary-Treasurer's office	250.00
Comm. on Training and Utiliztn.	100.00
Comm. on Place of Math	200.00
Subsidy *Duke Journal*	100.00
Chace Fund	430.93
Chauvenet Prize Fund	142.94
Life Membership Fund	752.97
Init. fees due to sections	822.00
Expense acct. Carus Mon. Fund.	100.00
	$4,483.84

If to the balance on 1935 business shown in the report, $7,344.61, there be added the bills receivable, $344.97, and there be subtracted the estimated bills payable, $4,483.84, there results an estimated final balance on 1935 business of approximately $3,200.00, which represents the accumulated surplus in current funds.

W. D. CAIRNS, *Secretary-Treasurer*

THE TENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The tenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Lafayette College, Easton, Pa., on Saturday, November 30, 1935, Professor Morris presiding.

The attendance was sixty-three, including the following twenty-seven members of the Association: J. A. Benner, William Beverley, S. S. Cairns, P. A. Caris, John Cawley, R. L. Charles, J. E. Davis, Tomlinson Fort, J. S. Gold, D. A. Hatch, V. V. Latshaw, D. H. Lehmer, W. F. Long, A. E. Meder, Richard Morris, C. A. Nelson, T. S. Peterson, A. G. Rau, G. E. Raynor, J. B. Reynolds, I. J. Schoenberg, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, R. M. Walter, C. R. Wilson.

At the business meeting the following officers were elected for next year: Chairman, J. W. Clawson, Ursinus College; Secretary, P. A. Caris, University

of Pennsylvania; Program Committee, Professors Oakley, Lehmer, Brinkmann. It was also agreed to hold the next meeting at the University of Pennsylvania on Saturday, November 28, 1936.

The following papers were presented:

1. "Collegiate curricula in mathematics in this section" by Dr. R. P. Bailey, Lafayette College, introduced by Professor Smith.

2. "Quantum mechanics" by Professor E. E. Witmer, University of Pennsylvania, introduced by Professor Shohat.

3. "A macroanalysis of some simple dynamical systems" by Professor G. A. Hedlund, Bryn Mawr College, introduced by Professor Fort.

4. "The teaching of mathematics in the Pennsylvania German schools" by Dean Albert G. Rau, Moravian College.

5. "Almost-periodic functions" by Professor Salomon Bochner, Princeton University, introduced by Professor Shohat.

Abstracts of the papers follow:

1. Doctor Bailey presented the results of a survey of the mathematics curricula of the colleges of the Philadelphia Section, calling attention in particular to the most prevalent types of curriculum organization and the major problems to which they give rise.

2. Professor Witmer divided classical or nineteenth century physics into two parts—classical macroscopic physics and atomic physics. The first part contained the major part of physics, consisting of mechanics, hydrodynamics, theory of elasticity, theory of electricity and magnetism, etc., and was based on the idea that all the quantities appearing in it could assume a continuous range of values. In particular, size was purely relative and had no absolute significance. In contrast with this, atomic theory was based on the idea that the world was constructed of entities which could not be further subdivided, the idea of atomism or discontinuity. In particular, size in this theory is not relative, since the radii of the atom are preferred lengths. Likewise mass could not assume any value but only certain permitted values, the atomic weights. Thus classical physics contained an internal contradiction. About the end of the last century the idea of continuity led to conflict with experiment, which resulted in development of quantum theory by Planck. Nature exhibited a greater degree of discontinuity than even classical atomic theory had supposed. The subsequent development of quantum theory led to modern quantum mechanics, which may be regarded as a synthesis of macroscopic classical physics with the concept of atomism or discontinuity. Since many physical quantities now have a discrete *spectrum* in at least part of their range, this naturally suggests that we should approach our physical problem as an "eigen"-value problem and that is what quantum mechanics is.

3. The dynamical systems Professor Hedlund considered are Euclidean and non-Euclidean billiards with a single ball which maintains its velocity and is reflected from the sides at equal angles. The table in the Euclidean case is rectangular, in the non-Euclidean case a triangular table with one vertex at

infinity. It is shown that in the Euclidean case almost all motions are point-transitive (approximate all points arbitrarily closely) but no motions are element-transitive. In the non-Euclidean case the existence of element-transitive motions is shown. Let B be a sector of motions through the initial point P_0 and let L_S be the set of points on these motions obtained by laying off the distance s from P_0 along the motions of the set B. Consideration is given to the problem of what happens to L_S with increasing s. In the Euclidean case L_S is asymptotically point-transitive but not necessarily asymptotically element-transitive. In the non-Euclidean case it can be shown that this second property does hold.

4. Dean Rau's researches among early records and diaries show that the teachers in the Moravian schools, at least, were acquainted with algebra and geometry. After 1780 these subjects were part of the curriculum as shown by scrapbooks containing pupils' work. Conic sections were a part of the schedule of the Moravian College after 1820. Analytical geometry and calculus were on the schedule after 1860 but were discontinued from about 1870 to 1900. The general question of mathematics in other Pennsylvania German schools could not be examined for lack of evidence and is postponed for a later paper.

5. Professor Bochner gave an outline of the theory of periodic and almost-periodic functions as viewed from the general theory of expansions of general functions on groups. He emphasized that the modern viewpoint explains adequately why the exponentials form a complete system of pure periodic functions of one variable.

P. A. CARIS, *Secretary*

THE SIXTEENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The sixteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at James Millikin University, Decatur, Illinois, on Friday and Saturday, May 3–4, 1935. Professor E. B. Miller, chairman of the Section, presided.

The attendance was seventy-eight, including the following forty-four members of the Association: Beulah M. Armstrong, Edith I. Atkin, H. W. Bailey, R. W. Barnard, O. K. Bower, J. M. Cell, Laura E. Christman, A. B. Coble, C. E. Comstock, H. B. Curtis, D. R. Curtiss, W. M. Davis, Elinor B. Flagg, A. E. Gault, F. C. Gentry, R. M. Ginnings, G. D. Gore, Frances Harshbarger, M. C. Hartley, Gertrude Hendrix, Mabel M. Heren, W. R. Holloway, Mildred Hunt, E. C. Kiefer, W. C. Krathwohl, Mayme I. Logsdon, W. D. MacMillan, H. J. Miles, E. B. Miller, G. A. Miller, C. N. Mills, G. E. Moore, E. J. Moulton, Mary W. Newson, Sallie E. Pence, D. W. Querfeld, Mary B. Rumsey, H. A. Simmons, Norma K. Stelford, R. C. Stephens, C. A. Stone, E. H. Taylor, M. E. Wescott, F. E. Wood.

The following officers were elected for the coming year: Chairman, Professor E. C. Kiefer, James Millikin University; Vice-Chairman, Professor C. N. Mills,

INFINITE EXPONENTIALS

By D. F. BARROW, University of Georgia

1. A *continued exponential* shall be denoted by the symbol

$$E_{i=0}^{n} a_i$$

which is an abbreviation for

$$a_0^{a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_n}}}}}}$$

that is, each a_i is used as the exponent of the preceding. This symbol is selected because of its analogy with the symbols for a continued sum and a continued product, namely

$$\sum_{i=0}^{n} a_i \qquad \text{and} \qquad \prod_{i=0}^{n} a_i .$$

In the continued exponential, the various a_i shall be called *exponents*. A longer symbol which puts the exponents into evidence will sometimes be used when one or more of the exponents are irregular, namely

$$E(a_0, a_1, a_2, \cdots, a_n) \equiv E_{i=0}^{n} a_i .$$

The *order* of this exponential is n.

2. *Some obvious properties* of continued exponentials are

(a) $\quad E(a_0, a_1, \cdots, a_m, a_{m+1}, \cdots, a_n) = E[a_0, a_1, \cdots, a_m, E(a_{m+1}, \cdots, a_n)]$

(b) $\quad E(a_0, a_1, \cdots, a_m, 1, a_{m+2}, \cdots, a_n) = E(a_0, a_1, \cdots, a_m)$

(c) $\quad E(a_0, a_1. \cdots, a_m, 0, a_{m+2}, \cdots, a_n) = E(a_0, a_1, \cdots, a_{m-1}).$

3. *An infinite exponential* is naturally denoted by the symbol

$$E_{i=0}^{\infty} a_i \qquad \text{or} \qquad E(a_0, a_1, \cdots).$$

We shall study only the case where *all the a_i are positive or zero*, so that none of the exponentials considered later is negative or complex.

If the limit, as $n \to \infty$, of $E_{i=0}^{n} a_i$ exists and equals say k, we shall say that the infinite exponential $E_{i=0}^{\infty} a_i$ *converges and has the value k*; but if the above limit does not exist, the infinite exponential shall be said to *diverge*.

The infinite exponential $E_{i=m}^{\infty} a_i$ shall be called the mth *residual* of $E_{i=0}^{\infty} a_i$.

If every residual of an infinite exponential converges, then the infinite exponential shall be called *properly convergent*; and if an infinite exponential con-

verges but at least one of its residuals diverges, the infinite exponential shall be called *improperly convergent.*

4. *Some obvious theorems* about infinite exponentials are:

THEOREM 1. *If any one of the residuals converges, the infinite exponential converges.*

THEOREM 2. *If any particular residual diverges, then every later residual diverges.*

If $E_{i=0}^{n}a_i$ grows infinite with n, the corresponding infinite exponential shall be called *properly divergent*; but if $E_{i=0}^{n}a_i$ never exceeds a fixed bound as n grows infinite, and yet fails to approach a limit, then the infinite exponential shall be called *improperly divergent,* or *oscillating.*

THEOREM 3. *If some exponent in an infinite exponential is less than unity, the exponential must converge or oscillate.* For if a_m is less than unity, the mth residual would not be greater than unity, and so could not grow infinite.

THEOREM 4. *If all the exponents in an infinite exponential are greater than unity, the value of $E_{i=0}^{n}a_i$ is monotone increasing with n.*

These two theorems show that the two types of divergence defined above are all inclusive, that is, there is no type of unbounded oscillation.

5. *The nth margerin function of x* shall be defined by $E(a_0, a_1, \cdots, a_n, x)$ and its properties are developed here to be used in proving convergence theorems for infinite exponentials.

6. *A graphical process* for evaluating the margerin function (or any continued exponential) may be described as follows:

Plot the graphs of the curves

$$y = a_0^x, \; y = a_1^x, \cdots, y = a_n^x .$$

(They are indicated in fig. 1 by the curves C_0, C_1, C_2, C_3, C_4, and the other curves are to be disregarded for the moment). Draw also the forty-five degree *turner line.*

Then choose a point P upon the turner whose abscissa is x, from P draw a vertical line to meet C_n in the point Q_n, from Q_n draw a horizontal line to meet the turner in P_n, from P_n draw a vertical line to meet C_{n-1} in Q_{n-1}, from Q_{n-1} draw a horizontal line to meet the turner in P_{n-1}, and continue until we have used each of the curves in turn beginning with C_n and ending with C_0, and the final point P_0, on the turner, will have an abscissa equal to the desired value of the nth margerin function.

If we start with an infinitesimal segment PP' of the turner whose horizontal projection is Δx, and carry its points through the above graphical process, we shall end with another infinitesimal segment P_0P_0', whose horizontal projection we shall call Δx_0; and moreover at each step of the process, the length of the

segment is multiplied by the slope of the C-curve taken at the corresponding Q-point. Then if we let m_i denote the slope of C_i at the point Q_i, we shall have

$$\Delta x_0 = m_0 \cdot m_1 \cdots m_n \cdot \Delta x$$

and hence the slope of the margerin function is given by

$$\frac{d}{dx} E(a_0, a_1, \cdots, a_n, x) = m_0 \cdot m_1 \cdots m_n.$$

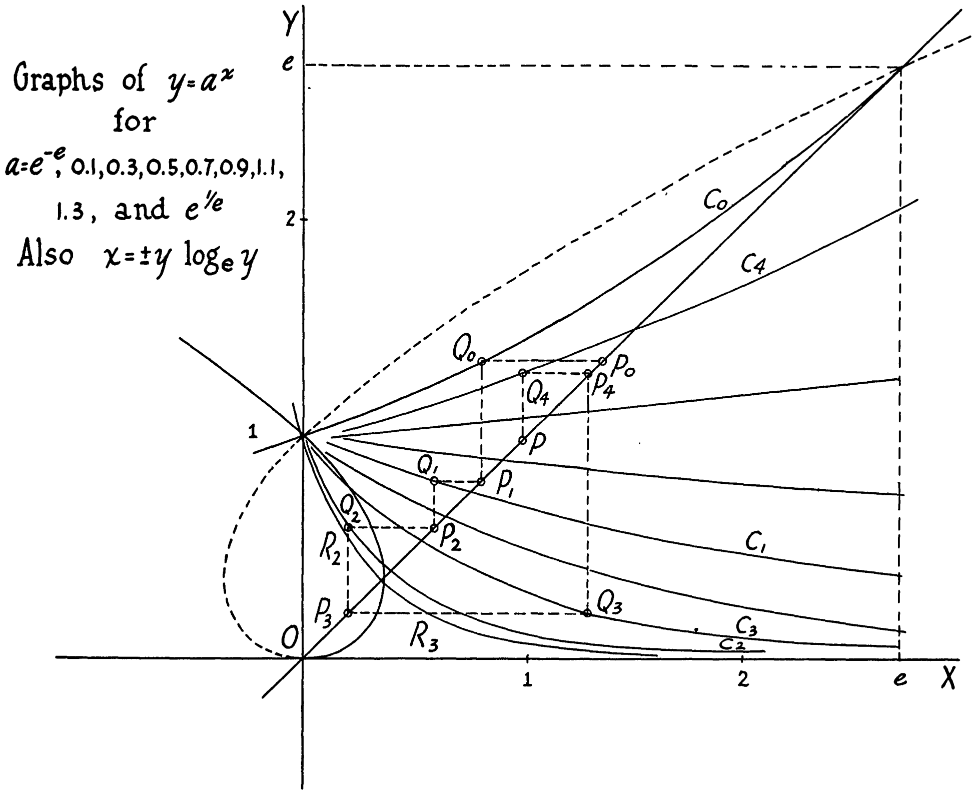


FIG. 1

The foregoing discussion makes it desirable to mark off the region of the plane in which the curves of the family $y = a^x$ have slopes numerically greater than unity, and the region in which they have slopes numerically less than unity. We shall call the former region the *expanding region*, and the latter the *contracting region*. These two regions are separated by a curve called the *neutral locus*, at each point of which some member of the family has a slope of plus one or minus one.

To get the neutral locus we set

$$\frac{dy}{dx} = a^x \log a = \pm 1$$

from which the equation of the neutral locus is found to be

$$x = \pm y \log y.$$

Fig. 1 shows part of the family and the neutral locus, and it is easy to see the expanding and contracting regions.

7. *The problem of the convergence of infinite exponentials all of whose exponents are equal* is now easily solved.* Consider $E(a, a, \cdots)$. If one approach the question of the convergence of this infinite exponential by evaluating graphically the series

$$E(a),\ E(a, a),\ E(a, a, a), \cdots$$

he will use only the graph of $y = a^x$ and the turner (see figs. 2, 3, 4, 5), and will be led to announce the following theorem:

THEOREM 5. (a) *If* $e^{1/e} < a$, *then* $E(a, a, a, \cdots)$ *is properly divergent.* (b) *If* $e^{-e} \leqq a \leqq e^{1/e}$, *then* $E(a, a, \cdots)$ *is convergent and has the value* k *which satisfies* $k = a^k$ *and* $1/e \leqq k \leqq e$. (c) *If* $0 < a < e^{-e}$, *then* $E(a, a, \cdots)$ *is improperly divergent and as the number of exponents increases in* $E(a, a, \cdots, a)$ *this exponential tends to oscillate between two values* k_1 *and* k_2 *never assuming a value between them, where* k_1 *and* k_2 *are determined by satisfying*

$$k_1 = a^{k_2} \textit{ and } k_2 = a^{k_1} \quad \textit{and} \quad 0 < k_1 < \frac{1}{e} < k_2 < 1.$$

Figures 2, 3, and 4 illustrate parts (a) and (b) of the above theorem, and are self explanatory. Fig. 5 illustrates part (c). In this figure, the points $Q_n, Q_{n-2}, \cdots$ approach from the left a limiting position Q, and $Q_{n-1}, Q_{n-3}, \cdots$ approach from the right a limiting position Q'. The points Q and Q' are located as the only points on the graph of $y = a^x$ which are symmetrical with respect to the turner; and two such points will exist only when $a < e^{-e}$. Theorem 5 brings into prominence two numbers of fundamental importance in our problem, namely

$$e^{1/e} = 1.444668 \cdots$$

$$e^{-e} = .06598803 \cdots$$

which we shall call *the upper and lower limits of the interval of convergence.* The exponential curve $y = e^{x/e}$ is tangent to the turner at the point (e, e) and $y = e^{-ex}$ cuts the turner at right angles at the point $(1/e, 1/e)$. The members of the exponential family in between these two cut the turner at slopes numerically less

* Since completing the paper, it has come to the attention of the author that Prof. Edmund Landau has treated the case of equal exponents in his university lectures, and obtained the results given in theorem 5; but there appears to be nothing published upon the subject.

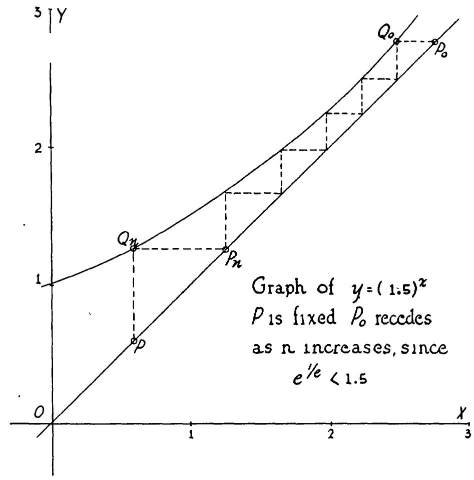


FIG. 2

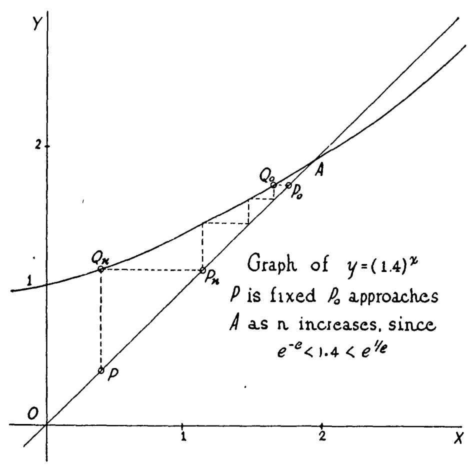


FIG. 3

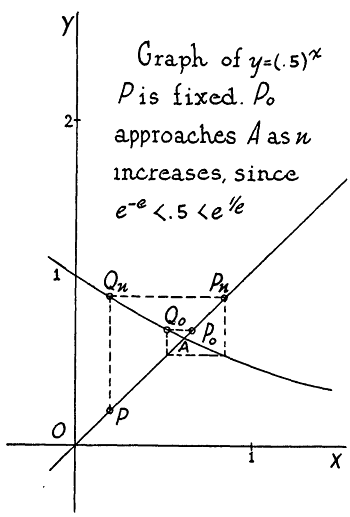


FIG. 4

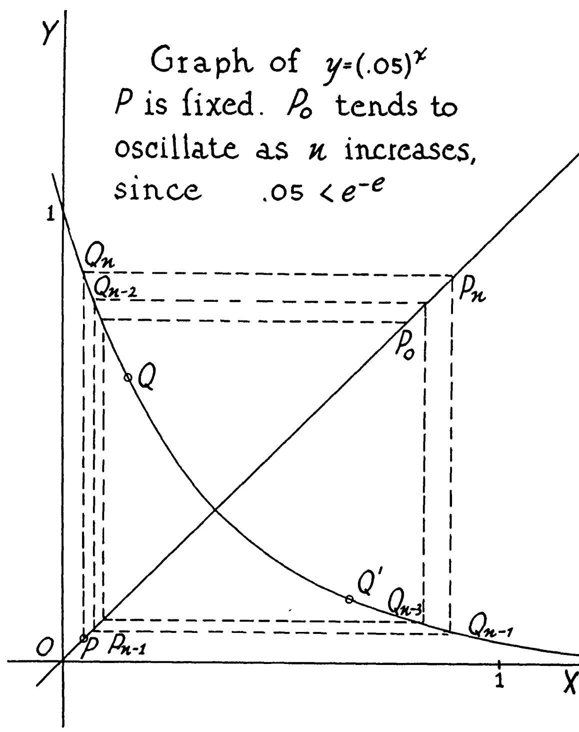


FIG. 5

than unity, and this is the reason why the exponential in theorem 5 (b) converges.

8. *Lemmas concerning the margerin function.*

LEMMA 1. *The margerin function $E(a_0, a_1, \cdots, a_n, x)$ is a monotone increasing or decreasing function of x according as an even or an odd number of the a_i are less than unity.* For

$$\frac{d}{dx} E(a_0, a_1, \cdots, a_n, x)$$
$$= E(a_0, a_1, \cdots, a_n, x)\cdot E(a_1, a_2, \cdots, a_n, x) \cdots E(a_n, x)\cdot \log a_0 \cdot \log a_1 \cdots \log a_n$$

and all factors in this product are positive except the logarithms of those a_i which are less than unity.

LEMMA 2. *If all the a_i lie in the interval of convergence, that is if*

$$e^{-e} \leqq a_i \leqq e^{1/e}, \qquad i = 0, 1, 2, \cdots$$

and if $0 \leqq x \leqq e$, *then*

$$0 \leqq E(a_0, a_1, \cdots, a_n, x) \leqq e.$$

The graphical process, fig. 1, will make this self evident.

LEMMA 3. *If all the a_i are in the interval of convergence, that is if*

$$e^{-e} \leqq a_i \leqq e^{1/e}, \qquad i = 0, 1, 2, \cdots$$

then the difference between the largest and smallest values of $E(a_0, a_1, \cdots, a_n, x)$ in the interval $0 \leqq x \leqq e$ approaches zero as n grows infinite.

Since the margerin function is monotone increasing or decreasing, we can get the difference between its largest and smallest values by integrating its derivative over the interval. Using the form of this derivative given in Art 6, we are trying to prove

$$\lim_{n\to\infty} \int_0^e m_0 \cdot m_1 \cdots m_n \cdot dx = 0.$$

Now referring to fig. 1, and remembering that each m_i is the slope of a member of the family at the point Q_i, we note the following facts. If a point such as Q_1 or Q_4 fall in the contracting region, the corresponding slope m_1, or m_4 is numerically less than unity. But if a point like Q_2 fall in the expanding region, the value of m_2, is numerically greater than unity. However the preceding point Q_3 must fall in the contracting region, as is easily seen, and we shall show that the product $m_2 m_3$ is less than unity. Let R_2 and R_3 be the points where the lines P_3Q_2 and P_3Q_3 meet the curve $y = e^{-ex}$, and we easily see that the slopes of this curve at R_2 and R_3 have greater numerical values than m_2 and m_3 respectively; and furthermore, by a little analytic geometry, we see that the product of the slopes of this curve at R_2 and R_3 is less than unity. This prod-

uct may, however approach unity as P_3 approaches the point where the turner cuts the last mentioned exponential curve.

Thus the integrand consists (with the possible exception of a finite last factor) of factors and pairs of factors each of which is less than unity. We wish to show that, as n grows infinite, the integrand approaches zero at all points of the interval with the possible exception of one point where it cannot exceed unity. The least favorable case occurs when all the a_i are equal to e^{-e} and x is near the value $1/e$, or else when all the a_i are equal to $e^{1/e}$, and x is near the value e. For at these points the extreme curves in fig. 1 are meeting the turner with slopes of -1 and 1 respectively. But even in these cases a small segment will be indefinitely shortened by the graphical process since each of its end points must approach the point where the exponential curve meets the turner. Since the integrand may be made arbitrarily small by increasing n, except perhaps at one point, the value of the integral may be made arbitrarily small, and our lemma is proved.

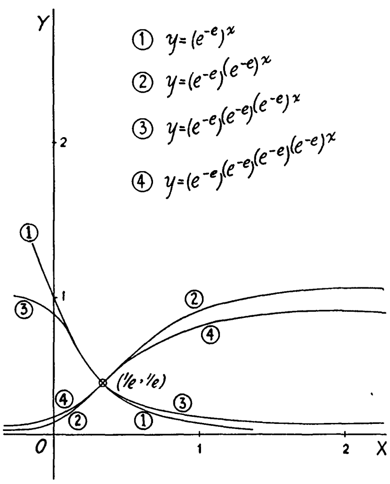


FIG. 6

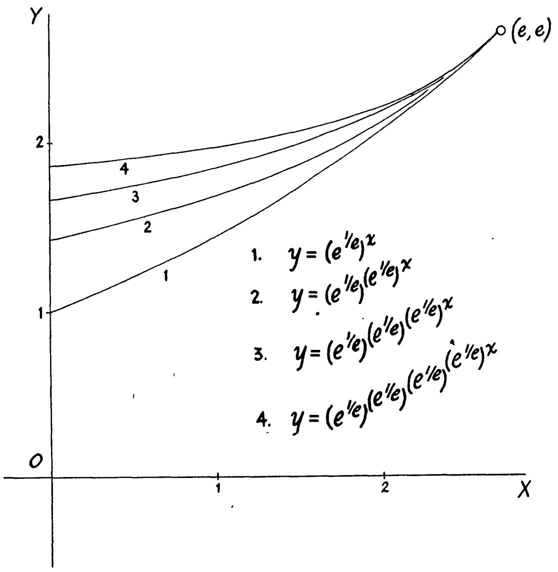


FIG. 7

To clarify this discussion, Fig. 6 shows the graphs of the first few margerin functions when all of the a_i are equal to the lower limit e^{-e}, and Fig. 7 shows them when all the a_i are equal to the upper limit $e^{1/e}$. One can see how the difference between the largest and smallest values of the margerin function approaches zero, although the slope at one point is 1 or -1.

9. *Fundamental theorems on the convergence of infinite exponentials.*

THEOREM 6. *The infinite exponential $E_{i=0}^{\infty} a_i$ converges if m can be found such that*

$$e^{-e} \leqq a_i \leqq e^{1/e}, \qquad i = m, m+1, m+2, \cdots .$$

Stated in words the theorem says that if beyond a certain point all the exponents lie in the interval of convergence, then the infinite exponential converges. By theorem 1 it will be sufficient to prove the convergence of the mth residual $E_{i=m}^{\infty} a_i$. By lemma 3, we can choose n so large that the total variation of $E(a_m, a_{m+1}, \cdots, a_{m+n}, x)$ as x ranges from 0 to e shall be less than ϵ, an arbitrarily small positive number. Then by lemma 2 the value of the finite exponential $E_{i=m}^{m+n} a_i$ will be altered by less than ϵ if we make n larger. This proves the theorem.

Theorem 6 is somewhat analogous to the fact that an infinite product of positive factors converges when the factors are all less than or equal to unity. And the further fact that the infinite product converges even when the factors all exceed unity by small amounts which form a convergent series suggests that we investigate whether the infinite exponential may not converge when all the exponents exceed the upper limit $e^{1/e}$ or are less than the lower limit e^{-e} by small amounts. The results were surprising in that the convergence of the "small amounts" as a series was not sufficient for the convergence of the infinite exponential.

THEOREM 7. *The infinite exponential* $E_{i=0}^{\infty}(e^{1/e}+\epsilon_i)$, *where the* ϵ_i *are all positive or zero*, (*a*) *will converge if*

$$\lim_{n\to\infty} \epsilon_n n^2 < \frac{e^{1/e}}{2e}$$

(*b*) *and will diverge if*

$$\lim_{n\to\infty} \epsilon_n n^2 > \frac{e^{1/e}}{2e}\cdot$$

LEMMA 4. *Any properly convergent infinite exponential can be put in the form* $E(b_0^{1/b_1}, b_1^{1/b_2}, \cdots)$, *which we shall call the b-form.* For we need only choose each b_i equal to the ith residual.

LEMMA 5. *Any exponential in the b-form converges if each b_i is greater than unity.* To prove this consider the finite exponential $E(b_0^{1/b_1}, b_1^{1/b_2}, \cdots, b_n^{1/b_{n+1}})$. It steadily increases with n since each exponent exceeds unity. Furthermore it never exceeds the value b_0, for if we increase the last exponent by replacing b_{n+1} by 1, the whole symbol evidently telescopes down to the value b_0.

Therefore to prove theorem 7, it is necessary and sufficient that it can be put in the b-form with each b_i greater than unity, that is, that

$$e^{1/e} + \epsilon_n = b_n^{1/b_{n+1}}.$$

Now since the maximum value of $x^{1/x}$ is $e^{1/e}$, in order that ϵ_n be positive for all values of n it is necessary that $b_n > b_{n+1} > e$. We accordingly write:

$$e^{1/e} + \epsilon_n = (e + \delta_n)^{1/(e+\delta_{n+1})}.$$

Expanding by Taylor's series we have

$$\epsilon_n = e^{1/e}\left\{\frac{1}{e^2}(\delta_n - \delta_{n+1}) - \frac{1}{2e^3}\left[2\delta_{n+1}(\delta_n - \delta_{n+1}) + \delta_n^2\right] + \frac{1}{2e^4}(\delta_n - \delta_{n+1})^2 + \cdots\right\}.$$

The principal part of ϵ_n is contained in

$$e^{1/e}\left[\frac{1}{e^2}(\delta_n - \delta_{n+1}) - \frac{\delta_n^2}{2e^3}\right].$$

To make the principal part of ϵ_n *positive* and *of as low order as possible* we must have the principal part of δ_n equal to k/n where k is some constant. So we give δ_n this value, and find that the principal part of ϵ_n has a maximum of $e^{1/e}/(2en^2)$ when k equals e.

Now to prove part (a) of theorem 7 we say that an exponential in the b-form can be found as above, and its exponents will be larger than $e^{1/e}+\epsilon_n$ so the given exponential converges by comparison. But in part (b) of theorem 7 no exponential in the b-form can be found whose exponents are as large as $e^{1/e}+\epsilon_n$, so this one cannot be convergent.

THEOREM 8. *That the infinite exponential* $E_{i=0}^{\infty}(e^{-e}-\epsilon_i)$ *shall converge properly, where* ϵ_i *are all positive or zero, and* $\epsilon_i \geqq \epsilon_{i+1}$, *it is necessary that* $\lim_{i\to\infty}\epsilon_i = 0$ *and it is sufficient that* $\lim_{i\to\infty} i^q \epsilon_i = 0$ *where* $q > 1$.

A proof of this theorem may be carried through by a careful analysis of the graphical process, and is omitted because it is lengthy and tedious.

10. *When the exponents of an infinite exponential are functions of* x, the exponential will define a function of x for all values of x which render it convergent. We shall study one such case, namely

$$f(x) = E(e^{k_0 x}, e^{k_1 x}, \cdots).$$

When x is zero this function is equal to unity. Assuming that there are other values for which the exponential converges, it can be developed formally into the following series:

$$\begin{aligned} E(e^{k_0 x}, e^{k_1 x}, \cdots) = 1 &+ k_0 x + (k_0^2 + 2k_0k_1)\frac{x^2}{2!} \\ &+ (k_0^3 + 6k_0^2k_1 + 3k_0k_1^2 + 6k_0k_1k_2)\frac{x^3}{3!} \\ &+ (k_0^4 + 12k_0^3k_1 + 24k_0^2k_1^2 + 4k_0k_1^3 + 24k_0^2k_1k_2 \\ &+ 24k_0k_1^2k_2 + 12k_0k_1k_2^2 + 24k_0k_1k_2k_3)\frac{x^4}{4!} \end{aligned}$$

$$+ (k_0^5 + 20k_0^4 k_1 + 90k_0^3 k_1^2 + 80k_0^2 k_1^3 + 5k_0 k_1^4 + 60k_0^3 k_1 k_2$$
$$+ 240k_0^2 k_1^2 k_2 + 60k_0^2 k_1 k_2^2 + 60k_0 k_1^3 k_2 + 120k_0 k_1^2 k_2^2$$
$$+ 20k_0 k_1 k_2^3 + 120k_0^2 k_1 k_2 k_3 + 120k_0 k_1^2 k_2 k_3 + 120k_0 k_1 k_2^2 k_3$$
$$+ 60k_0 k_1 k_2 k_3^2 + 120k_0 k_1 k_2 k_3 k_4)\frac{x^5}{5!} + \cdots .$$

The general expression for a term in any one of these parentheses is

$$(\alpha_0 + \alpha_1 + \cdots + \alpha_p)!\frac{\alpha_0^{\alpha_1} \cdot \alpha_1^{\alpha_2} \cdots \alpha_{p-1}^{\alpha_p}}{\alpha_0!\alpha_1!\alpha_2! \cdots \alpha_p!} k_0^{\alpha_0} k_1^{\alpha_1} k_2^{\alpha_2} \cdots k_p^{\alpha_p} .$$

If all the k_i are equal to unity we obtain

$$E(e^x, e^x, \cdots) = 1 + x + \frac{3}{2!}x^2 + \frac{16}{3!}x^3 + \cdots + \frac{(n+1)^{n-1}}{n!}x^n + \cdots$$

and the ratio test shows that this series converges when $-1/e<x<1/e$. But theorem 6 shows that the exponential converges when $-e \leqq x \leqq 1/e$.

This function satisfies the transcendental equation

$$\log y = xy$$

and may be regarded as a means of solving this equation explicitly for y in terms of x. Since the ratio of each term to the preceding is always less than ex, it follows that the remainder of the series beyond the nth term is less that the geometric series

$$\frac{(n+1)^{n-1}}{n!} x^n(1 + ex + e^2x^2 + \cdots) = \frac{(n+1)^{n-1}}{n!} \frac{x^n}{1 - ex}.$$

11. *An arbitrary function of x whose value is unity when x is zero can be developed into an infinite exponential of the type considered in article 10* by equating the terms in its series development to the terms in the development of the exponential and determining the values of the k_i one after another. For instance:

$$\cos x = E(e^{-x^2/2}, e^{x^2/6}, e^{11x^2/60}, \cdots)$$
$$\sin x = xE(e^{-x^2/6}, e^{x^2/30}, e^{59x^2/1260}, \cdots).$$

The author was keenly disappointed in not finding either simple values for the k_i or a general expression for the nth exponent, especially since these functions have such elegant expansions as infinite series and infinite products.

12. *The solution for y in terms of x* of the transcendental equation*

* H. L. Slobin, this MONTHLY, Oct. 1931, p. 444, solved this equation in terms of a parameter. He was interested in finding positive, rational, unequal values of x and y that satisfy the equation, one obvious pair being 2 and 4.

$$y^x = x^y$$

may be carried out by first writing it in the form

$$y = (x^{1/x})^y$$

and then substituting the whole right side of this equation for the y that stands in the exponent, getting

$$y = (x^{1/x})^{(x^{1/x})^y}.$$

Continued substitution gives the infinite exponential

$$y = E(x^{1/x}, x^{1/x}, \cdots)$$

which converges for all values of x greater than $1/e$.

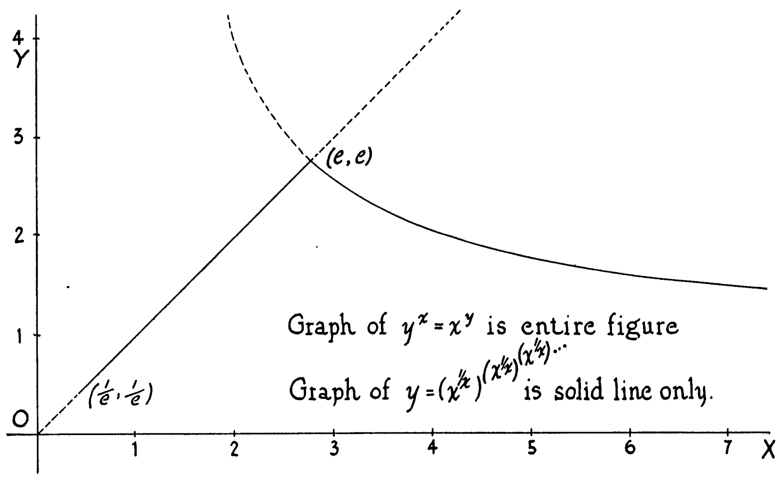


FIG. 8

The graph of this infinite exponential has a ninety degree corner at the point (e, e). The solid line in Fig. 8 is the graph of the infinite exponential while the whole curve is the graph of the original equation for positive values of x and y.

ISOGONAL AND ISOTOMIC CONJUGATES AND THEIR PROJECTIVE GENERALIZATION

By P. H. DAUS, University of California at Los Angeles

1. *Introduction*. Let P be a point in the plane of the triangle $A_1A_2A_3$, and designate the sides of the triangle by a_i and the lines A_iP by p_i. If p_1' is determined so that angle $a_3p_1 =$ angle $p_1'a_2$, sense taken into account, then p_1 and p_1' are called *isogonal conjugate lines*. The following fundamental incidence properties follow immediately from the trigonometric form of the theorems of Ceva and Menelaus.

If three lines one through each vertex are concurrent, their isogonal conjugates are concurrent.

The points of concurrency are called *isogonal conjugate points.*

*If three lines one through each vertex cut the opposite sides in three collinear points, their isogonal conjugates cut the opposite sides in three collinear points.**

It is the purpose of this note to examine the subject from a projective point of view, considering a projective generalization of isogonal conjugates, which in turn throws some light upon the original transformation. We likewise consider the similar topic of isotomic conjugates in §§4 and 5.

2. *Isogonal Conjugates.* We first recall additional properties of isogonal conjugates. In general, to each point P corresponds a unique point P', and the transformation is involutory. The triangle of reference is exceptional in that any point on a side corresponds to the opposite vertex, so that the point that corresponds to a vertex is indeterminate. However, as a moving point approaches a vertex along a given direction, its isogonal conjugate approaches a limiting position on the opposite side, which thus enables us to restore the 1-1 correspondence. The four points of intersection of the angle bisectors are the only self-conjugate points.

From a simple consideration of angles it is readily shown that points on the circumcircle correspond to points on the line at infinity and conversely. If the trilinear coordinates of P are (x_1, x_2, x_3), where x_i is the directed perpendicular distance from P to a_i, and if P' has coordinates (x_1', x_2', x_3'), then an elementary application of similar triangles shows that

$$x_1x_1' = x_2x_2' = x_3x_3' = \text{const.}$$

It follows that our transformation is a quadratic Cremona transformation;† the line $u_1x_1+u_2x_2+u_3x_3=0$ is transformed into the conic $u_1/x_1+u_2/x_2+u_3/x_3=0$, which circumscribes the triangle of reference. Since it is well known that in this particular system of trilinear coordinates the equation of the line at infinity is $a_1x_1+a_2x_2+a_3x_3=0$, a_i being the lengths of the sides, and that the circumcircle has the equation $a_1/x_1+a_2/x_2+a_3/x_3=0$, we have verified the relation between them in this point-to-point transformation. The transform of the general line will be an ellipse, parabola, or hyperbola according as the line cuts the circumcircle in no, one or two points.

---

* For further details of these and the properties of §2, the reader may consult R. A. Johnson, *Modern Geometry*, 1929, pp. 153–159; F. Morley and F. V. Morley, *Inversive Geometry*, 1933, pp. 196–197, 249–255; and J. Clemow, *A note on isogonal conjugates*, Mathematical Gazette, vol. 18 (1934), pp. 289–293. Corresponding properties for isotomic conjugates will be found in Johnson, loc. cit., and an analytical treatment in D.N.Y. Sommerville, *Analytical Conics*, especially p. 152.

† For further information concerning quadratic Cremona transformations the reader may consult H. P. Hudson, *Cremona Transformations*, Chap. 3; or Karl Doehlemann, *Geometrische Transformationen*, Chap. 10–12.

3. *Projective Isogonal Conjugates.* The projective definition of angle suggests the following generalization. Let I and J be any two points in the plane of the triangle $A_1A_2A_3$. We define the *projective isogonal* correspondence between the lines p_1 and p_1' by the following relations between cross ratios:

$$(1) \qquad (p_1a_3,\ i_1j_1) = (a_2p_1',\ i_1j_1) = (p_1'a_2,\ j_1i_1),$$

where i_1 and j_1 indicate the lines A_1I and A_1J, with similar definitions for the other vertices. Equations (1) define an involution at A_1 with (i_1, j_1) and (a_2, a_3) as pairs of lines, having, say, e_1 and g_1 as fixed elements.

Let p_1 and p_2 meet at P, and suppose that P moves along a line u. Then $u(P) \,\overline{\wedge}\, A_1(p_1) \,\overline{\wedge}\, A_1(p_1')$, and similarly $u(p) \,\overline{\wedge}\, A_2(p_2) \,\overline{\wedge}\, A_2(p_2')$, so that P', the intersection of p_1' and p_2', describes a conic u'. If P moves on a line through A_3, we see by placing P at A_3 and then on a_3 that the pencils $A_1(p_1')$ and $A_2(p_2')$ are perspective, and P' describes a line through A_3, giving our first incidence relation.

If three lines one through each vertex are concurrent, their projective isogonal conjugates are concurrent.

The conic u' can be generated by projective pencils at any two vertices. The transformation we have set up is a quadratic involution. The four points of intersection of e_i and g_i other than A_i are the fixed points. The points of the triangle of reference are exceptional, the point A_i and the line a_i being paired by the transformation. To every other line corresponds a conic through the vertices A_i. The points I and J are interchanged by the transformation, so that the line IJ corresponds to the conic c determined by the five points A_i, I, J.

If the line u cuts a_1 at P, so that the line $A_1A_2 = p_2'$, then p_1' is tangent to the conic u' at A_1. Similarly, if u cuts a_2 and a_3 at Q and R, the conic u' is tangent to q_2' and r_3'. Since by Pascal's theorem the tangents meet the opposite sides in three collinear points, we have the second incidence relation.

If three lines one through each vertex cut the opposite sides in three collinear points, their projective isogonal conjugates cut the opposite sides in three collinear points.

The above properties may be verified by considering the equations of the transformation. If the given points are $I(i_1, i_2, i_3)$ and $J(j_1, j_2, j_3)$, then the transformation is simply

$$(2) \qquad \rho x_1' = i_1j_1x_2x_3, \qquad \rho x_2' = i_2j_2x_3x_1, \qquad \rho x_3' = i_3j_3x_1x_2.$$

If we specialize the points I and J, we obtain special transformations. We consider the two most important cases. (a) *Isogonal conjugates.* If I and J become the circular points at infinity, the conic c becomes the circumcircle, IJ the line at infinity, equation (1) implies the equality of angles, and the transformation is that of isogonal conjugates. Not only do we see that the line at infinity corresponds to the circumcircle, a conic to a quartic with a double

point at A_i, but, since I and J are interchanged, that a circle is transformed into a circular quartic, and in particular, a circle through two vertices corresponds to a second circle through these vertices. (b) *Inversion.* If A_1 and A_2 become the circular points at infinity, a line corresponds to a circle through A_3; and a circle not through A_3 corresponds to another such circle, and in addition, to the lines a_1 and a_2, whose only real point is A_3. The transformation is an inversion with respect to A_3 and radius equal to $(A_3I\cdot A_3J)^{1/2}$, followed by a reflection on the bisector of the angle IA_3J. The points I and J are interchanged and the only real fixed points are the intersection of the circle of inversion and the axis of reflection. These facts may be verified analytically by considering a homogeneous rectangular system of coordinates with $A_1(1,\ i,\ 0)$, $A_2(1,\ -i,\ 0)$, $A_3(1;\ 0,\ 0)$, $i=\sqrt{-1}$, and the coordinates of I and J so selected that the bisector is the x axis. It is readily apparent from symmetry that if p_3 has the equation $y=mx$, then p_3' has the equation $y=-mx$; an application of (1) to the pencils at A_1 and their projection on the y axis yields the distance property of inversion.

4. *Isotomic Conjugates.* Let P be any point in the plane of the triangle $A_1A_2A_3$, and let $p_1\equiv A_1P$ cut a_1 in P_1. If P_1' is determined so that $A_3P_1=P_1'A_2$, then p_1 and p_1' are called *isotomic conjugate lines.* Incidence theorems analogous to those of §1, with *isogonal* replaced by *isotomic*, follow immediately from the theorems of Menelaus and Ceva.

Let A_iP_i meet at P and define the areal coordinates of $P(K_1,\ K_2,\ K_3)$ as the areas of the triangles PA_2A_3, PA_3A_1, PA_1A_2, sense being taken into account. Now $K_i=(PP_i/A_iP_i)K=r_iK$, where K is the area of $A_1A_2A_3$, so that $K_1{:}K_2{:}K_3=r_1{:}r_2{:}r_3$. Let P_1, P_2, P_3, divide the sides A_2A_3, A_3A_1, A_1A_2, respectively, in the ratios k, l, m, so that $klm=1$. By successive applications of the theorem of Menelaus, we can express r_i in terms of k, l, m, and find

$$K_1{:}K_2{:}K_3 = l{:}ml{:}kml.$$

The isotomic conjugates of A_iP_i will meet at P' called the *isotomic conjugate* of P. We designate the symbols for the isotomic conjugate by primes. Then, since $k'=1/k$, $l'=1/l$, $m'=1/m$, it follows that

$$K_1K_1' = K_2K_2' = K_3K_3' = \text{const.} \tag{3}$$

In terms of trilinear coordinates this becomes

$$a_1^2\,x_1x_1' = a_2^2\,x_2x_2' = a_3^2\,x_3x_3'\,, \tag{4}$$

where a_i represent the lengths of the sides.

It is now seen that the transformation by isotomic conjugates is a quadratic involution closely allied to that for isogonal conjugates. The median point G and the exmedian points E_i whose areal coordinates are $(\pm 1,\ \pm 1,\ \pm 1)$ are invariant. If G_1 is the median point of the triangle $E_1A_2A_3$ (it has coordinates $(-1,\ 2,\ 2)$), with corresponding definitions for G_2 and G_3, then it may be veri-

fied that the line at infinity $K_1+K_2+K_3=0$ is transformed into the conic $K_2K_3+K_3K_1+K_1K_2=0$ through the six points A_i, G_i.

5. *Projective Isotomic Conjugates.* Let G be any point in the plane of the triangle $A_1A_2A_3$, and let g be its trilinear polar. That is, if A_iG meet a_i in B_i, and if C_i is the harmonic conjugate of B_i with respect to the vertices on that side, then the three points C_i lie on g. Not only do the lines A_iB_i meet at G, but also A_1B_1, A_2C_2, A_3C_3 meet at E_1, the lines A_1C_1, A_2B_2, A_3C_3 meet at E_2, and A_1C_1, A_2C_2, A_3B_3 at E_3. Further, let G_i be the trilinear pole of g for the triangle $E_iA_jA_k$.

Let P be any point in the plane and let A_iP cut a_i in P_i. Let P_i' be the harmonic conjugate of P_i with respect to B_i and C_i. As P moves along a line u we can readily see as in §3 that P' appears as the intersection of corresponding lines of projective pencils at A_1 and A_2. We may call this a transformation by *projective isotomic* conjugates. The vertices and sides are exceptional and the points G, E_i are the four invariant points. To a line in the plane corresponds a conic through the vertices and to the special line g corresponds a conic g' through the six points A_i, G_i, and inscribed in the triangle $E_1E_2E_3$. That g' passes through G_i can be obtained from the maze of sets of harmonic points in the diagram showing that $(GE_i, G_iD_i)=-1$, where D_i is the intersection, of A_iG and g, for G and E_i are fixed points in the involution. That g' is tangent to E_2E_3 at A_1 follows from the fact that C_1 is on the invariant line E_2E_3.

It is to be noted that the transformation by projective isotomic and isogonal conjugates are the same, but in this section our attention has been fixed upon the fixed elements rather than on a pair of corresponding points. Indeed, if we use a general system of projective coordinates with $G(g_1, g_2, g_3)$, the equations of the transformation can be written

$$(5) \qquad \rho x_1' = g_1^2 x_2x_3, \qquad \rho x_2' = g_2^2 x_3x_1, \qquad \rho x_3' = g_3^2 x_1x_2.$$

This will be identical with (2) if

$$i_1j_1 : i_2j_2 : i_3j_3 = g_1^2 : g_2^2 : g_3^2,$$

which shows that I and J may be chosen as any pair of corresponding points of (5).

MAXIMA AND MINIMA OF FINITE SUMS

By TOMLINSON FORT, Lehigh University

The calculus of variations is built about the idea of maxima and minima of definite integrals. In 1915 R. B. Robbins* made an approach to the calculus of variations by first deriving certain conditions for maxima and minima of finite sums and then passing to the limit. He was thus able to arrive at a number of

* American Journal of Mathematics, Vol. XXXVII, p. 367.

classical results. In the present paper a study is made of finite sums but no effort is made to pass to the calculus of variations. The problem of the finite sum is considered to be of interest on its own account. A sufficient condition is obtained which seems simpler than the corresponding result of Robbins. A simple sum is then treated in detail and finally a necessary form is obtained for the minimum surface formed by revolving a broken straight line about an axis. The whole paper is quite elementary; but it seems likely that an extensive theory of maxima and minima of finite sums is possible and that applications could be made to numerous useful problems.

1. *Maxima and Minima of Functions of More than One Variable.* I call attention to some theorems relative to maxima and minima in order that the reader may have them freshly in mind. Given a function $f(x_1, \cdots, x_n)$ all of whose third derivatives exist in the neighborhood in question:

I. A necessary condition for a maximum or minimum is that

$$\frac{\partial f}{\partial x_i} = 0; \; i = 1, \cdots, n. \tag{1}$$

II. If (1) is satisfied at a point $(x_1^{(1)}, \cdots, x_n^{(1)})$ then a sufficient condition for a minimum at that point is that the form

$$\left[\left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}\right)^2 f(x_1, \cdots, x_n)\right]_{(x_1^{(1)}, \cdots, x_n^{(1)})} \tag{2}$$

be positive definite. A sufficient condition for a maximum is that it be negative definite.

2. *The Finite Sum.* Consider the sum

$$\phi = \sum_{i=0}^{n-1} F(x_i, y, \Delta y), \tag{1}$$

where y is to be determined as a function of x_i so as to make ϕ a maximum or minimum, where $x_{i+1} > x_i$ and where $\Delta y = y(x_{i+1}) - y(x_i)$. We shall assume at least the existence of all third derivatives of F in all neighborhoods considered.

Evidently x_i is a function of i. An analytic form for this function can be written down in a variety of ways. A satisfactory way is by means of the Lagrange interpolation formula. Such a transformation on x_i reduces (1) to the form

$$\phi = \sum_{i=0}^{n-1} f(i, y, p), \tag{2}$$

where now y is to be determined as a function of i and $p = \Delta y = y(i+1) - y(i)$. The function $y(i)$ can in turn be written as a function of x_i if desired. The Lagrange interpolation formula is again an adequate instrument. Partly on

account of this transformation we shall consider only the sum (2), although the major reason is for simplicity in writing. The reader will have no difficulty in carrying through the reasoning for (1) if he so desires.

For convenience we shall usually replace $f(i, y, p)$ simply by $f(i)$.

Let us apply to (2) the first necessary condition for a minimum. We have

$$\frac{\partial \phi}{\partial y(i)} = f_y(i) - f_p(i) + f_p(i-1) = 0 \tag{3}$$

which can be written

$$f_y(i) - \Delta f_p(i-1) = 0.$$

This holds when $0 < i \leqq n-1$. We can rewrite (3)

$$f_y(i+1) - \Delta f_p(i) = 0, \tag{4}$$

where now $0 \leqq i \leqq n-2$. This in turn can be written

$$\Delta[f_y(i) - f_p(i)] + f_y(i) = 0. \tag{4'}$$

THEOREM. *The satisfaction of equation* (4) *is a necessary condition that* ϕ *be a maximum or minimum.*

In case $y(0)$ and/or $y(n)$ are variable as well as the y's at intermediate points we have in addition the relations

$$\begin{aligned} f_y(0) - f_p(0) &= 0 \text{ and/or} \\ f_p(n-1) &= 0. \end{aligned} \tag{5}$$

These are a kind of boundary condition which result from the fact that $f(-1)$ does not occur in the sum ϕ and consequently that the last term of (3) is lacking when $i=0$ and/or that $f(n)$ does not occur in ϕ.

The similarity of (4) to Euler's equation in the calculus of variations will be immediate to those who have studied that subject and it is interesting to derive it by the method of variations.

A function which satisfies (4) if $y(0)$ and $y(n)$ are fixed or (4) and (5) in the contrary case will be called a critical function.

The sufficiency condition, namely that form (2) be definite as is quoted under article 2, results as follows* when applied to ϕ. Let $y(i)$ be a critical function and let us assume that $y(0)$ and $y(n)$ are constants. Let

$$a_{ij} = \frac{\partial^2 \phi}{\partial y(i) \partial y(j)}.$$

Then form (2) can be written

* See Kowalewski, *Determinantentheorie*, ed. 1909, p. 239 where the general quadratic form is discussed.

$$
\begin{aligned}
P(h_1, \cdots, h_n) = {} & a_{11}h_1^2 + a_{12}h_1h_2 + 0 + 0 + 0 + \cdots + 0 + 0 + 0 \\
& + a_{21}h_1h_2 + a_{22}h_2^2 + a_{23}h_2h_3 + 0 + 0 + \cdots + 0 + 0 + 0 \\
& + 0 + a_{32}h_2h_3 + a_{33}h_3^2 + a_{34}h_3h_4 + 0 + \cdots + 0 + 0 + 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + 0 + 0 + 0 + 0 + 0 + \cdots + 0 + a_{nn-1}h_{n-1}h_n + a_{nn}h_n^2,
\end{aligned} \tag{6}
$$

with $a_{ij} = a_{ji}$.

Let

$$D_0 = 1, D_1 = a_{11}, D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \cdots,$$

$$D_j = \begin{vmatrix} a_{11} & a_{12} & 0 & \cdots 0 & 0 \\ a_{12} & a_{22} & a_{23} & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots a_{j,j-1} & a_{jj} \end{vmatrix}.$$

THEOREM. *A sufficient condition that the bilinear form* (6) *be positive definite is that the sequence $D_0 = 1$, D_1, D_2, $\cdots$, D_n have all of its terms positive. A sufficient condition that the form be negative definite is that the terms of the sequence alternate in sign.*

Proof: We note first that

$$D_j = a_{jj}D_{j-1} - a_{j,j-1}^2 D_{j-2}. \tag{7}$$

We next write

$$P(h_1, \cdots, h_n) = a_{11}h_1^2 + 2a_{12}h_1h_2 + a_{22}h_2^2 + \cdots + a_{nn}h_n^2.$$

Assuming that no D is zero, completing the square preceding from the beginning and using relation (7), we arrive at the following identity:

$$
\begin{aligned}
P(h_1, \cdots, h_n) = {} & \frac{1}{D_1}(a_{11}^2 h_1^2 + 2a_{11}a_{12}h_1h_2 + a_{12}^2 h_2^2) \\
& + \frac{D_1}{D_2}\left(\frac{D_2^2}{D_1^2} h_2^2 + 2\frac{D_2}{D_1} a_{23}h_2h_3 + a_{23}^2 h_3^2\right) + \cdots \\
& + \frac{D_{n-2}}{D_{n-1}}\left(\frac{D_{n-1}^2}{D_{n-2}^2} h_{n-1}^2 + 2\frac{D_{n-1}}{D_{n-2}} a_{n-1,n}h_{n-1}h_n + a_{n-1,n}^2 h_n^2\right) \\
& + \frac{D_n}{D_{n-1}} h_n^2.
\end{aligned}
$$

The conclusion is drawn immediately from this relation.

Our condition states that if a solution of the recurrent relation (7) *subject to the initial conditions*

$$D(0) = 1, D(1) = a_{11} = \phi_{y(1)y(1)}$$

is positive throughout the interval over which $y(i)$ *is variable then* $y(i)$ *renders* (2) *a minimum. If, on the other hand,* D_j *alternates in sign throughout this interval* $y(i)$ *renders* (2) *a maximum.*

By differentiation of (2) we have

$$a_{ii} = \phi_{y(i)y(i)} = f_{pp}(i) + f_{pp}(i-1) - 2f_{yp}(i) + f_{yy}(i) \tag{8}$$

$$a_{i\,i-1} = \phi_{y(i)y(i-1)} = f_{yp}(i-1) - f_{pp}(i-1). \tag{9}$$

3. *A Simple Example.* Consider the sum

$$\sum_{i=0}^{n-1} (4y^2 + 3p^2) \tag{10}$$

Here $f(i) = 4y^2 + 3p^2$, $f_y = 8y$, $f_p = 6p$, $f_{yy} = 8$, $f_{pp} = 6$, $f_{yp} = 0$. Equation (4′) takes the form

$$\Delta(8y - 6p) + 8y = 0$$

which can be written

$$3y(i+2) - 10y(i+1) + 3y(i) = 0. \tag{11}$$

To solve this recurrent relation we form the auxiliary equation,

$$3m^2 - 10m + 3 = 0,$$

which yields $m = 3, 1/3$. The general solution of (11) is then

$$y = c_1 3^i + c_2 \frac{1}{3^i}. \tag{12}$$

Let us impose the boundary conditions

$$\begin{aligned} i &= 0 \qquad & y &= A \\ i &= n \qquad & y &= B. \end{aligned}$$

Under these conditions equations (5) do not enter. They are necessary only under the conditions that $y(0)$ and/or $y(n)$ are variable. Solving for c_1 and c_2 we find

$$c_1 = \frac{A - B3^n}{1 - 3^{2n}}, \qquad c_2 = \frac{3^n B - A3^{2n}}{1 - 3^{2n}}. \tag{13}$$

By (8) and (9) we find

$$\phi_{y(i)y(i)} = 20, \quad \phi_{y(i)y(i-1)} = -6.$$

Equation (7) takes the form

$$D(j+2) - 20D(j+1) + 36D(j) = 0,$$

from which

$$D(j) = C_1 18^i + C_2 2^i.$$

Moreover

$$D(0) = 1, \quad D(1) = \phi_{y(1)y(1)} = 20.$$

By means of these initial values we determine C_1 and C_2 obtaining

$$D(j) = \frac{1}{16}(18^{j+1} - 2^{j+1}).$$

This is positive when j is positive and consequently ϕ is a minimum when y is given by (12) and (13).

3. *Minimum Surface of Revolution.* Connect the points $(0, A)$ and (n, B) in the (i, y)-plane with a broken straight line, each segment of the broken line extending from a point $(i, y(i))$ to a point $(i+1, y(i+1))$. The broken line is then revolved about the i-axis. We wish to determine the function $y(i)$ which renders the surface generated a minimum. Let A and B be positive.

We are to minimize

$$\sum_{i=0}^{n-1} (y(i+1) + y(i))(1 + p^2)^{1/2}. \tag{14}$$

Here $f = (p + 2y)(1 + p^2)^{1/2}$. Equation (4′) becomes

$$\Delta \frac{1 - 2py}{(1 + p^2)^{1/2}} = -2(1 + p^2)^{1/2} \tag{15}$$

which we can write

$$\frac{1 - 2p(i+1)y(i+1)}{(1 + (p(i+1))^2)^{1/2}} = -\frac{1 + 2py(i+1)}{(1 + p^2)^{1/2}}. \tag{15′}$$

Call the right hand member of this equation $-2c$ and solve for $p(i+1)$. We get

$$p(i+1) = \frac{y(i+1) \pm c(1 + 4((y(i+1))^2 - c^2))^{1/2}}{2((y(i+1))^2 - c^2)}$$

provided $\{y(i+1)\}^2 - c^2 \neq 0$. Replace $p(i+1)$ by $y(i+2) - y(i+1)$ and we get

$$y(i+2) = \frac{N \pm S}{D}, \qquad \text{where}$$

$$N = 2(y(i+1))^3 + 2y(i+1)(y(i))^2 + y(i+1)$$
$$S = y(i+1) + 2(y(i+1))^3 - 2y(i+1)(y(i))^2 + y(i)$$
$$D = 4y(i)y(i+1) - 1.$$

We readily show that the vanishing of D is necessary and sufficient for the vanishing of $\{y(i+1)\}^2 - c^2$ which for the moment we have assumed not to vanish.

Substituting for N, S and D

$$y(i+2) = \frac{2y(i+1) + 4(y(i+1))^3 + y(i)}{4y(i)y(i+1) - 1} \quad \text{or} \tag{16}$$

$$y(i+2) = y(i). \tag{17}$$

By a rather tedious substitution, we find that $y(i+2)$ as given by (16) identically satisfies (15).

Substituting (17) in (15) yields $1+2p(i)\ y(i+1)=0$ which is in general not true. However, if we assume it to be true, (16) reduces to (17). In other words, (17) never gives $y(i+2)$ unless it is also given by (16). If $4y(i)\ y(i+1)-1=0$ substitution in (15) yields $1+2p(i)\ y(i+1)=0$. This is inconsistent with $4y(i)\ y(i+1)-1=0$ for real values of $y(i)$ and $y(i+1)$. We conclude that the satisfaction of (16) is necessary as well as sufficient for the satisfaction of (15) by real values. If $4y(i)\ y(i+1)-1$ would be negative y would necessarily change sign since in this case from (16) if $y(i)$ and $y(i+1)$ are positive or zero $y(i+2)$ is negative. But a function which generates the surface of revolution which is an absolute minimum can not be both positive and negative, as can be proved by elementary geometry.

We consequently conclude that *for the minimum surface of revolution in question it is necessary that y satisfy* (16) *and that* $4y(i)\ y(i+1)-1>0$.

For equation (16) we have the following existence theorem:

Given any two initial values $y(0)$ and $y(1)$, it is possible to successively determine $y(2)\ y(3), \cdots, y(n)$ provided that at no point $4y(i)\ y(i+1)-1=0$, $i=0, 1, \cdots, n-2$. If at some such point $4y(i)\ y(i+1)-1=0$ there is no solution with the given initial values.

Since $4y(i)\ y(i+1)-1=0$ is a rational algebraic equation in $y(0)$ and $y(1)$, the initial values for which there is no solution are correspondingly restricted.

We have remarked that all solutions of (16) are solutions of (15) and that (15) has no other solutions. We consequently have an existence theorem for (15).

We next show by actual substitution that

$$y = \frac{1}{2\sinh\dfrac{1}{2a}}\cosh\frac{i-k}{a} \text{ is a solution of (15).} \tag{18}$$

$$p = \frac{1}{2\sinh\dfrac{1}{2a}}\left(\cosh\frac{i+1-k}{a} - \cosh\frac{i-k}{a}\right)$$

$$= \sinh\left(\frac{i-k}{a} + \frac{1}{2a}\right).$$

$$(1+p^2)^{1/2} = \cosh\left(\frac{i-k}{a} + \frac{1}{2a}\right)$$

$$\frac{1-2py}{(1+p^2)^{1/2}} = -\frac{1}{\sinh\dfrac{1}{2a}}\sinh\frac{i-k}{a}$$

$$\Delta\frac{1-2py}{(1+p^2)^{1/2}} = -2\cosh\left(\frac{i-k}{a} + \frac{1}{2a}\right),$$

verifying the solution.

We next ask the question: does (18) give all solutions of (15) which are positive at all points considered and which are described in our existence theorem above as determined by point to point solution of (16)?

Suppose $4AC-1>0$ and

$$(19)\qquad \begin{aligned} y(0) &= A = \frac{1}{2\sinh\dfrac{1}{2a}}\cosh\frac{k}{a} \\ y(1) &= C = \frac{1}{2\sinh\dfrac{1}{2a}}\cosh\left(\frac{1}{a} - \frac{k}{a}\right). \end{aligned}$$

We wish to solve these two equations for a and k.

$$(20)\qquad C - A = -\sinh\left(\frac{k}{a} - \frac{1}{2a}\right) = -\sinh\frac{k}{a}\cosh\frac{1}{2a} + \cosh\frac{k}{a}\sinh\frac{1}{2a}.$$

Replace cosh (k/a) in this equation by $2A$ sinh $(1/2a)$ and sinh (k/a) by $\pm\{4A^2\sinh^2(1/2a)-1\}^{1/2}$. Solving for sinh $(1/2a)$ we find

$$(21)\qquad \sinh\frac{1}{2a} = \pm\left[\frac{(C-A)^2+1}{4AC-1}\right]^{1/2}.$$

We first discard the minus sign before the radical, as we are only interested in solutions which are always positive. We next notice that $4AC-1>0$, inasmuch as $4y(i+1)\,y(i)-1>0$ in particular when $i=0$. Equation (21) consequently determines a positive value for sinh $(1/2a)$ and consequently a positive value of a. To determine k replace sinh $(1/2a)$ by the above value in (20),

cosh $(1/2a)$ by $\{1+\sinh^2(1/2a)\}^{1/2}$, cosh (k/a) by $2A$ sinh $(1/2a)$ and solve for sinh (k/a). A unique value is determined for k. These values satisfy both equations (19) as again is verified by substitution.

Formula (18) *then gives us the general solution of* (15) *which remains positive throughout the interval in question and consequently is a necessary form for the function* $y(i)$ *in order that the surface of revolution be a minimum.*

The next problem is to see under what conditions it is possible to pass a catenary with equation of the type (18) through the two points $(0, A)$ and (n, B). This problem for the analogous case of the smooth curve has been extensively studied.* We are, however, more interested here in our sufficiency condition as applied to the present problem. If recurrent relation (7) is set up for (14) it will be found to be quite complicated and it is likely that in this case some other method of investigating sufficiency is necessary.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

NEW EQUILATERAL POLYHEDRA†

By MICHAEL GOLDBERG, Washington, D. C.

In answer to a query in which the questioner sought polyhedra bounded by sixty-four congruent faces, I submitted three polyhedra which occurred to me. When I consulted Hess‡ and Brückner§ to see if there were any others I was surprised to find that not only were there no more given, but that one of my polyhedra was not listed. It seems that Hessel|| the original investigator of "gleichflächige Polyeder" did not discover this polyhedron and that later writers on polyhedra have repeated this error of omission.

The new polyhedron is formed by crowning with pyramids the bases of a skew prism. Therefore, the polyhedron is one of an infinite series, the number of faces being always a multiple of four. A simple way of constructing it is to divide a sphere into $4n$ congruent spherical isosceles triangles whose vertex angles are $2\pi/n$ and whose base angles are $\pi(n-1)/2n$. The vertices of the triangles of this spherical net can also serve as the vertices of the plane triangles bounding

---

* See Harris Hancock: Annals of Mathematics, 1st Series, vol. 10, p. 159.

† I have been informed that Mr. A. H. Wheeler of Worcester, Mass. has independently discovered the $4n$ series in answer to the same query.

‡ E. Hess, *Lehre von der Kugeltheilung*, Leipzig, (1883).

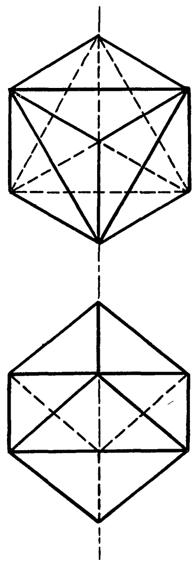
§ M. Brückner, *Vielecke und Vielfläche*, Leipzig, (1900).

|| J. Hessel, *Übersicht der gleicheckigen Polyeder*, Marburg, (1871).

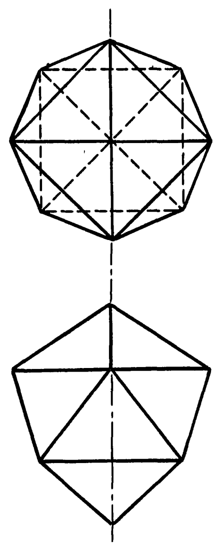
the polyhedron. Note that when thus constructed the polyhedron is both circumscribable and inscribable by spheres.

For $n=3$ the polyhedron degenerates into a cube and for $n=5$ the polyhedron is the regular icosahedron. For all other values of n the polyhedron is new.

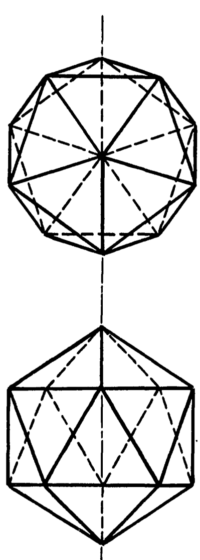
The $4n$ spherical triangles described above are congruent on the sphere. If



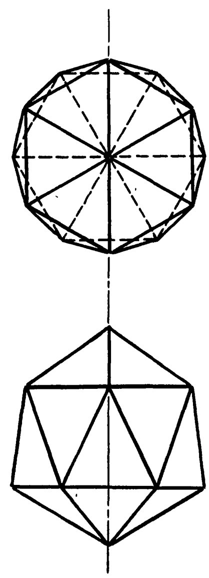
$4n=12$



$4n=16$



$4n=20$



$4n=24$

each of these isosceles triangles is cut in two by a line from the vertex to the base, $8n$ *symmetric* triangles are formed. However, the plane triangles they determine are *congruent*. Thus there is formed a new infinite series of equilateral polyhedra possessing $8n$ faces. These polyhedra are topologically different from the $4n$ series since they possess vertices at which four edges meet.

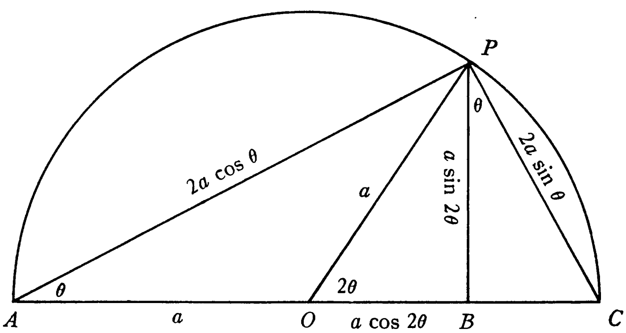
The figures show the plan and elevation of the first four polyhedra of the $4n$ series.

The Trigonometric Functions of Half or Double an Angle

By Roscoe Woods, State University of Iowa

The following simple method for deriving the formulas connecting the trigonometric functions of double or half an angle is not featured in texts on trigonometry.

For convenience we consider acute angles only. The figure needs no explanation.



$$AB = a(1 + \cos 2\theta), \text{ and } BC = a(1 - \cos 2\theta).$$

From triangle ABP, we have by definition,

$$\sin\theta = a \sin 2\theta/2a\cos\theta, \quad \text{or,} \quad 2\sin\theta\cos\theta = \sin 2\theta,$$

$$\cos\theta = a(1 + \cos 2\theta)/2a\cos\theta, \quad \text{or,} \quad 2\cos^2\theta = 1 + \cos 2\theta,$$

$$\tan\theta = a\sin 2\theta/a(1 + \cos 2\theta), \quad \text{or,} \quad \tan\theta = \sin 2\theta/(1 + \cos 2\theta).$$

From triangle PBC, we have by definition,

$$\sin\theta = a(1 - \cos 2\theta)/2a\sin\theta, \quad \text{or,} \quad 2\sin^2\theta = 1 - \cos 2\theta,$$

$$\cos\theta = a\sin 2\theta/2a\sin\theta, \quad \text{or,} \quad 2\sin\theta\cos\theta = \sin 2\theta,$$

$$\tan\theta = a(1 - \cos 2\theta)/a\sin 2\theta, \quad \text{or,} \quad \tan\theta = (1 - \cos 2\theta)/\sin 2\theta.$$

By substitution the relation

$$AP^2 - AB^2 = BP^2 = PC^2 - BC^2$$

becomes

$$\cos^2\theta - \sin^2\theta = \cos 2\theta.$$

Editorial Note. The author writes: "I have looked through about fifty texts on trigonometry and have seen the figure in only four, viz., Cróckett, p. 73; Smail, p. 105; Granville-Smith-Mikesh, p. 97; Hobson (2nd Ed.), p. 53. The authors use the figure to prove some of the formulas for the half and double angles, but do it in such a way that it appears very hard."

An Application of Sylvester's Dialytic Method to Elliptic Functions

By W. D. Dickinson, Jr., University of Arkansas

The following demonstrates an interesting application of Sylvester's dialytic method of elimination. It is the purpose of this note to determine the first order differential equations satisfied by the higher derivatives of Weierstrass' elliptic $\wp$ function, these higher derivatives being, of course, themselves elliptic functions.

The relationship between the $\wp$ function and its first derivative is given by

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{1}$$

Differentiating and simplifying we obtain

$$\wp''(z) = 6\wp^2(z) - g_2/2 \tag{2}$$

$$\wp'''(z) = 12\wp(z)\wp'(z) \tag{3}$$

$$\wp^{\mathrm{IV}}(z) = 12[\wp'^2(z) + \wp(z)\wp''(z)] \tag{4}$$

etc.

Each derivative then can be expressed in terms of $\wp$ and $\wp'$ thus $\wp^{(2n)}(z) = P_{n+1}$ and $\wp^{(2n+1)}(z) = Q_n\wp'(z)$ where P_{n+1} and Q_n are polynomials in $\wp(z)$ of degrees $n+1$ and n respectively.

From the above we see that the even derivatives are already expressed as a function of the $\wp$ function. If we square the odd derivatives and substitute for $\wp'^2$ we have $[\wp^{(2n+1)}(z)]^2 = P_{2n+3}$ where P_{2n+3} is a polynomial of degree $2n+3$. Applying Sylvester's method then to eliminating the $\wp$ function between any two successive derivatives we get the desired differential equations.

Setting up the resultant for equations (1) and (2) we have*

* Cf. Whittaker and Watson, *Modern Analysis*, Ex. 11, p. 457.

$$\begin{vmatrix} \wp'^2+g_3 & -g_2 & 0 & 4 & 0 \\ 0 & -\wp'^2-g_3 & -g_2 & 0 & 4 \\ \wp''+\frac{g_2}{2} & 0 & 6 & 0 & 0 \\ 0 & -\wp''-\frac{g_2}{2} & 0 & 6 & 0 \\ 0 & 0 & -\wp''-\frac{g_2}{2} & 0 & 6 \end{vmatrix} = 0$$

which is the first order differential equation satisfied by $\wp'$. In like manner the resultant for (2) and (3) gives

$$\begin{vmatrix} \wp''+\frac{g_2}{2} & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & -\wp''-\frac{g_2}{2} & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -\wp''-\frac{g_2}{2} & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -\wp''-\frac{g_2}{2} & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & -\wp''-\frac{g_2}{2} & 0 & 6 \\ \wp'''^2 & 0 & -144g_3 & -144g_2 & 0 & 576 & 0 \\ 0 & -\wp'''^2 & 0 & -144g_3 & -144g_2 & 0 & 576 \end{vmatrix} = 0$$

the first order differential equation for $\wp''$.

In general, the method will give as the first order differential equation satisfied by the $2n$th derivative, a resultant determinant of order

$$(n + 1) + (2n + 3) = 3n + 4,$$

and for the equation satisfied by the $(2n+1)$th derivative we get a resultant determinant of order

$$(2n + 3) + (n + 2) = 3n + 5.$$

The method outlined may also be used to get the algebraical relationship between any two of the derivatives.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Statistical Method in Education. By C. W. Odell. New York, Appleton-Century, 1935. xx+457 pages. $3.50.

As is common with such texts, this book "is designed to meet the statistical needs of those interested in the field of education by presenting the subject in a manner as non-mathematical as possible." In its twenty-three chapters it treats tabulation and classification of data, graphical representation, the normal curve, frequency distributions, averages, variability, correlation, and reliability. Its three appendices present a general bibliography, tables of the normal curve, and the values and logarithms of certain constants. There is an index of authors and an index of topics. The binding is good, the typography is excellent, and in general the format is attractive.

The author follows the traditional approach, taking up first the descriptive and computational phases of his subject, and later introducing the concept of estimation, with which the modern science of statistics is more fundamentally concerned.

The first few chapters conform to the author's intention to present the material in a manner for non-mathematicians to understand, but the later parts of the book present concepts, procedures, and formulae without justifying them, and in many cases without completely setting forth the conditions under which they are properly to be applied.

Judged by very high standards, the text is unsatisfactory in many respects. Perhaps the most serious fault is the appearance of statements which either are in error or hold only under certain conditions which are not set forth with the statements. On page 119, for example, the mean deviation is said to include approximately 57.5 per cent of the cases within the points on the scale to which it reaches when laid off in both directions from the mean or median. No qualifying statement appears concerning the form of distribution in which this relationship holds. On page 132 kurtosis is left out of account in describing conditions under which the median deviation is closely approximated by $.6745\sigma$. On page 137 it is implied that the median deviation of a given distribution may be expressed in terms of the median deviations of the component parts of the distribution. At the bottom of page 137 the median deviation is said to equal a given fraction of the standard deviation, and to possess most of its qualities. This implies that other measures of dispersion are not "given fractions" of the standard deviation in the same sense. Furthermore, the statement concerning the median deviation does not hold as a generality. A clearly erroneous statement appears in the footnote on page 137 where the author asserts that the point of inflection of the normal curve is the point at which the slant of the curve

forms an angle of exactly 45 degrees with the baseline. Another occurs on page 143 where the author says that correlation cannot be computed unless there are available two or more measures for each of a number of individuals. This statement is refuted by the existence of a large number of studies of correlation among twins, and among other pairs of individuals. At the top of page 327 the relationship between the median deviation and the standard deviation is again stated without qualification as to the form of the distribution. On page 328 the author recommends using the median deviation itself in the formula for the probable error of the mean, when it is really $.6745\sigma$, the estimated median deviation, which should generally be employed. On page 348 the author implies that the correlation coefficient is normally distributed in samples, regardless of the value of the coefficient in the universe.

Another unfortunate shortcoming of the book concerns the care the author has apparently exercised in examining the literature in his field. On page 203 Holzinger's criticism of a formula by Kelley is referred to as a refutation of some of the points raised much later by Dickey, who also criticized Kelley's formula. On page 255 Fisher's criticism of the Blakeman criterion is mentioned, but Fisher's solution to the problem attacked by Blakeman is nowhere presented. On page 335 the author presents uncritically a series of unsound recommendations concerning the correction of standard error formulae for small numbers of cases. Nowhere in connection with the treatment of standard error formulae does the author make recommendation for the use of universe values wherever they are available for substitution in such formulae. On page 354 the author refers in commendatory terms to the actually questionable contribution of Lincoln concerning the insignificance of significant differences. On page 386 the chi-square test for goodness of fit is referred to Holzinger. And on page 400 the author says that the only common measure of kurtosis is that suggested by Kelley, the quartile deviation divided by the 10–90 percentile range.

Another undesirable feature of the book is the incidence of recommendations for practices which are discordant with logic or with the best accepted practice in the same connection. In the chapter on tabulation and classification, for example, only one method of assigning class limits is presented, and that method does not conform to logic when applied to observations which are reported to the nearest unit. This treatment is revised later in the text, so that the student must learn new rules for situations he has supposedly mastered. And the later revision is accompanied by the unsound recommendation on page 77 that unless it is specifically stated in connection with the data in question, a given measure should generally be taken as meaning from that up to the next higher measure. The author's justification for this recommendation is the questionable statement on page 75 that this assumption agrees with the scoring system used on most standardized and other objective tests, since, if a pupil responds to 50 items on such a test correctly and almost but not quite correctly to another, his score is 50 and not 51. "Therefore a score of any given size, such as 50, denotes performance equal at least to that much and perhaps almost equal to the next pos-

sible score." Here the author ignores the fact that if an unspeeded power test of the multiple choice type is scored so as to discount for chance success on the items, a student's expectancy of improving his score by working on a marginal item is exactly equal to his expectancy of lowering his score by his performance on that item. Another example of undesirable practice is the use of a column diagram on page 36 where there is no implied discontinuity in the data. On page 48 we are told that the percentile curve is the same as the cumulative frequency curve except that the axes are interchanged. And on the bottom of page 333 the author recommends adherence to an erroneous practice because it has become so common. The practice referred to is the use of $\sqrt{N}$ in the denominator of standard error formulae.

One of the most unfortunate features of the text is the frequent occurrence of confusing or inconsistent presentation or terminology. On page 26 the word "percentile" is used unnecessarily in a sense quite different from its conventional use and from its use elsewhere in the text. On pages 69, 71, and 348, the word "error" is used in three discrepant ways. On page 66 reliability and correlation are stated to be "types of measures or facts necessary to the summarization or description of a distribution." On pages 71 and 348 appear discrepant uses of the expression "true mean." On page 88 the true mode is defined in a manner inconsistent with the treatment of true measures on page 128. On page 89 a frequency polygon is presented with the mode indicated at some distance from the highest point. At the top of page 108 the reported values for the first and third quartiles are not consistent with the author's recommendation concerning the interpretation of a score. On page 116 the author computes a value for the range by a method inconsistent with his definition of the range. On page 100 the author says the median is not so rigidly defined as the mean, and then a few lines farther on gives what appears to be a completely rigorous definition of the median. On page 131 the author recommends varying the method of computing the standard deviation when the number of cases is small. On page 132 the author says the median deviation is often erroneously called the probable error. In the same paragraph he says the two quantities are numerically equal and are found in the same way. Surely the alert student may well ask why the two terms should not be used interchangeably if they are always numerically equal and are found in the same way. On page 137 the standard deviation is referred to as a function, instead of a parameter, of the normal probability curve. On page 132 the author implies normality as a prerequisite of a distribution in which the median deviation and the quartile deviation have the same value.

Certain other disturbing features of the book are mere ineptitudes. Readers with an affection for the niceties of language may be displeased by the author's statement on page 39 that a histogram assumes something, and the assertion on page 119 that the mean deviation assumes something else. On page 290 it is a formula that assumes. On page 56 the expansion of "the binomial" is mentioned without any previous treatment of binomials or any accompanying statement as to what binomial is meant. After defining Q as a point, the author states on

page 117 that Q includes 25 per cent of the measures. In general, throughout the book where the author might have used the word "subtends," he has used "includes" in the sense just illustrated. On page 197 the author says that since $k^2+r^2=1$, the sum of k^2 and r^2 "represents the total situation." The thoughtful student may well inquire what situation is being referred to. On page 407 the author lists as three assumptions what are really two assumptions and a conclusion therefrom.

The minor slips appearing in the text can of course be corrected, and are not fundamental to an evaluation of the work. Among such slips is the listing on page 6 of the name of a widely advertised golden hair wash as the misspelled name of a calculating machine. On page 115 certain measures are referred to as being "further" away from the average than others. In Figure 33, on page 133, the range of two median deviations is not symmetrically disposed with reference to the mean. At the bottom of page 203 appears a footnote without anywhere a reference to the author's name. In the paragraph heading on page 229, the Greek letter rho is misprinted as a lower case p. The second sentence in the second paragraph on page 326 has a clause with a plural subject and a singular predicate.

The book possesses several good qualities. It is desirable in such a text that considerable attention be devoted to the concept of reliability. And we find two chapters given to a discussion of the problem of reliability and errors of estimate and of measurement. As a whole, the book contains more material than most of those in its field. The author looks at his material from the very practical standpoint favored by educators. Certain parts of the book are definitely better than many of its kind, as, for example, the discussion in connection with the interpretation of the correlation coefficient and of the probable error of a statistical concept. There is a remarkable freedom from misprints throughout the text, and the printing of the formulae is unusually satisfying.

But the book will almost certainly irritate as much as instruct the mathematically trained reader in search of tuition in statistics. And the thoughtful student will surely be seriously confused at times, whatever his background.

P. J. RULON

The Design of Experiments. By R. A. Fisher. Edinburgh, Oliver and Boyd, 1935. v+252 pages. 12/6.

One of the most interesting and important developments in statistical methods in recent years has been the discovery of the analysis of variance and the exploration of its possibilities. Since the credit for this is due almost exclusively to Professor Fisher, the appearance of this book by him largely devoted to the subject, will be greeted with the greatest interest by theoretical and practical statisticians alike.

Like his previous work, *Statistical Methods for Research Workers*, the present book is intended to be a guide to statistical workers so that they may properly apply and interpret the methods dealt with. The mathematical deriva-

tions that seem to the mathematician essential to complete understanding are not given, but on the other hand the exposition of the nature of a statistical experiment, its possibilities and limitations, and its correct formulation and interpretation is brilliant and will repay careful study by all but the most mature students of mathematical statistics. In this connection the first three chapters, and particularly the second on "The Principles of Experimentation Illustrated by a Psycho-physical Experiment," are especially recommended.

The main body of the book is devoted to the exposition of experimentation by the use of randomized blocks, dealing with increasingly complicated set-ups which are illustrated with agricultural experiments. The casual reader will gain a good idea of the present development and power of the analysis of variance. The student will be well repaid for the close study that will be required for mastery of the contents and he will be well-advised to have previously familiarized himself with the corresponding sections in the author's *Statistical Methods for Research Workers*, and in Tippett's *The Methods of Statistics*, and with Snedecor's valuable *Analysis of Variance.* The remainder of the book is mainly devoted to some applications of sampling theory and the author's theory of estimation with applications from biology and genetics. It seemed to the reviewer that a good part of this would be particularly difficult going for readers who do not already have considerable familiarity with the ideas discussed.

As in all his writings, the author's brilliant and concise style pays a high compliment to his reader's intelligence. But also as in his previous writings the serious student of statistics cannot afford to neglect what is being said.

C. C. CRAIG

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

CLUB TOPICS

FOUR COLOR PROBLEM

Geographers wish to use as few colors as possible in making maps. Yet they insist that no two districts which have a line boundary in common shall be colored alike. It is supposed that map makers had realized the necessity of four colors for this purpose some centuries ago, but it remained for Francis Guthrie to first consider, as mathematical problems, whether four colors are necessary and whether four colors are sufficient. Guthrie communicated the problem to DeMorgan about 1850. There have been many papers and much discussion since that time. The problem is still unsolved. We present a list of some of these discussions.

Proceedings of the London Mathematical Society, vol. 9, p. 148, vol. 10, p. 229.
Proceedings of the Royal Geographical Society, N. S., vol. 1, p. 229.
Proceedings of the Royal Society of Edinburgh, vol. 10, p. 728.
Philosophical Magazine, Series 5, vol. 17, p. 41.

Nature, vol. 21, p. 399.

A. B. Kempe, *On the geographical problem of four colours*, American Journal, vol. 2, p. 193.

P. J. Heawood, *Map-colour theorem*, Quarterly Journal of Mathematics, vol. 24, p. 332.

G. D. Birkhoff, *The reducibility of maps*, American Journal, vol. 35, p. 115.

P. Franklin, *The four color problem*, American Journal, vol. 44, p. 225.

W. W. R. Ball, *Mathematical Recreations and Problems*, p. 48.

H. R. Brahana, *The four color problem*, this MONTHLY, vol. 30, p. 234.

C. N. Reynolds, *On the problem of coloring maps in four colors*, I and II, Annals of Mathematics, vol. 28, p. 1 and p. 477.

G. D. Birkhoff, *On the number of ways of coloring a map*, Proceedings of the Edinburgh Mathematical Society, Series 2, vol. 2, p. 83.

P. J. Heawood, *On extended congruences connected with the four color map theorem*, Proceedings of the London Mathematical Society, vol. 33, p. 253.

H. Whitney, *A theorem on graphs*, Annals of Mathematics, vol. 32, p. 378.

C. N. Reynolds, *Circuits upon polyhedra*, Annals of Mathematics, vol. 33, p. 367.

H. Whitney, *The coloring of graphs*, Annals of Mathematics, vol. 33, p. 688.

I. Kittell, *A group of operations on a partially colored map*, Bulletin of the American Mathematical Society, vol. 41, p. 407.

Errera, (Thesis) *Du coloriage des cartes*, Brussels, 1921, Gauthier-Villars, Paris.

KAPPA MU EPSILON

Five years ago, Dr. Kathryn Wyant, at Northeastern Teachers College, Tahlequah, Oklahoma, began the organization of a national mathematics fraternity in the four-year colleges. Her object in grouping together undergraduate clubs in these colleges was four-fold; first, to arouse interest of students in mathematics; second, to develop friendships among those interested in the science and teaching of mathematics; third, to develop an appreciation of the beauty as well as the practicality of mathematics; fourth, to provide recognition of achievements of undergraduates in this field.

A report of the present activities of the first chapter was given in our last issue. At present this fraternity has fourteen chapters scattered through a dozen states. These are active clubs and the growth of the organization is indicative of the growing interest in mathematics in the four-year colleges.

Dr. Wyant is now on the faculty of Athens College, Athens, Alabama. Professor J. A. G. Shirk, Kansas State Teachers College, Pittsburg, Kansas, is at present the national president of Kappa Mu Epsilon.

CLUB REPORTS

1934–35

Unless otherwise stated, all officers mentioned in these reports are for the school year 1934–35 and all programs were given during that year.

Pi Mu Epsilon of The University of Oklahoma

This very active chapter had forty-one members in 1934–35 and held seventeen meetings. These included business meetings, a picnic, and an initiation banquet. At the close of business and program meetings mathematical puzzles were presented and informal discussions encouraged during a social hour.

Officers for the year were Mildred Dolezal, Director; H. Feldstein, Vice Director; W. Kitchens, Secretary; Mary C. Simpson, Treasurer; J. Laudermilk, Librarian; Dora McFarland, Corresponding Secretary.

The chapter has established a Pi Mu Epsilon Scholarship for excellence in elementary mathematics. The written examination covers analytic geometry and calculus. The scholarship carries

with it the remission of fees for one semester at the University of Oklahoma. Thomas Wiancho was the 1935 winner.

Prizes are offered for the best talks given at regular meetings. R. E. Smith was given Sokolnikoff's *Higher Mathematics for Engineers and Physicists*, for "the best example of an advanced discussion which necessitated simple treatment." Rosemary Hudson received a copy of Smith's *Source Book of Mathematics* for "the best example of a more elementary discussion." Their papers are listed with the regular programs which follow: "Homogeneous coordinates" by W. Kitchens; "Line coordinates" by Mary Simpson; "Trigonometric series from the view point of Klein" by Thelma Sherry; "Introduction to invariants" by Mary Tappan; "Newton's Interpolation formula" by Patience Sewell; "The cubic equation" by E. Skipper; "The quartic equation" by Rosella D. Scott; "Graphical and nomographical solution of the cubic" by B. L. Hirsh; "A theorem of projective geometry" by Ruth Rice; "Non-Euclidean Geometry" by Rosemary Hudson, prize winner; "Differential equations as applied to structural beams" by W. Simpson; "Diophantine recreations" by Harriet Slemmer; "Vector Notations" by R. Van Dyne; "An application of calculus to astronomy" by Barbara Ellis; "Elementary wave mechanics" by R. E. Ellis, prize winner.

Mathematics Club of the University of Colorado

This club is organized to stimulate among the members a spirit of inquiring interest in mathematics and to promote friendship and cooperation. Professor A. J. Kempner was sponsor for the club. The Presidents were W. Swan and R. Lee, the Vice Presidents were R. Lee and E. Coyer. B. Gordon was Secretary and M. Clark, Treasurer.

Ten regular meetings were held and the year closed with a steak fry on Flagstaff Mountain. The programs were as follows: "Blair-Leighton equation and photography" by Dr. H. Blair; "Samples and specimens" by Professor William Hazard; "Interesting facts in mathematics" by C. Merrill; "Miscellaneous facts in mathematics" by J. Britton; "Geometrical progressions" by Professor A. J. Kempner; "Squaring the circle" by F. Fletcher; "The abacus" by M. Clark; "Mathematical fallacies" by E. Coyer; "Review of Abbott's Flatland" by J. Conner; "The theorem of Pythagoras" by R. Burling; "Magic squares" by R. Lee.

Delta x of the University of Toledo

This club has a membership of seventy-three and mixes its clever social meetings with serious minded papers. At a chili supper they listened to "The semi-regular polyhedra" by Aubrey Mather, the Christmas meeting had festivities and "Al Khowarizmi's solution of the quadratic equation" by Ella May Rike. The get-acquainted roast and the closing picnic seem to have ignored serious matters.

Other papers presented at regular meetings included the following: "Graeffe's root-squaring method" by K. Whelan; "Reversible numbers" by Eleanore Jablinski; "The theory of relativity" by V. Georgeff; "Mathematics in photography" by W. Stevens; "Illustrations of the method of finite differences" by Professor M. M. Lemme and Professor J. B. Winslow.

Kappa Mu Epsilon of Nebraska State Teachers College

This chapter held several social meetings in connection with other groups in the college and in addition held bi-monthly meetings with the following papers: "Interesting numbers" by J. Jones; "Mathematical recreations" by C. Sorensen; "Number curiosities" by Elaine Gildersleeve; "What mathematics means to me" by D. Davis; "Methods of curve fitting" by Dr. J. Brock; "Review of E. T. Bell's Mathematics up to date" by Donnabelle Stuart; "The trigonograph" by L. Dannals; "Magazine reviews" by J. Larson; "Canterbury puzzles" by L. Erxleben; "An excursion in numbers" by J. Hall; "Mathematics in the secondary schools of England and France" by Irene Daws; "Mathematics in the physical sciences" by J. Jones; "Mathematics in engineering" by F. Ostapoff; "Space in mathematics" by F. Victor; "Revolving numbers" by J. McKinley; "Mathematical requirements in our schools" by L. Gildersleeve; "Differential equations" by J. Jones; "Wave motion" by L. Dannals.

Pi Mu Epsilon of Syracuse University

This chapter was founded in November, 1903, the first of the thirty-one chapters of this undergraduate mathematical fraternity, and one of the earliest undergraduate mathematics clubs in this country. There were thirty-two new members initiated last year. Officers were Professor May Harwood, Director; Jane Scudder, Vice Director; G. Moore, Treasurer; Ellen Olson, Secretary. Two initiations were held, the first at the home of Professor Alan Campbell, the second at Drumlins.

Papers at the regular monthly meetings were as follows: "Cubic equations" by E. Eachus; "Filtration" by G. Moore; "Practical engineering problems" by M. Nienalski; "Aerial photography" by Professor E. F. Church; "Mechanical integration" by E. Eachus; "Eclipses" by Professor L. Lindsey.

The Mathematics Club of Oberlin College

Officers were Zoe E. Schnabel, President; R. Whitney, Vice President; Jean Humphrey, Social Chairman; Lucile Worden, Program Chairman; Frances Sherman, Secretary-Treasurer.

Ten regular meetings were held in addition to a St. Patrick's Day party, a tea for Professor M. Emily Sinclair, and a banquet at which Professor M. H. Ingraham of the University of Wisconsin was the guest speaker.

Topics taken from Dantzig's *Number, the Language of Sicence* formed the core of the programs for regular meetings. The year's program was: "Advanced preparatory schools in England, France and Italy" by Professor W. D. Cairns; "The empty column" by Frances Sherman; "Number lore" by Ruth Binning; "The last number" by Margaret Chapman; "Pascal's Theorem" by Jane Reynolds; "Large numbers and the factor stencil" by Zoe E. Schnabel; "Symbolism" by Villa Bailey; "The algebra of vectors" by J. Molnar; "Stellar distances and parallaxes" by K. Leach; "Nomography" by H. Wright; "Non-Euclidean geometry" by H. Polster and J. Friedman; "Logarithms and slide rules" by J. Lupton and W. Ward; "Geometric probability" by Jean Humphrey; "Probable error" by Betty Madsen.

The Mathematics Club of the University of Kansas

Officers were M. Sholander, President; Helen Welch, Vice President; R. LaShelle, Secretary-Treasurer; Minnie McDaniel, Social Chairman; Professor H. E. Jordan, Faculty Adviser.

A social hour followed each regular meeting and the year closed with a picnic at State Lake near Tonganoxie. Eleven regular meetings were held. The programs were as follows: "Mathematics and optics" by G. Omer; "Magic squares" by W. Utermohlen; "Some relations between logic and mathematics" by Anna McCracken; "Short cuts in mathematical computation" by M. Sholand; "Common trends in the histories of mathematics and psychology" by Professor H. H. Wheeler; "Coordinate systems" by E. Emery; "The mathematics of the psychological field" by Professor J. F. Brown; "Nomographic charts" by Professor H. E. Jordan; "Paper folding" by G. Ulmer; "Mathematical explanation of ghosts" by Dr. G. Beck; "So figures don't lie" by J. Lumpkin; "Examples of mathematical economics" by W. Simmons.

The Mathematics Club of Boston University

Officers were T. Mariner, President; M. Peridier, Vice President; Eunice Leavitt, Secretary; Celia Epstein, Treasurer; Elizabeth Heath, Assistant Secretary; Elnore Lundgren, Assistant Treasurer; Professor R. E. Bruce, Faculty Adviser.

Ten meetings were held with programs as follows: "Mathematical oddities" by Dr. R. M. Frye; "Cycloidal pendulum" by Professor L. B. Taylor; "Magic squares" by H. Lisman; "Trisection of the angle" by Mildred Florian; "1152 varieties of algebra" by Dr. R. M. Frye; "Mathematical astronomy" by Professor L. A. Brigham; "The fourth dimension" by P. Doe; "Duodecimal number system" by J. Miller; "Simple mathematics in piloting" by T. Mariner; "Constructions with limited means" by Professor Marion E. Stark of Wellesley College.

Graduate Club of the University of Cincinnati

During the second semester the club held three meetings with these programs: "Knotted curves in space" by Dr. C. W. Mendel; "The summation of series by elementary methods" by Professor A. I. Barnett; "Fundamentals of vector analysis" by Dr. N. Koch.

Officers for 1935–36 are E. Yelton, President; H. Reingold, Vice President; Jane Rudolph, Secretary; K. Stover, Treasurer.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 199. *Proposed by W. E. Buker, Leetsdale High School, Pa.*

From an arbitrary point within a triangle, draw three lines to the sides of the triangle which shall trisect its area. (Although this problem is not new, it seems to be missing from the more readily accessible texts.)

E 200. *Proposed by N. A. Court, University of Oklahoma.*

The square of the distance between a fixed point F, and a variable point P, is equal to the sum (or the difference) of the powers of the two points with respect to a given sphere S. Find the locus of P. (The corresponding problem in the plane was solved analytically in the Educational Times, Reprints, vol. I (1864), p. 6, by A. Cayley.)

E 201. *Proposed by A. A. Bennett, Brown University.*

Show that

$$\sum_{0}^{3} \operatorname{arccot}(3n-1) = 0.$$

E 202. *Proposed by W. B. Clarke, San Jose, California.*

While three auto mechanics were eating their lunch, they noticed that the center of the trademark on a tire of the nearest automobile was directly over the center of an oil spot on the floor. Measurement showed that it was two inches above the floor, and ten inches below the center of the hub.

A discussion arose as to how far they would have to roll the car straight ahead so that the centers of the trademark, hub and oil spot would again be in line. To settle it, they moved the machine and made measurements, but an argument arose. Bill maintained that the machine had been rolled exactly four

feet six inches; Slim claimed it was just short of that, but Doc insisted it was a trifle more.

So they decided to roll the car ahead again until the three points were once more in line and see if they couldn't agree on that. This time Bill's measurement was seven feet nine inches from the oil spot. As before, Slim thought it was a trifle less and Doc the least bit more.

Who was right each time, and what were the distances, to the nearest sixteenth of an inch?

E 203. *Proposed by J. F. Berry, New York City.*

Determine a point in the plane of two given circles, from which tangents to the circles shall form a given angle and have a given ratio.

E 204. *Proposed by J. E. Trevor, Cornell University.*

Let $abcd$ be any four-digit number, with a not zero. The sum of the digits in the product, $99\cdot abcd$, is either 18, 27 or 36. Find the necessary and sufficient conditions on a, b, c and d, under which each of these three values is produced.

SOLUTIONS

E 171 [1935, 508]. *Proposed by J. E. Trevor, Cornell University.*

A chord of constant length slides around in a circle with fixed diameter. The midpoint of the chord and the projections of its ends upon the fixed diameter form the vertices of a triangle. Prove that this triangle is isosceles and never changes its shape.

Solution by L. M. Kelly, Lawrence, Mass.

If the chord, AB and its midpoint M, project onto the diameter at C, D and N respectively, then N is the midpoint of CD, and triangle MCD is isosceles. Let E be the symmetric of A with respect to the diameter. Then $CM \| EB$, and since angle AEB is measured by half the arc AB, angle ACM is constant. Hence angles DCM and MDC are constant, and the triangle MCD is invariant in shape.

Also solved by F. A. Alfieri, J. A. Benner, David Blackwell, W. E. Buker, J. E. Burnam, Wm. Douglas, W. R. Hardman, Evelyn Hesseltine, Ruth Kasper, D. L. MacKay, Leon Recht, J. Rosenbaum, J. T. Rule, C. V. L. Smith, E. P. Starke, Virginia Sweeney, C. W. Trigg, M. J. Turner, Simon Vatriquant and the proposer.

E 172 [1935, 508]. *Proposed by V. Thébault, Le Mans, France.*

Determine a perfect square containing each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 just once, and having its four digits at the right form an arithmetic progression. Show that the solution is unique.

Solution by J. E. Burnam, Hardin-Simmons University, Abilene, Texas

Let S^2 be the required number. The digit-sum is 45, so that S^2 is divisible by 9 and S by 3. Then the digit-sum of S is divisible by 3. Since S^2 is between

1023456789 and 9876543210, S must be between 31991 and 99831. The last four (right-hand) digits of S^2 are determined by the last four digits of S.

The last two digits of all squares are found among the last two digits of the first twenty-five squares. Consideration of these shows that the only possible last four digits of S^2 must be one of the arithmetic series, 3456, 0369, 6789, 9876, and 4321, which will now be examined.

If S^2 ends in 3456, the last two digits of S may be 16, 34, 66 or 84. If the last three digits of S are $x16$, then for $(x16)^2$ to end in 456, x must be 1 or 6. Trying the 1, call the last four digits of S, $y116$, and square. Here y must be 0 or 5. Try $S=z0116$, remembering that the digit-sum is a multiple of 3, but squaring introduces duplicated digits, so this case is ruled out. Try $y=5$ in the same way, and again find no admissible z. Hence S does not end in 116. An investigation of 616 along similar lines rules it out also. The terminations 34, 66 and 84 meet the same fate, and so S^2 does not end in 3456.

A continuation of this process of trial and error eventually produces the single solution, $S=87639$, with $S^2=7680594321$.

Also solved by W. E. Buker, Mary L. Constable, E. P. Starke, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 173 [1935, 508]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

An exact division was made, and then the even digits were all replaced by x's and the odd digits were all replaced by y's, getting this pattern:

$$
\begin{array}{l}
y\;y\;y\;)\;y\;x\;x\;x\;y\;x\;x\;(\;x\;y\;y\;x\\
\qquad\quad\;\; \underline{y\;x\;x\;x}\\
\qquad\qquad\quad x\;x\;y\\
\qquad\qquad\quad \underline{y\;x\;y}\\
\qquad\qquad\qquad y\;x\;x\\
\qquad\qquad\qquad \underline{x\;y\;y}\\
\qquad\qquad\qquad y\;x\;y\;x\\
\qquad\qquad\qquad \underline{y\;x\;y\;x}
\end{array}
$$

Reconstruct the original long division problem, and show that the solution is unique.

Solution by W. R. Talbot, Lincoln University, Mo.

The products, yxy and xyy, of yyy by two odd digits, tell us that the two odd digits are different, that neither is unity, and that neither is 9 since an even multiple of yyy contains four digits. Since the products of yyy by the two even digits of the quotient give distinct four-digit products, the two even digits in the quotient must be distinct and larger than the two distinct odd digits in the quotient. Hence the two even digits are 6 and 8, and the odd digits are 3 and 5.

Since $5(yyy)<1000<6(yyy)$, the divisor must start with 17 or 19. But

8(19y) cannot start with yx, so the divisor starts with 17. Now 5(17y) starts with 8 and must be xyy, while 3(17y) starts with 5 and so must be yxy. To get products of these types, $5\cdot y$ must give an even amount to carry, and $3\cdot y$ an odd amount. The only odd value of y meeting these two conditions is 5, so the divisor is 175. We have already seen that the quotient is 8356, so the dividend is 1462300, and a performance of the division readily verifies this unique solution.

Also solved by David Blackwell, W. E. Buker, J. E. Burnam, Mary L. Constable, Wm. Douglas, J. A. Duerksen, Daniel Finkel, Cornelius Gouwens, H. E. H. Greenleaf, Evelyn Hesseltine, L. S. Johnston, Leon Recht, J. T. Rule, E. P. Starke, E. E. Strock, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 174 [1935, 509]. *Proposed by Roy MacKay, Eastern New Mexico Junior College.*

Defining a median triangle as the triangle determined by an edge of a tetrahedron and the midpoint of the opposite edge, show that the sum of the areas of the median triangles is less than three-halves of, but greater than the total, area of the tetrahedron.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

With the tetrahedron A-BCD, and on the edges meeting at A, construct the parallelepiped $ACED$-$BFGH$. A plane through AB and GE divides the parallelepiped into two prisms and bisects CD at K. The sum of the areas of the two lateral faces, AH and HE, of the triangular prism, is greater than the area of the third face, AG. But face $AH=2\Delta ABD$, face $HE=$face $AF=2\Delta ABC$, and face $AG=4\Delta ABK$. It follows that $\Delta ABD+\Delta ABC>2\Delta ABK$. Similarly, each of the other five median triangles is less than half the sum of the including faces. Upon adding the six inequalities and simplifying, we find that the sum of the areas of the median triangles is less than three-halves the sum of the faces.

The medians of a tetrahedron intersect at a point three-fourths the length of each median from its vertex. The triangle determined by an edge of the tetrahedron and the two medians from its extremities, has an area equal to one-half that of the median triangle in whose plane it lies. These small, half-median, triangles constitute the faces of four tetrahedra whose bases are the faces of the given tetrahedron. Now the sum of the areas of three faces of a tetrahedron is greater than the area of the fourth face. Upon adding the four inequalities thus obtained from the four small tetrahedra, we find that the sum of the areas of the median triangles is greater than the sum of the areas of the faces.

Also solved by Ruth Kasper, Simon Vatriquant and the proposer.

E 175 [1935, 509]. *Proposed by C. E. Sharp, Jr., Clayton, Mo.*

If a conic of eccentricity e is revolved about its major axis, and the resulting quadric surface cut by a plane which makes an angle A with the axis of

revolution, prove that the eccentricity of the resulting conic section is $e \cos A$.

Solution by E. P. Starke, Rutgers University.

A conic in the XY-plane with focus $(p, 0)$, directrix $y=0$, and eccentricity e, is given by

$$y^2 + x^2(1 - e^2) - 2px + p^2 = 0. \tag{1}$$

The surface obtained upon revolving (1) about the X-axis is

$$y^2 + z^2 + x^2(1 - e^2) - 2px + p^2 = 0. \tag{2}$$

Let the given cutting plane be

$$z = x \tan A + c, \qquad (A \neq 90^\circ). \tag{3}$$

The required conic is determined by (2) and (3) together. If we eliminate z between these equations, we obtain the projection of the conic on the XY-plane, the equation of which is

$$y^2 + x^2(\sec^2 A - e^2) + 2x(c \tan A - p) + p^2 + c^2 = 0.$$

Choose for new coordinate axes the intersections of (3) with the XZ- and YZ-planes. Then if (u, v) represents any point on the required conic, we have $x = u \cos A$, $y = v$. So now our conic has the equation

$$v^2 + u^2(1 - e^2 \cos^2 A) + 2u(c \sin A - p \cos A) + p^2 + c^2 = 0.$$

Since this is of the same form as (1), it represents a conic of eccentricity $e \cos A$.

The case of $A = 90^\circ$ omitted in (3) is obvious, since the section then is a circle of eccentricity 0, while $e \cos 90^\circ$ is also zero.

Also solved by Leon Recht, O. M. Rogers, Simon Vatriquant and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3774. *Proposed by J. Rosenbaum, Hartford Federal College.*

If $A_1A_2 \cdots A_n$ is a regular polygon with unit radius, the locus of a point P such that

$$\sum_{i=1}^{n} (PA_i)^{2r} = K,$$

is a circle, provided r is a positive integer less than n and K is not less than n.

Show also that when

$$K = \frac{1\cdot 3\cdot 5 \cdots (2r-1)2^r n}{r!},$$

the locus is the circumcircle of the polygon.

3775. *Proposed by D. T. Sigley, The University of Kansas City.*

Prove that the order g of an abstract finite group G which involves $6k$ (k a positive integer) sets of conjugate operators is of the form $6n$.

3776. *Proposed by E. P. Starke, Rutgers University.*

Determine all triangles whose sides are relatively prime integers and such that one angle is double another.

3777. *Proposed by A. D. Wallace, University of Virginia.*

Show that the total curvature along a characteristic line of a surface is equal to the square of the ratio of the arc length on the spherical image to the arc length on the surface.

Note. Along an asymptotic line the total curvature is minus the square of the ratio.

SOLUTIONS

3697 [1934, 453]. *Proposed by J. Rosenbaum, Milford, Conn.*

In an orthocentric tetrahedron the centroid bisects the line segment joining the circumcenter with the orthocenter.

This is another analogy between an orthocentric tetrahedron and a triangle, for in a triangle the centroid trisects the segment joining the circumcenter with the orthocenter.

I. *Solution by Leon Recht, Student, College of the City of New York.*

(1) In orthocentric tetrahedron $ABCD$, F, the foot of the altitude from A, is the orthocenter of face BCD. (This MONTHLY, vol. 42, 1935, p. 53). For altitudes AF, BK determine a plane perpendicular to CD at W. F is a point in BW, which is an altitude of BCD. Similarly, F lies in the other altitudes of the face.

(2) A plane, X, through altitude AF and median AM, will intersect BCD in a line which also contains the circumcenter, S, of the face.

(3) The tetrahedron's circumcenter, O, lies on a line perpendicular to BCD at S. Since OS is parallel to AH, plane X contains O, as well as (by construction) centroid G and orthocenter H.

(4) Similarly, O, G, H are contained in a plane through any other vertex of

the tetrahedron. Hence they are contained in the intersection of two planes, i.e., are collinear.

(5) Draw GQ perpendicular to SF. $MQ:QF=MG:GA=1:3$. But $SM:MF=1:2$ (Altshiller Court, *College Geometry*, p. 95) Hence $SQ=QF$ and therefore, $OG=GH$.

II. *Solution by J. E. LaFon, University of Oklahoma.*

In an orthocentric tetrahedron the opposite edges are orthogonal and the altitudes of the tetrahedron meet in a point called its "orthocenter." Let A, B, C, D be the vertices of the tetrahedron and O, its orthocenter. CD is perpendicular to AB and AO and therefore perpendicular to plane ABO. Thus the plane ABO is one of the Monge planes. Similarly the other five Monge planes contain O, whence it coincides with the Monge point. The proposition follows from the following theorem due to Gaspar Monge: The six planes through the mid-points of the edges of a tetrahedron and perpendicular to the edges respectively opposite (Monge planes) have a point in common (Monge point). This point is the symmetric of the circumcenter with respect to the centroid.

III. *Solution by R. Goormaghtigh, Bruges, Belgium.*

The theorem is known in a more general form:

In a tetrahedron the centroid bisects the line segment joining the circumcenter with the center of the altitude-hyperboloid. See Rouché et de Comberousse, *Traité de Géométrie*, 7th edition, vol. 2, p. 664.

Let $A_1A_2A_3A_4$ be the tetrahedron, h_1, h_2, h_3, h_4 the altitudes, P_4 the projection of A_4 on $A_1A_2A_3$, H_4 the orthocenter of $A_1A_2A_3$, p_4 the perpendicular erected at H_4 on the plane $A_1A_2A_3$.

Since h_1 is perpendicular to $A_2A_3A_4$, and p_4 is perpendicular to $A_1A_2A_3$, the three straight lines h_1, p_4 and A_1H_4 are perpendicular to A_2A_3, and therefore they are in a plane perpendicular to A_2A_3. Hence the three altitudes h_1, h_2, h_3 cut the straight line p_4, which therefore belongs to the altitude-hyperboloid (H).

The two straight lines h_4 and p_4 being parallel and belonging to (H), the center H of (H) is on the parallel q_4 to h_4 and p_4 drawn through the mid-point Q_4 of P_4H_4. If G_4 is the centroid and O_4 the circumcenter of the triangle $A_1A_2A_3$, and G the centroid of the tetrahedron, the projection g_4 of G on $A_1A_2A_3$ lies on G_4P_4 and $g_4P_4=3G_4g_4$; therefore g_4 bisects O_4Q_4 and q_4 passes through the image of O with respect to G, which proves the theorem.

The preceding proof establishes also the following property:

The perpendiculars erected on the faces of the tetrahedron at the orthocenters of the faces belong to the altitude-hyperboloid.

Solved also by Frank Ayres, Jr., Harry Langman, Roy MacKay, and the proposer.

Editorial Note. Ayres's solution used the theorem that in an orthocentric tetrahedron the mid-points of the edges and the feet of the common perpendiculars to opposite edges lie on a sphere with its center at the centroid of the tetrahedron. See this MONTHLY, 1934, p. 502, and 1935, (b) p. 53.

The theorem in II regarding any tetrahedron is easily proved by taking the centroid as origin of vectors to the vertices. Using the relation $\mathbf{a}_i+\mathbf{a}_j=-(\mathbf{a}_k+\mathbf{a}_l)$, we may write the equation of the Monge plane through the mid-point of A_iA_j and that of the perpendicular bisector of A_kA_l, respectively, as

$$(a)\quad \left(\mathbf{r}+\frac{\mathbf{a}_k+\mathbf{a}_l}{2}\right)\cdot(\mathbf{a}_k-\mathbf{a}_l)=0; \qquad (b)\quad \left(\mathbf{r}-\frac{\mathbf{a}_k+\mathbf{a}_l}{2}\right)\cdot(\mathbf{a}_k-\mathbf{a}_l)=0.$$

We know that the six equations (b) are satisfied by one and only one vector $\bar{\mathbf{r}}$, the vector to the circumcenter; hence the six equations (a) have one and only one solution, $-\bar{\mathbf{r}}$. This proves the theorem.

We may now use this theorem to prove in another way some of the results in III. Denote the middle points of the edges through A_4 by M_{41}, M_{42}, M_{43}; and the orthocenter of this M triangle by H_4'; and the orthocenter of the face opposite A_4 by H_4. The M triangle is similar to the face triangle with A_4 as center of similitude, and hence H_4' bisects A_4H_4. The Monge planes for the edges through A_4 intersect in $H_4'Q_4=q_4$, where Q_4 is the projection of H_4' on the face opposite A_4. Hence Q_4 is the mid-point of H_4P_4, where P_4 is the projection of A_4 on the opposite face. Hence q_1, q_2, q_3, q_4 intersect in the Monge point H. It is easily seen as in III that each of the four p lines intersects each of the four altitudes, the h lines. Consider the trihedral angle formed by the edges through A_4. The altitude planes of this trihedral, the planes through the edges perpendicular to the corresponding opposite faces, intersect in a straight line o, the orthocentric line for the trihedral. Also the altitudes h_1, h_2, h_3, intersect o_4, and, obviously, h_4 does so also. Thus the altitudes are each cut by each of the eight lines in the two systems p and o. If we suppose that no two of the altitudes intersect, they belong to a system of generators of a ruled surface of the second order; and, since the altitudes are not parallel to the same plane, this surface must be an hyperboloid of one sheet having a second system of generators to which the eight lines p and o belong. Thus no two of these latter eight lines intersect. The lines p_4 and h_4 meet at infinity on the surface and determine a plane tangent at that point both to the surface and to the asymptotic cone. This plane being tangent to the cone along an element passing through the center of the surface, the element must be parallel to p_4 and h_4 and half way between them. Hence this element is q_4. Thus the four q lines are elements of the cone passing through the center of the surface, and the latter is the Monge point H.

If two altitudes intersect, the surface degenerates into two planes and then the other two altitudes intersect. This may be proved directly in a simple manner. Returning to the non-degenerate surface, the face $A_1A_2A_3$ cuts it in a conic K_4 passing through its vertices and H_4. It is easily proved that a conic passing through the vertices and orthocenter of a triangle must be an equilateral hyper-

bola; and, conversely, if any three points are taken on an equilateral hyperbola the orthocenter of the triangle lies also on the curve. The parallel plane through H cuts out a similar hyperbola with asymptotes parallel, respectively, to those of the first section. Since q_4 is an element of the cone perpendicular to the face considered, we see that the cone is of a special type, containing three elements forming an orthogonal system. The reference cited in III shows that, if any cone of the second order has three elements forming a mutually orthogonal system, the orthocentric line of the trihedral formed by any three elements is also an element. Conversely, if any cone of the second order has one system of four elements such as the latter, it has an infinite number of such systems, and an infinite number of systems of three mutually orthogonal elements. A very simple geometrical proof is given.

3698 [1934, 453]. *Proposed by Clifford W. Mendel and Gaylord M. Merriman, The University of Cincinnati.*

Prove the projective theorem: Given, in a projective plane, a complete quadrangle and a line l not passing through any vertex. On each side of the quadrangle consider the point which is the harmonic conjugate, with respect to the vertices on this side, of the point of intersection of the side with l. Among the six points thus determined call two points *opposite* if they lie on the opposite sides of the quadrangle. Prove that the three lines joining pairs of opposite points are concurrent, and that on each of these lines this point of concurrency is the harmonic conjugate, with respect to the pair of opposite points determining the line, of the point of intersection of the line with l.

I. *Solution by A. Pelletier, Montreal, Canada.*

Let the four vertices of the quadrilateral be denoted by A_i; the intersection of l with A_iA_j by M_{ij}; and the harmonic conjugate of M_{ij} with respect to A_i and A_j by M_{ij}'. Then the two harmonic ranges A_i, M_{ij}, A_j, $M_{i\,j}'$ and A_i, M_{ik}, A_k, M_{ik}' have the point A_i in common; therefore $M_{ij}M_{ik}$, A_jA_k, $M_{ij}'M_{ik}'$ meet in M_{jk} on l. By the same reasoning $M_{lj}'M_{lk}'$ passes through M_{jk}. Similarly, we prove that $M_{ji}'M_{jl}'$ and $M_{ki}'M_{kl}'$ pass through M_{il}. Hence M_{ij}', M_{ik}', M_{lj}', M_{lk}' are the vertices of a complete quadrilateral with the two diagonal points M_{jk} and M_{il} on l. If $M_{ij}'M_{kl}'$ is cut by l in K, then it is cut by $M_{ik}'M_{lj}'$ and $M_{il}'M_{kj}'$ in K', where M_{ij}', K', M_{kl}', K are harmonic. This proves the theorem.

II. *Solution by S. Vatriquant, Brussels, Belgium.*

A simple homographic transformation in the plane of the figure may be used to carry the line l to infinity. For the transformed figure the statement of the problem becomes a known elementary theorem: The straight line joining the mid-points of the two diagonals of a quadrilateral and the two joining the mid-points of opposite sides are concurrent in the mid-point of each of the three segments. Since the inverse transformation does not alter harmonic properties the theorem of the problem is true.

Solved also by D. C. Duncan, A. S. Householder, J. E. La Fon, and the proposer.

Editorial Note. The remaining solutions used coordinates. A proof similar to I may be obtained by using the theorem that any two tangents to a conic are cut by a variable tangent in two projective ranges of points. A conic is tangent to A_iA_j, A_jA_k, A_kA_l, A_lA_i, and l. Since A_i, M_{ij}', A_j, M_{ij}, and A_l, M_{lk}', A_k, M_{lk}, are harmonic and therefore projective, $M_{ij}'M_{kl}'$ must be also a tangent. So also is $M_{il}'M_{jk}'$. If $M_{ij}'M_{kl}'$ is cut by $M_{il}'M_{jk}'$ and l in K' and K, respectively then M_{ij}', K', M_{kl}', K are harmonic. Using another set of five lines we see that $M_{ik}'M_{jl}'$ must also pass through K'. In the proposers' proof, using homogeneous coordinates, was the remark that the six conjugate points are the vertices of a Brianchon hexagon, and that the lines joining them in proper order circumscribe a conic. It was also stated that if l is at infinity there results the elementary geometry theorem which is the basis for solution II.

A projective transformation of the kind used in II was called an homology on page 118 in the note to solution of 3650 [1935, 115]. Here we take the axis a parallel to l and choose a convenient point for the pole F; the transformation is then determined by the fact that l goes to infinity and by the construction there described for finding the transform of a given point. The truth of the elementary theorem is obvious on considering the centroid of equal weights at the four vertices in three simple ways. For the completed quadrilateral in the elementary theorem it is also known that the mid-points of the three diagonals lie in a straight line. This will lead to an additional harmonic relation for the original figure.

3699 [1934, 521]. *Proposed by V. Thébault, Le Mans, France.*

In the tetrahedron $ABCD$ let G be its centroid and set $a=BC$, $b=CA$, $c=AB$, $a'=DA$, $b'=DB$, $c'=DC$. If the tetrahedron $GABC$ is trirectangular at G, show that

$$\text{(a)} \qquad 2(b^2+c^2) = a^2+a'^2, \quad 2(c^2+a^2) = b^2+b'^2,$$
$$2(a^2+b^2) = c^2+c'^2.$$

(b) The edges DA, DB, DC are equal, respectively, to the medians of triangle $A_2B_2C_2$, the anticomplement of ABC; and the tetrahedron $DA_2B_2C_2$ has equal opposite edges.

Note by the Editors. Anticomplement means here that the side A_2B_2 passes through C and is parallel to AB; and, similarly, for the other two sides.

I. *Solution by Roy MacKay, Eastern New Mexico Junior College.*

Let M, N be the midpoints of the edges AD, BC, respectively. The bimedian MN is bisected at G and since GN is a median to the hypotenuse of the right triangle GBC, $MG=GN=a/2$. Moreover, since AG is perpendicular to the plane of the triangle GBC, angle AGM is right and by the Pythagorean theorem

$(a'/2)^2=(a/2)^2+AG^2$ or $4AG^2=a'^2-a^2$. Again, from the right triangles AGC, CGB, and AGB, $2(b^2+c^2)=2(CG^2+2AG^2+BG^2)=2a^2+4AG^2=a^2+a'^2$. Similarly the other equalities of (a) may be proved.

Now the vertices of the triangle ABC are the midpoints of the sides of the anticomplement triangle and the sides of the latter are double the corresponding sides of the former. Since the square of a median of a triangle is half the sum of the squares of the sides concurrent with it diminished by one-fourth the square of the third side,

$$AA_2^2 = \tfrac{1}{2}A_2B_2^2 + \tfrac{1}{2}A_2C_2^2 - \tfrac{1}{4}B_2C_2^2 = 2(c^2 + b^2) - a^2 = a'^2, \quad AA_2 = a'.$$

Similarly $BB_2=b'$ and $CC_2=c'$.

Finally, denote the edges DA, DB, DC by a'', b'', c'', respectively. Then, since AD is a median of the triangle DB_2C_2,

$$a'^2 = \tfrac{1}{2}b''^2 + \tfrac{1}{2}c''^2 - \tfrac{1}{4}(2a)^2, \quad \text{or} \quad b''^2 + c''^2 = 2(a'^2 + a^2) = 4(b^2 + c^2).$$

Similarly, $c''^2+a''^2=4(c^2+a^2)$ and $a''^2+b''^2=4(a^2+b^2)$; whence

$$a'' = 2a, \quad b'' = 2b, \quad c'' = 2c.$$

Therefore opposite edges of the tetrahedron $DA_2B_2C_2$ are equal.

II. *Solution by J. H. Nelson, A. & M. College of Texas, College Station, Texas.*

The solution of this problem is readily obtained by the use of Vector Analysis. Let the origin of vectors be at G, and let GA, GB, GC, GD be represented by the vectors α, β, γ, δ, respectively. Then

(1) $$\alpha + \beta + \gamma + \delta = 0,$$

(2) $$\alpha\cdot\beta = \beta\cdot\gamma = \gamma\cdot\alpha = 0.$$

(a) By the use of (1), the first equation of (a) may be written

$$2(\gamma - \alpha)^2 + 2(\alpha - \beta)^2 = (\beta - \gamma)^2 + (2\alpha + \beta + \gamma)^2.$$

Expanding and dropping terms seen by (2) to be zero:

$$2(\gamma^2 + \alpha^2) + 2(\alpha^2 + \beta^2) = \beta^2 + \gamma^2 + 4\alpha^2 + \beta^2 + \gamma^2.$$

Hence this equation and, by analogy, the other similar equations are seen to be true. (b) The median AA_2 of triangle $A_2B_2C_2$ may be written vectorially

$$2[\tfrac{1}{2}(\beta + \gamma) - \alpha] = -2\alpha + \beta + \gamma.$$

But $a'=|\alpha-\delta|=|2\alpha+\beta+\gamma|=|-2\alpha+\beta+\gamma|$; where the last equality is true since α, β, γ are mutually perpendicular. Hence a' is equal to one median of triangle $A_2B_2C_2$. By analogy b' and c' are equal to the other two medians respectively. In vector notation,

$$DA_2 = |\gamma - \delta + (\beta - \alpha)| = |2\beta + 2\gamma|.$$

Also $B_2C_2=|2(\beta+\gamma)|$, and therefore $DA_2=B_2C_2$. By analogy $DB_2=A_2C_2$ and $DC_2=A_2B_2$; and the tetrahedron $DA_2B_2C_2$ has equal opposite edges.

Solved also by A. D. Bradley, R. Goormaghtigh, A. S. Householder, J. E. La Fon, A. Pelletier, F. Underwood, and the proposer.

Editorial Note. The solution by Householder made use of vectors. In Goormaghtigh's solution it was noted that the first part of the problem had been previously given by Neuberg, Mathesis, 1922, p. 34.

3701 [1934, 521]. *Proposed by Harry Langman, Brooklyn, N. Y.*

If the C's denote binomial coefficients, show that

$$(a)\qquad \sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{i+1}=\frac{1}{n},$$

$$(b)\qquad n\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{(i+1)^2}=\sum_{i=1}^{n}\frac{1}{i}.$$

I. *Solution by J. F. Locke, State Teachers College, Memphis, Tennessee.*

For the proof of (a) we have

$$\frac{1}{n}=\int_0^1(1-x)^{n-1}dx=\int_0^1\sum_{i=0}^{n-1}(-1)^i{}_{n-1}C_ix^idx=\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{i+1}.$$

For (b) we have first

$$\sum_{i=0}^{n-1}(1-x)^i=\frac{1-(1-x)^n}{x}=\sum_{i=0}^{n-1}(-1)^i{}_nC_{i+1}x^i=n\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_ix^i}{i+1}.$$

Integration of each side from 0 to 1 gives

$$\sum_{i=1}^{n}\frac{1}{i}=n\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{(i+1)^2}.$$

II. *Solution by C. W. Trigg, Cumnock College, Los Angeles.*

Denote the left side of (a) by S; then

$$(1)\qquad \begin{aligned}1-nS&=1+\sum_{i=0}^{n-1}(-1)^{i+1}{}_nC_{i+1}=\sum_{j=0}^{n}(-1)^j{}_nC_j,\\&=(1-1)^n=0.\end{aligned}$$

This proves (a). Denote the left side of (b) by $f(n)$, then we have

$$(2)\qquad \begin{aligned}f(n)-f(n-1)&=\sum_{i=0}^{n-2}(-1)^i\frac{{}_{n-1}C_i}{i+1}+(-1)^{n-1}\frac{1}{n},\\&=\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{i+1}=\frac{1}{n},\end{aligned}$$

where the identity

$$n\,{}_{n-1}C_i - (n-1)\,{}_{n-2}C_i = (i+1)\,{}_{n-1}C_i \tag{3}$$

has been used for reduction, and also (a). We now have

$$f(n) = \sum_{j=1}^{n} [f(j) - f(j-1)] = \sum_{i=1}^{n} \frac{1}{i}, \tag{4}$$

where $f(1)=1$ and $f(0)=0$. This proves (b).

See School Science and Mathematics, vol. 33, p. 99 (1933).

Solved also by Frank Ayres, Jr., J. A. Bullard, Hansraj Gupta, Roy MacKay, F. L. Manning, E. G. Olds, A. Ollivier, A. Pelletier, E. P. Starke, B. Townes and F. Underwood.

Editorial Note. The remaining solutions were somewhat similar to the above, some using two integrations for (b) and others using mathematical induction. It may be of interest to note that the right side of (b) occurs in the solution of 3652 [1935, 118]; and on page 123 of that solution it is expressed in a form different from (b).

The solution by Olds extended the formulas of the problem, and his results will be indicated briefly. Set

$$(n, k) = n\sum_{i=0}^{n-1} (-1)^i \frac{{}_{n-1}C_i}{(i+1)^k}, \tag{5}$$

$$(0, k) = 0,\ (1, k) = 1;\ \text{and, for } n \geqq 2,\ (n, 0) = 0.$$

By use of the identity (3) in II and by similar methods, we easily find the formula

$$(n, k) - (n-1, k) = \frac{1}{n}(n, k-1). \tag{6}$$

Then by summation of both sides of (6), we have

$$(n, k) = \sum_{j=1}^{n} \frac{1}{j}(j, k-1). \tag{7}$$

Hence $(n, 1)=1$, and using this result (7) gives

$$(n, 2) = \sum_{i=1}^{n} \frac{1}{i}.$$

These are (a) and (b) of the problem. We now readily obtain by repeated use of (7) the value of (n, k), or

$$n\sum_{i=0}^{n-1} (-1)^i \frac{{}_{n-1}C_i}{(i+1)^k} = \sum_{i_1=1}^{n} \frac{1}{i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2} \sum_{i_3=1}^{i_2} \frac{1}{i_3} \cdots \sum_{i_{k-1}=1}^{i_{k-2}} \frac{1}{i_{k-1}},$$

where the first summation is with respect to i_{k-1}. This includes the results of the problem.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The following lectures were given at the Galois Institute of Mathematics, Long Island University, during December 1935: by Professor Felix Bernstein, formerly of the University of Göttingen, now of Columbia University, "The Pythagoreans and the Pythagorean Theorem"; by Professor R. Courant, also of Göttingen, now at New York University, "The geometrical theory of functions of a complex variable."

Professor W. W. Hart has resigned from his position at the University of Wisconsin, effective at the end of the first semester of this year. He has been appointed to an honorary lectureship in Educational Methods at the University of Wisconsin for the second semester of the current year. He will hereafter devote himself to writing texts for and pertaining to secondary and elementary mathematics.

Professor J. R. Musselman is on leave of absence from Western Reserve University for the second semester of the current year. He will spend some weeks at the Institute for Advanced Study, following this with travel in Belgium, Denmark, and Norway, including the International Congress of Mathematicians at Oslo.

Professor R. G. D. Richardson, professor of mathematics and dean of the Graduate School of Brown University, has been appointed official representative of the University, which is president of the American Association of Universities.

The following appointments to instructorships have been announced:
University of Chicago: Dr. Ralph Hull
Kent State University: Dr. Foster Brooks.

Professor William Elwood Byerly, professor emeritus of mathematics at Harvard University, died at Swarthmore, Pennsylvania, on December 20, 1935, in his eighty-seventh year. He was the first holder of the doctorate from Harvard University, taking the degree in 1873. He taught at Cornell University from 1873 to 1876 and at Harvard University from 1876 to 1913.

Professor T. Hayashi, professor emeritus of mathematics, Tohoku Imperial University, died October 4, 1935.

Professor Emilie N. Martin, professor emeritus at Mount Holyoke College, died February 8, 1936. She had taught mathematics at Mount Holyoke for thirty-two years. She was a charter member of the Mathematical Association.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Summer Meeting, Harvard University, Aug. 31, 1936.

Twenty-first Annual Meeting, Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, May 8–9.
INDIANA, North Manchester, May.
IOWA, Iowa City, April 3–4.
KANSAS, March.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov. 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S. C., April 17–18.
SOUTHERN CALIFORNIA, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, May 2.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 4, APRIL

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE PH.D. DEGREE AND MATHEMATICAL RESEARCH*

By R. G. D. RICHARDSON, Brown University

Recommendations regarding the training and utilization of advanced students of mathematics must be based on specific information concerning the present situation and on a knowledge of how the past has contributed to its upbuilding. No group of persons can be entirely certain that sound deductions are possible from data as incomplete as those now available. However, queries often raised regarding quality of personnel and regarding supply and demand can be answered with considerable assurance. In so complicated a problem any light that can be shed is undoubtedly welcome, and it is proposed in this report to set forth a variety of facts and to venture partial and tentative answers to certain questions.

We propose questions such as the following: How many doctor's degrees have been conferred on American mathematicians here in the United States and Canada? How many in Europe? What proportion of the present teachers of mathematics in colleges and universities have such degrees? Is there a sufficient number of competent students of mathematics now being enlisted and subjected to proper training by our graduate schools? What percentage of the Ph.D.'s have published considerable research? Is the record of publication improving with the newer crop of Ph.D.'s? What universities have the distinction of the largest average amount of publication by those to whom they have granted degrees? How does the record of the National Research Fellows stand?

Beginning with 1900, the Bulletin of the American Mathematical Society has made a practice of publishing an annual list of the doctor's degrees conferred in mathematics. In addition, each university has, on request, been kind enough recently to furnish lists of all doctor's degrees in mathematics conferred since the beginning of its graduate work and including the year 1934. From these sources† accurate data have been assembled. The corresponding information regarding Americans and Canadians taking degrees abroad is not so complete.

Each year, beginning with 1890, the Bulletin of the American Mathematical Society has printed an *Annual List of Papers Read Before the Society and Subsequently Published*. These lists have been drawn on for the period 1890–1933, and other sources for the period before 1890. It has been possible to make a complete tabulation for the sixty years ending with 1933,‡ giving the number of such papers and the number of pages printed. It should be pointed out, however, that papers of American authorship not presented to the Society are

* A report to the Commission on the Training and Utilization of Advanced Students of Mathematics presented at its session on December 31, 1935.

† Beginning with 1934, a list of doctors and thesis titles in all fields of advanced study is being published annually by H. W. Wilson Company under the title *Doctoral Dissertations Accepted by American Universities* compiled for the National Research Council and the American Council of Learned Societies by the Association of Research Libraries.

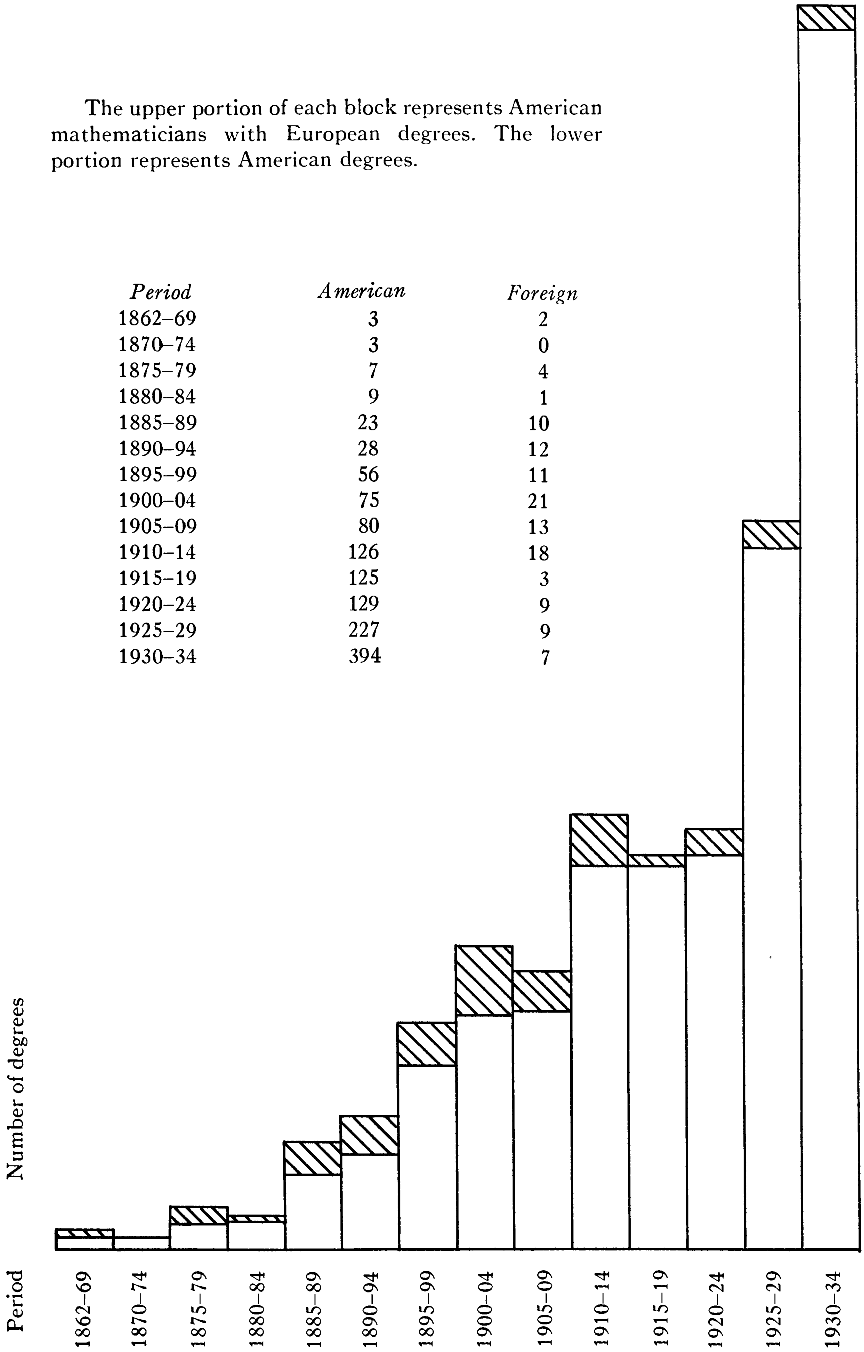
‡ For the National Research Fellows the tabulation has been extended to include 1934.

GRAPH I

DISTRIBUTION OF DEGREES CONFERRED IN MATHEMATICS 1862–1934

The upper portion of each block represents American mathematicians with European degrees. The lower portion represents American degrees.

Period	*American*	*Foreign*
1862–69	3	2
1870–74	3	0
1875–79	7	4
1880–84	9	1
1885–89	23	10
1890–94	28	12
1895–99	56	11
1900–04	75	21
1905–09	80	13
1910–14	126	18
1915–19	125	3
1920–24	129	9
1925–29	227	9
1930–34	394	7



omitted from the tabulations; and this is true whether these papers appeared in American or foreign journals. Some mathematicians, especially among the more eminent, publish papers not read before the Society; but a study of the total publication of mathematical research for a typical year (1932) shows that only about 22% of such publication (1,158 pages out of a total of 5,290) is not included in our data. The data assembled seem, therefore, fairly adequate for the purpose in hand.

For each person it has been possible to compute an index of publication which is the annual average, measured in pages, of printed research during the period subsequent to receiving the degree, but in no case exceeding twenty-five years. The average of the indices for the group of doctors beginning with 1862 is 4.73, which then establishes a norm by which comparisons can be made. Otherwise stated, a doctor publishes yearly, on the average, 4.73 pages of research which has been presented to the Society. (If the year 1932 referred to above is a fair sample, this figure would be increased to 6 pages by including papers not presented to the Society. And it should be expressly understood that the list includes only papers published in research journals, and that separate monographs and books, which are so important in the development of research, are excluded from the count.) Reckoning eight dollars as the cost of printing a page, the average annual cost of publication per doctor is approximately fifty dollars, a relatively small item in the cost of production of research.

As a rough check on this figure of 4.73 pages, we note that the American Journal, Annals, Bulletin, and Transactions print about 3,500 pages of research annually and that other journals add perhaps 1,600 pages more. This makes a total of 5,100 pages as against approximately 1,200 doctors.

Number of Ph.D.'s Conferred in the United States and Canada

A careful investigation indicates that the number of Ph.D.'s in mathematics conferred by institutions in the United States and Canada during the period 1862–1934 is 1,286, 168 of which were conferred on women. Graph I exhibits the number of degrees conferred during five-year periods, this number following in general an exponential curve, though interrupted during the world war. The totals by years for the period 1930–34 are 84, 79, 69, 71, 92, respectively.

The figures naturally vary somewhat with the inclusion or exclusion of degrees taken in applied fields such as mathematical physics, mathematical astronomy, mechanics, theory of statistics, etc.; but they are accurate enough for our purposes.

Besides this number of American degrees, the information available indicates that there have been, during the period 1862–1930, 114 degrees conferred by foreign universities on mathematicians* who have been active in America.†

* The University of Göttingen accounts for 34 (Henry Blumberg, Maxime Bôcher, Oskar Bolza, Ann L. Bosworth, W. D. Cairns, A. R. Crathorne, H. B. Curry, Edgar Dehn, T. M. Focke, D. C. Gillespie, Charles Haseman, M. W. Haskell, E. R. Hedrick, W. A. Hurwitz, Dunham Jackson, O. D. Kellogg, A. J. Kempner, S. D. Killam, Luise Lange, Heinrich Maschke, Max Mason,

Included in this list are 40 mathematicians of European birth who subsequently took up residence in America and thoroughly established themselves by residence of several years.

Yale University, the first institution in America to confer the Ph.D. degree in course (1861), awarded the degree in mathematics in 1862 to J. H. Worrall. William Watson, later professor at Harvard University, received from Jena in 1862 the first foreign degree of which we have record. The earliest degrees conferred on women were granted to Winifred H. Edgerton by Columbia University in 1886 and to Charlotte A. Scott by the University of London in 1885.

As will be noted from Table I, more than one-sixth of the 1,286 degrees conferred in America have been awarded by the University of Chicago alone. Six institutions—Chicago, together with Cornell, Harvard,* Illinois, Johns Hopkins, and Yale—are responsible for more than half. Of the remaining 53 institutions, there have been 7 which have each conferred 25 or more degrees (California, Clark, Columbia, Michigan, Pennsylvania, Princeton, and Wisconsin) and 14 that have each conferred from 10 to 24 degrees (Brown, Bryn Mawr, Catholic, Cincinnati, Indiana, State University of Iowa, Massachusetts Institute, Minnesota, Missouri, Ohio State, Pittsburgh, Rice, Texas, and Virginia). The remaining 94 degrees were given by 32 institutions. During the past ten years only one-half the universities on this list (29 out of 59) have conferred five or more degrees and can thus be considered an important present factor. In the five years 1930–34, Chicago maintains its lead in the number of degrees conferred with 52, while Michigan is second with 35, followed by Cornell and Harvard each with 28.

C. A. Noble, L. W. Reid, W. B. Smith, Virgil Snyder, Elijah Swift, J. H. Tanner, E. J. Townsend, E. B. Van Vleck, Paul Wernicke, W. D. A. Westfall, H. S. White, Mary F. Winston, F. S. Woods); the others are distributed as follows: Berlin 4 (Harris Hancock, Ludwik Silberstein, A. G. Webster, E. J. Wilczynski), Bonn 2 (J. L. Coolidge, W. C. Graustein), Bordeaux 1 (P. L. Saurel), Cambridge 4 (E. W. Brown, R. C. Maclaurin, Frank Morley, H. M. Tory), Charlottenburg 1 (E. P. Wigner), Dublin 2 (James McMahon, J. L. Synge), Edinburgh 2 (Alexander Macfarlane, J. H. M. Wedderburn), Erlangen 4 (Henry Benner, W. F. Osgood, E. D. Roe, H. W. Tyler), Fribourg 1 (R. F. Schnepp), Ghent 1 (N. A. Court), Greifswald 1 (F. J. Dohmen), Heidelberg 2 (R. R. Fleet, T. E. Hart), Jena 1 (William Watson), Kazan 1 (G. Y. Rainich), Kiel 1 (Arthur Schultze), Königsberg 1 (J. B. Chittenden), Leiden 1 (D. J. Struik), Leipzig 9 (H. F. Blichfeldt, C. L. Bouton, H. B. Fine, A. G. Hall, J. M. Page, B. O. Peirce, D. A. Rothrock, W. E. Story, A. F. Wintner), London 1 (Charlotte A. Scott), Munich 9 (G. N. Armstrong, C. H. Ashton, P. S. Epstein, L. A. Howland, W. W. Küstermann, H. W. March, G. W. Myers, W. S. Seidel, Edwin R. Smith), Oslo 1 (Oystein Ore), Oxford 2 (W. R. Burwell, F. V. Morley), Paris 2 (Pierre Boutroux, Jacques Chapelon), Rome 1 (Oscar Zariski), St. Anthony 1 (J. A. Caparó y Peréz), St. Petersburg 4 (Alexander Chessin, J. A. Shohat, J. D. Tamarkin, J. V. Uspensky), Salzburg 1 (Eberhard Hopf), Stockholm 1 (Einar Hille), Strasbourg 5 (R. C. Archibald, J. W. Bradshaw, Myrtie Collier, E. V. Huntington, L. C. Karpinski), Szeged 1 (Tibor Radó), Tübingen 2 (S. C. Davisson, J. E. Manchester), Upsala 1 (T. H. Gronwall), Utrecht 1 (Hermance Mullemeister), Vienna 3 (F. W. Doermann, J. L. Gibson, James Pierpont), and Zurich 5 (W. H. Butts, S. R. Epsteen, Lulu Hofmann, William Marshall, A. B. Pierce).

† When persons have taken two doctors' degrees, the first only is listed.

* Throughout this paper, statistics for Harvard include Radcliffe.

TABLE I

NUMBER OF PH.D. DEGREES IN MATHEMATICS CONFERRED BY AMERICAN UNIVERSITIES

Institution	1862–69	70–79	80–89	90–94	95–99	00–04	05–09	10–14	15–19	20–24	25–29	30–34	Total
Boston U.								3					3
Brown											1	13	14
Bryn Mawr				1	1	2		1	1	2	3	2	13
Calif. Inst. of Tech.											5	3	8
California						2	1	5	7	8	11	11	45
Catholic									3		3	9	15
Chicago					4	15	20	25	31	41	49	52	237
Cincinnati								1		1	1	7	10
Clark				4	7	7	2	4	2				26
U. of Colorado							1					1	2
Columbia U.			4	1	3	6	5	11	10	5	6	11	62
Cornell U.		1	3	1	1	7	8	8	8	8	16	28	89
Cumberland				1									1
Dartmouth		1											1
Duke												3	3
Fordham											2		2
Geo. Washington				1								2	3
Harvard and Radcliffe		2	3		5	5	7	11	15	11	16	28	103
Haverford						1							1
Illinois						1		6	8	10	25	23	73
Indiana								2	3		1	5	11
Iowa State												1	1
State U. of Iowa										3	5	9	17
Johns Hopkins		2	12	7	9	8	10	14	7	8	15	11	103
Kansas					1			2			2	1	6
Kentucky												1	1
Lafayette			1	1									2
Marquette												2	2
Mass. Inst. of Tech.										1	4	12	17
Michigan								4	5	3	8	35	55
Minnesota										2	4	4	10
Missouri								1	1	2	4	2	10
Moravian									1				1
Nebraska					2	1					1		4
New York									1		2	2	5
U. of N. Carolina						1						1	2
Ohio State U.												16	16
Ohio Wesleyan				1									1
Otterbein				1									1
U. of Pennsylvania				2	7	3	6	5	7	2	9	11	52
Pittsburgh											1	12	13
Princeton			2	1		1	2	8	3	9	4	18	48
Purdue				1									1
Rensselaer										1		2	3
Rice									1	1	6	5	13
St. Louis U.												2	2
Stanford						1		1			2	3	7
Syracuse U.			1	1		1		2	1	3			9
U. of Texas									1	1	5	4	11
U. of Toronto									1		1	6	8
Tulane				1									1
Vanderbilt			1				1						2
U. of Virginia			1		3	1	3		1			2	11
Washington U.				1								2	3
U. of Washington											2	3	5
West Virginia U.												2	2
Wisconsin					2	1	2	1	3	5	5	20	39
Wooster			1										1
Yale	3	4	3	2	11	11	12	11	4	2	8	8	79
Totals	3	10	32	28	56	75	80	126	125	129	227	395	1286

In 1933 a committee of the American Council on Education made a study of institutions conferring Ph.D. degrees. As the result of a ballot in each field, a group of institutions was designated as approved, and from that list the most distinguished were singled out by stars. Of the 395 degrees in mathematics granted in 1930–34, 157 were conferred by institutions starred in this field, 187 by other institutions approved in the field, and 51 (about 12%) by institutions not on the approved list.

Of the 32 institutions now members of the Association of American Universities, 28 have conferred the degree in mathematics during the five years 1930–34, the total being 348, or 88% of all such degrees conferred.

Foreign Influence on American Mathematics

Grateful acknowledgment should be made of America's tremendous debt to foreign universities. The original inspiration for work in a large number of the fields now cultivated here had its source on the other side of the Atlantic, especially in Germany. As will be noted from Graph I, the number of foreign degrees during the period 1885–1914 was a considerable proportion of the whole; not only is this group important numerically but, as will be seen later in this discussion, it is significant scientifically. During the period in question, many of the ablest of our students went abroad to obtain degrees. Harvard in particular followed a very generous policy in awarding fellowships to persons wishing to take degrees in Europe. Up to and including 1913, Harvard sent abroad 26 mathematical fellows, of whom 16 took European degrees.*

Beginning with 1914, however, the number of Americans taking their degrees abroad declined rapidly. By that time the custom had been established of taking degrees in this country, even when the students wished to pursue further study abroad. The number of American mathematicians attending foreign universities in recent years is by no means so great as it was a quarter of a century ago, but scholars in this country are still indebted to universities in Europe for much stimulation. There seems to be no doubt that by our connection with foreign countries we have avoided and are avoiding the inbreeding which is apt to lead to sterility.

During the past decade the international fellowships, supported by the Rockefeller Foundation, have been of great importance in establishing contacts between American and European mathematicians. Other recent factors are the importation of some European scholars through the founding of the Institute for Advanced Study, and the recent infiltration of other displaced German scholars into our mathematical group. Besides the 40 foreign-born mathematicians referred to above as having entered the United States and Canada before 1930, there have been approximately 20 mathematicians who have more recently transferred to these countries from Europe, thus further raising the volume of our research activity.

* This includes B. O. Peirce and A. G. Webster whose fields bordered on physics.

While the tide of students venturing forth to Europe has abated since the world war, the number of foreign students arriving on our shores has rapidly increased, drawn partly by the fame of our leaders in scholarship and partly by the hope that a career might open for them here. Leadership in many fields is definitely passing to America, and the tides of students to and from Europe need to be regulated by an informed opinion with a view to best serving the cause of American scholarship.

Number of College Teachers of Mathematics

Data collected in the autumn of 1935, based on information furnished by the institutions themselves, indicate, as tabulated in Table II, that the number of persons teaching mathematics in colleges, universities, junior colleges, and degree-granting normal colleges in the United States (with its outlying possessions) and Canada is approximately 4,500. This includes some persons who are teaching descriptive geometry, mechanics, and methods in mathematics, as well as some who are teaching part time or who are largely in administrative work or who are emeriti; but on a conservative estimate, 4,000 persons are actually engaged full time in the teaching of mathematics of college freshman grade or higher. Similar figures were collected in 1932, and there seems to have been a considerable increase in the interim, due chiefly to the growth of junior colleges.

Table II is a statistical study by states of the number of teachers of mathematics in junior colleges, teachers colleges, and other colleges and universities, of the number of men and women teachers, of the number of teachers holding doctor's degrees, and of the number of those who are members of the American Mathematical Society or of the Mathematical Association of America or of both. The best information obtainable indicates that probably somewhat less than 1,300 of the present teachers of mathematics have the Ph.D. degree! Many of the 1,400* listed as having obtained degrees in America or abroad are deceased or have entered fields of work such as government service, banking, or industry. It should be remarked also that mathematics has furnished more than its share of administrative officers to colleges and universities. There have been some doctors who have drifted out of mathematics into other fields of science, and probably more who have correspondingly drifted in.

Of the 4,444 teachers of mathematics listed, 1,292, or 29%, hold the degree of Ph.D., while slightly less (1,263) are members of the American Mathematical Society and slightly more (1,333) are members of the Mathematical Association. In the Summary at the end of Table II the states have been grouped by sections of the country, and we note that 35% of the teachers in the northeast section from Illinois to Maine hold the doctor's degree. Only about 22% of those in the south central states from Kentucky to Texas hold the doctor's degree. In the remainder of the country about 26% hold that degree.

* This figure does not include the 1935 crop of Ph.D.'s, most of whom are doubtless included in Table II.

TABLE II

STATISTICS REGARDING TEACHERS OF MATHEMATICS IN AMERICAN AND CANADIAN COLLEGES AND UNIVERSITIES, 1935

	Number of Teachers						Holders of Ph.D.		Membership in					
	(a)	(b)	(c)	(d)	(e)	(f)	No.	%	A.M.S.	%	M.A.A.	%	Both	%
Alabama	66	5	11	50	48	18	13	20	7	11	13	20	5	7
Alaska	3	3	—	—	3	0	0	0	0	0	0	0	0	0
Arizona	10	2	2	6	9	1	6	60	4	40	5	50	3	30
Arkansas	44	9	5	30	34	10	13	29	9	20	5	11	3	7
California	262	126	17	119	231	31	64	24	60	23	55	21	40	15
Canal Zone	2	2	—	—	2	0	1	50	0	0	0	0	0	0
Colorado	60	6	10	44	48	12	17	28	14	23	23	38	13	22
Connecticut	45	7	3	35	38	7	27	60	26	58	22	49	21	47
Delaware	7	—	—	7	6	1	2	28	5	71	4	57	3	43
Dist. of Columbia	48	13	8	27	30	18	18	37	16	33	11	23	8	17
Florida	38	5	—	33	37	1	12	31	10	26	13	34	9	24
Georgia	84	23	2	59	70	14	20	24	16	19	23	27	9	11
Hawaiian Islands	5	—	—	5	4	1	0	0	1	20	0	0	0	0
Idaho	19	9	—	10	19	0	3	16	3	16	1	5	1	5
Illinois	222	41	16	165	183	39	87	39	81	36	80	36	53	24
Indiana	96	4	8	84	87	9	33	34	25	26	42	44	15	16
Iowa	128	42	4	82	104	24	34	26	33	26	38	30	22	17
Kansas	92	24	10	58	71	21	22	24	25	27	41	44	21	23
Kentucky	77	22	19	36	56	21	19	24	14	18	28	36	12	15
Louisiana	49	7	4	38	37	12	11	22	9	18	21	43	8	16
Maine	25	2	1	22	23	2	6	24	6	24	10	40	5	20
Maryland	92	3	2	87	80	12	29	31	28	30	35	38	20	22
Massachusetts	170	14	16	140	139	31	53	31	66	39	43	25	38	22
Michigan	157	27	15	115	144	13	49	31	44	28	52	33	32	20
Minnesota	83	19	10	54	65	18	26	31	16	19	31	37	13	16
Mississippi	52	22	5	25	37	15	8	15	7	13	12	23	5	10
Missouri	122	31	21	70	97	25	30	24	27	22	29	24	12	10
Montana	19	2	1	16	18	1	4	21	7	37	5	26	4	21
Nebraska	55	5	10	40	45	10	14	25	13	24	19	34	12	22
Nevada	3	—	—	3	2	1	1	33	2	67	1	33	1	33
New Hampshire	28	1	3	24	26	2	13	46	11	39	11	39	9	32
New Jersey	107	17	7	83	102	5	35	33	44	41	29	27	24	22
New Mexico	20	3	4	13	19	1	6	30	8	40	7	35	5	25
New York	402	9	8	385	341	61	143	35	179	44	139	34	112	28
North Carolina	124	30	5	89	97	27	28	23	26	21	27	22	13	10
North Dakota	30	5	10	15	25	5	0	0	1	3	6	20	1	3
Ohio	195	4	5	186	167	28	74	38	67	34	90	46	52	27
Oklahoma	100	32	20	48	84	16	13	13	10	10	27	27	8	8
Oregon	31	4	—	27	26	5	6	19	7	22	9	29	5	16
Pennsylvania	283	12	28	243	239	44	106	37	110	39	104	37	67	24
Philippine Islands	38	—	—	38	31	7	4	10	0	0	0	0	0	0
Porto Rico	4	—	—	4	4	0	3	75	4	100	2	50	2	50
Rhode Island	32	1	2	29	29	3	13	41	19	59	14	44	13	41
South Carolina	50	3	3	44	40	10	7	14	5	10	6	12	2	4
South Dakota	27	5	2	20	24	3	6	22	7	26	6	22	3	11
Tennessee	82	10	12	60	69	13	21	26	9	11	14	17	4	5
Texas	202	62	18	122	169	33	47	23	42	21	55	27	27	13
Utah	28	13	—	15	26	2	6	21	6	21	4	14	3	11
Vermont	26	2	—	24	23	3	4	15	4	15	5	19	3	11
Virginia	98	20	6	72	74	24	23	23	26	26	29	29	21	21
Washington	47	10	5	32	44	3	19	40	18	38	13	28	11	23
West Virginia	40	6	9	25	34	6	17	42	9	22	7	17	6	15

TABLE II—*Continued*

Wisconsin	108	22	24	62	92	16	28	26	38	35	34	31	27	25
Wyoming	5	—	—	5	4	1	1	20	0	0	4	80	0	0
Canada	202	—	2	200	194	8	47	23	39	19	29	14	22	11
	4444	746	373	3325	3750	694	1292	29	1263	28	1333	30	828	19

SUMMARY

Geographic Division	Number of Teachers						Holders of Ph.D.		Membership in					
	(a)	(b)	(c)	(d)	(e)	(f)	No.	%	A.M.S.	%	M.A.A.	%	Both	%
New England	326	27	25	274	278	48	116	36	132	40	105	32	89	27
Middle Atlantic	792	38	43	711	682	110	284	36	333	42	272	34	203	26
East North Central	778	98	68	612	673	105	271	35	255	33	298	38	179	23
West North Central	537	131	67	339	431	106	132	25	122	23	170	32	84	16
South Atlantic	581	103	35	443	468	113	156	27	141	24	155	27	91	16
East South Central	277	59	47	171	210	67	61	22	37	13	67	24	26	9
West South Central	395	110	47	238	324	71	84	21	70	18	108	27	46	12
Mountain	164	35	17	112	145	19	44	27	44	27	50	30	30	18
Pacific	340	140	22	178	301	39	89	26	85	25	77	23	56	16
Territories	52	5	—	47	44	8	8	15	5	10	2	4	2	4
Canada	202	—	2	200	194	8	47	23	39	19	29	14	22	11

(a) Total; (b) Junior colleges; (c) Teachers colleges; (d) Other colleges and universities (e) Men; (f) Women.

New York State has the largest number of mathematics teachers of college grade, followed by Pennsylvania, California, Illinois, and Texas. Arizona and Connecticut are the states with the highest percentage of doctors on their faculties, with New Hampshire, West Virginia, Rhode Island, Washington, and Illinois following; at the other end of this scale is North Dakota with no doctors listed.

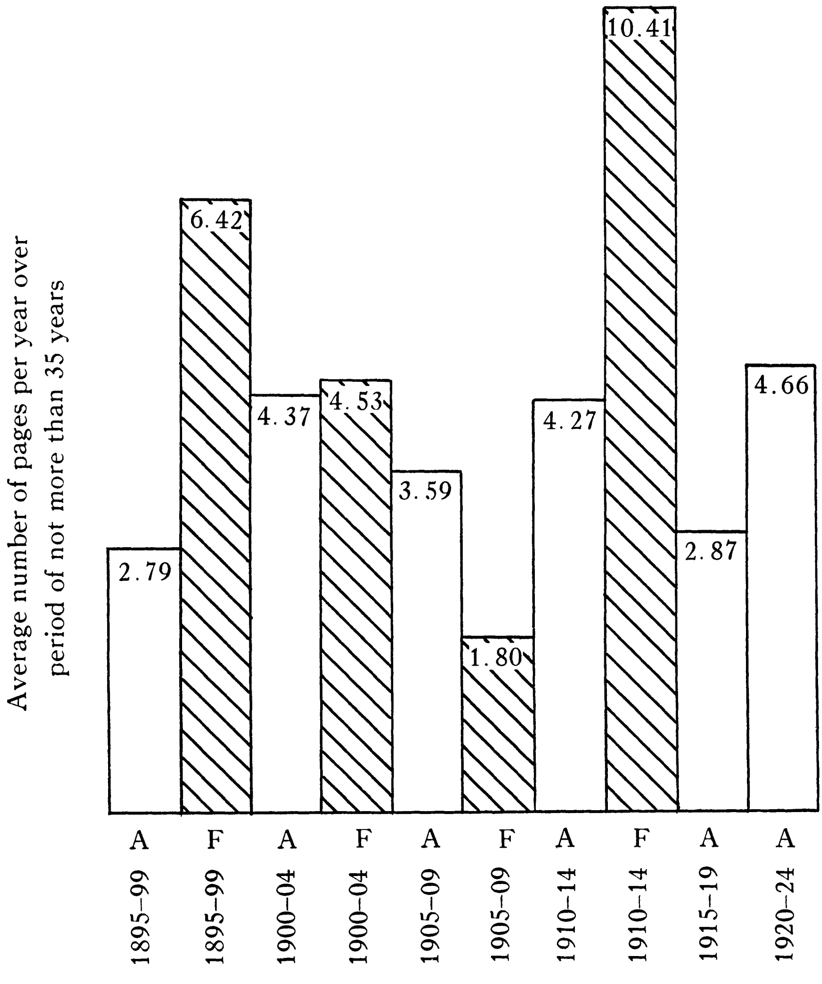
Whereas for all institutions above the rank of junior colleges (universities, colleges, and degree-granting teachers colleges) the proportion of persons with a doctor's degree is 33%, in the group of 285 colleges and universities on the approved list of the Association of American Universities it is 44%.

The decline of growth in the degree-granting institutions has been to a considerable extent offset by the establishment of emergency institutions of college freshman grade or of junior college grade. While the number of college teaching positions in mathematics probably doubled during the period 1918–1930, the unprecedented year-by-year increase in the college student population on which this increase was based was slackening rapidly even before 1929, and it seems improbable that, even if a period of economic stress had not ensued, there would have been many more academic students in the colleges of the country than there are at present. The decided drop in the number of additional appointments to the staffs of degree-granting institutions appears to have been inevitable. Without doubt the unemployment problem for college teachers is aggravated by the financial depression, but in main outlines it might have been foreseen by competent executives.

It is probably true that the number of persons attending institutions of grade beyond the high school will still increase greatly. Registration in junior colleges is increasing by leaps and bounds. The situation seems predictable, and it may be sufficiently accurate to estimate at 4,500–5,000 the number of professionally trained mathematicians who will be employed in college teaching during the next decade.

GRAPH II

COMPARISON OF PRODUCTIVITY FOR PERSONS TAKING AMERICAN AND FOREIGN DEGREES IN DIFFERENT PERIODS



Of the 60,000 persons employed in teaching all subjects in the junior colleges, colleges, and universities (not including professional schools), about 7.5% are in the departments of mathematics. In this group of institutions there are approximately 800,000 students, or about one teacher of mathematics to 175

students. In the junior colleges alone, there are approximately 6,000 teachers, of whom 12.5% are in mathematics; the total enrolment is approximately 110,000, or one teacher of mathematics to 150 students.

If we consider twenty-five years as an average period of service, it would seem that, as soon as the situation becomes a little more normal, there will be need for at least 175 new persons to be added to the staffs each year. We have arrived at the point where the universities are granting about half that number of Ph.D. degrees annually. Less than one-third of the present staffs have such degrees, and, if one-half the new appointments are made from those holding the doctorate, the situation will probably be as satisfactory as we can hope for at present. The standards in this regard need elevation, and doubtless the next decade or two will see rapid advancement. There are many college teachers with pitifully meager preparation; the institutions must look forward to their gradual replacement by well-trained men and women.

Proportion of Ph.D.'s Publishing Research Papers

It goes without saying that the number of papers and the number of pages printed is not an adequate criterion for measuring the influence of a person on mathematical thought. Often the ideas of a scholar appear in papers published

TABLE III

ANALYSIS OF NUMBERS OF PAPERS PUBLISHED BY AMERICAN MATHEMATICIANS

	Persons taking degrees 1862–1933		Persons taking degrees 1895–1924	
	Number	%	Number	%
No papers	549	46	232	39
1 paper	227	19	109	18
2 papers	100	8	58	10
3–5 papers	131	11	66	11
6–10 papers	70	6	41	7
11–20 papers	69	6	50	9
21–30 papers	20	2	17	3
More than 30 papers	22	2	17	3
Total	1188	100	590	100

by his pupils or colleagues. But a study of the amount of publication is the easiest (perhaps the only) means that is available from a statistical standpoint. There is a great deal of information contained in Graph II and Tables III–V concerning this fundamental matter of the amount of publication, and, in spite of the reservations just made, the data have real significance.

Table III gives the numbers and percentages of persons who have published no research articles, one, two, three, etc. research articles; the first division of the table gives figures for all Ph.D.'s granted in 1862–1933 and the second (we note an improvement) for those during the central period 1895–1924, which was selected as being far enough in the past so that men have had an opportunity to get something into print, and not extending back far enough so that lack of publication facility and of stimulus enters into the calculation.

A perusal of the figures in the various tables indicates that not more than one-third of the persons taking doctor's degrees have made as substantial contributions to research as would be evidenced by the publication of three or more research articles; and that not more than one-fifth have really been consistently productive in their contributions. About 60 (or 5%) of the doctors are responsible for half of the published pages of research.

TABLE IV

STATISTICS REGARDING THE NUMBER OF PAPERS PUBLISHED BY AMERICANS WHO TOOK DEGREES 1862–1933

Group	No papers	1 paper	2 papers	3–5 papers	6–10 papers	11–20 papers	21–30 papers	30+ papers
	%	%	%	%	%	%	%	%
American degree								
–1894	38	18	8	18	5	8	2	3
1895–1899	54	7	19	5	4	4	2	5
1900–1904	30	22	3	7	13	10	7	6
1905–1909	29	20	7	19	5	12	3	4
1910–1914	37	17	13	12	7	8	3	3
1915–1919	40	19	8	11	10	7	2	1
1920–1924	46	19	9	11	3	7	2	2
1925–1929	45	18	6	14	10	4	1	1
1930–1933	56	23	9	9	1	1	0	1
Foreign degree								
–1894	16	5	0	5	32	11	21	11
1895–1899	8	25	8	0	0	25	17	17
1900–1904	8	15	0	23	0	31	15	8
1905–1909	0	30	20	30	10	10	0	0
1910–1914	10	0	0	10	10	50	10	10

A more careful analysis of this situation in Table IV shows salient facts as follows: the percentage of persons doing no publishing has in recent years slightly increased; the percentage of persons publishing a large number of

papers has remained fairly constant in the group of those taking American degrees as well as of those taking European; the proportion of prolific authors is greater among those taking degrees abroad. Contrary to the general opinion, America seems in recent years to be adding to the quantity of personnel, but not improving the average quality as judged by the number of papers published.

It would be exceedingly interesting to know what proportion of the men and women with mathematical ability of high order are now being drawn into the graduate schools; in other words, how efficiently the nation is using this human material. Are there more persons of mathematical talent in the nation than can well be utilized? But such a study, which would have to begin with high school students, is entirely beyond the power of any single organization such as the Mathematical Association of America or the American Mathematical Society.

TABLE V

AMOUNT OF PUBLICATION BY GROUPS OF FIVE YEARS AFTER RECEIVING DEGREE

Period	First 5 years	Second 5 years	Third 5 years	Fourth 5 years	Fifth 5 years	Sixth 5 years	Seventh 5 years	Total	
American degree	pages	pages	pages	pages	pages	pages	pages	papers	pages
-1894	14.91	8.92	7.38	8.75	4.51	7.03	6.25	6.32	57.75
1895–1899	19.43	16.24	17.50	9.72	15.74	7.39	11.70	7.02	97.72
1900–1904	33.40	26.67	30.80	17.58	9.31	13.33		7.15	131.07
1905–1909	34.07	30.30	12.78	13.80	8.94			5.76	89.89
1910–1914	30.18	19.70	18.31	17.29				6.63	85.47
1915–1919	18.61	10.75	13.66					3.57	43.01
1920–1924	27.40	19.18						3.65	46.58
1925–1929	34.02							2.97	34.02
Foreign degree									
-1894	13.95	44.32	43.74	23.63	30.68	18.26	25.37	14.45	199.95
1895–1899	50.00	47.17	60.00	25.25	19.00	15.00	8.37	16.75	224.79
1900–1904	45.54	23.15	21.85	15.77	15.92	13.69		11.70	135.92
1905–1909	24.10	4.50	4.70	8.50	3.90			3.80	45.00
1910–1914	39.20	33.40	45.10	90.60				17.20	208.30

In Table V the amount of publication is analyzed by periods of five years after receiving the degree, and it should be noted that figures are given for a five-year period and that no average per year is tabulated. It should further be observed that in this table the figures pertain to pages instead of to papers published. As an example of how they are to be interpreted, we note that for all the doctors receiving American degrees for the five-year period 1895–99, the average number of pages which they published during the second five years

after receiving degrees was 16.24, which implies that the index, or average number of pages published annually, was 3.25.

For statistical purposes those papers which were printed before the degree was acquired are included in the first period. The average length of paper over the whole period is approximately 13.48 pages (4,916 papers and 66,268 pages*) and does not seem to vary significantly either with the age of the writer or with the date at which he took his degree. The first period of five years seems to be the most prolific one; doubtless this is due to the fact that men printing only one paper contribute during this period only. It is noteworthy that, even after more than thirty years subsequent to the conferring of the degree, there is on the part of many research workers not much slackening in the rate of publication.

The fifth edition of *American Men of Science* (1933) contains brief biographies of 1,242 living mathematicians who are listed because they "have contributed to the advancement of science." In the earlier editions there were listed an additional 168 names of people since deceased. Of this total of 1,410 mathematicians, 934 hold the doctor's degree; 104 of these degrees were awarded abroad; and of the recipients, 59 were born abroad Since *American Men of Science* contains primarily the names of persons now in residence in the United States and only incidentally a few of those working in Canada, the information has not so wide a basis as that in the study presented here, which includes Canada also.

On the basis of extensive ballots, Professor Cattell, in the various editions of *American Men of Science*, selected a total of 182 mathematicians (136 were listed as alive in the fifth edition) as representing the leaders in research, and stars were attached to their names. Of this group all but 19 have obtained the doctor's degree; 40 were born in Europe; and 56 received their doctor's degrees abroad. Our present study indicates that these 182 mathematicians have an average index of 13.03 as compared to 4.73 for the whole list of doctors, and that approximately 45,580 pages, or 69% of the whole mathematical publication of America, is due to them.

Comparison of Productivity as Regards Universities of Origin

Since it was the more enterprising and able of our students who went abroad during the twenty-year period beginning with 1895, it is to be expected that the productivity of the group taking their degrees in European universities would be greater than that of the corresponding group awarded American degrees. Graph II, which gives the average number of pages printed yearly during a period of not more than thirty-five years subsequent to attaining the degree, indicates that, for a period of twenty years before the world war (except for a lapse during 1905–09), the European universities attracted a considerable number of our ablest men as candidates for degrees. So far as the home product is concerned, we note that a peak was reached with the group of men taking degrees in 1920–24.

* This total is for Americans holding doctor's degrees and does not include articles presented to the Society by foreigners or non-doctors.

A department in a university waxes and wanes. At one period an outstanding professor attracts an able body of students and inspires them to continue their research; at another the quality of the staff and students is mediocre. If we consider a group of ten institutions, each conferring a large number of degrees, we note from Table VI that there is very considerable variation in the matter of productivity. This is true both as between the universities and in a given university as regards the four decades covered by the table. The figures in parentheses represent the number of degrees conferred in the period named in the heading of the column, and the other figures represent the average number of pages per person per year subsequently published. It is a matter of consider-

TABLE VI

COMPARATIVE STUDY OF AMOUNT OF PUBLICATION PER GRADUATE IN TEN INSTITUTIONS

Institution	1890–99	1900–09	1910–19	1920–29	Whole period
1	(4) 26.63	(35) 6.69	(56) 3.31	(90) 4.28	(185) 4.92
2	(16) 0.95	(18) 4.62	(21) 1.91	(23) 0.96	(78) 2.06
3	(5) 1.50	(12) 4.59	(26) 6.33	(27) 13.96	(70) 8.63
4	(13) 0.56	(23) 2.55	(15) 2.15	(10) 4.86	(61) 2.41
5	(2) 0.00	(15) 4.87	(16) 1.01	(24) 4.06	(57) 3.28
6		(1) 0.56	(14) 1.87	(35) 1.21	(50) 1.38
7	(4) 6.36	(11) 0.69	(21) 8.21	(11) 5.45	(47) 5.65
8	(9) 0.02	(9) 2.34	(12) 1.29	(11) 17.34	(41) 5.55
9	(1) 0.44	(3) 0.92	(11) 8.71	(13) 12.58	(28) 9.38
10			(9) 1.15	(11) 9.05	(20) 5.50
Average of the ten	3.01	4.22	3.77	5.83	4.62
General average of all Ph.D.'s		4.34	3.60	5.68	4.73

able significance that the persons taking degrees at these large institutions (and constituting more than three-quarters of the total during that period) do slightly less publication on the average than those taking their degrees at the remaining group of institutions. It is noteworthy also that the average of the indices for some of the institutions not included in this list of ten is higher than for any in the list.

Persons Without Degrees

Many persons who have not attained to doctor's degrees have published papers read before the Society. The number of such persons is about 125, and a few of them are ranked among our leading mathematicians.

National Research Fellows

It is to be expected that the publication rate for the National Research Fellows in mathematics will be greater than that for the whole group of per-

sons taking degrees, not only because they have been given special opportunities, but also because they are a picked body of men. Let us select as a basis for discussion the 49 Fellows starting on their fellowships during the first five-year period, 1925–29, in which the fellowships were awarded. The average number of pages published annually over a five-year period subsequent to their entering on the fellowship is 17.82, while for all 227 taking Ph.D.'s during the years 1925–29, it is 6.80. For the period of tenure (generally two years) and one year thereafter (to allow for lag in publication), the average for the Fellows is 25.89; this indicates what can be expected from young men if a proper selection for quality is imposed and opportunity for full-time study is made available.

For accurate comparison purposes, however, it is desirable to have more equitable bases than those just used. Let us eliminate from the 49 Fellows starting on their incumbencies in 1925–29 the 4 persons who took their degrees earlier than 1923 and who had already thus had opportunity to establish themselves in research careers. Let us allow a lapse of a year and a half after giving up the fellowship; the doctor is then professionally employed and has probably already published the main results of the research done while an incumbent. For the other group let us select those Ph.D.'s of the same period who were not awarded fellowships and let us allow a lapse of a year and a half in their cases also, so that the doctoral thesis will probably not be included in the publications counted; let us compare these groups during the subsequent years and down to the time of the last information available (including 1934). The figure for the Fellows is 10.06 and for the remaining Ph.D.'s is 2.87, which establishes a significant differential. However, using the same groups and the same periods, let us compare the bottom one-third of the Fellows with the top one-third of the non-fellows. The results are 0.93 for the 15 persons of the one group as against 8.49 for the 60 persons of the other, indicating that the selection of the Fellows carried with it a considerable factor of error. It should, however, be pointed out that not all the more talented students apply for fellowships.

Let us next make a comparison of the 30 most productive of the 45 Fellows considered above and the 30 most productive Ph.D.'s among the 180 non-fellows of the period 1925–29 by subjecting their records to the conditions of the preceding paragraph. The indices are 14.63 and 14.77; but it must be pointed out that these selections are for the upper two-thirds and upper one-sixth respectively.

Another fundamental comparison that can be made is that between the Fellows of the 1925–29 period and the corresponding persons of the 1920–24 period before the fellowships were available. Selecting the 30 from each group who have published the most, and using comparable periods beginning one and one-half years after fellowship and degree respectively and terminating with 1934 and 1929 respectively, the figure for the Fellows is 14.63 and for the others is 10.16. Since in this comparison the case for the Fellows is discounted by the fact that not all the best men are elected to fellowships, the figures indicate a decided stimulus from the fellowships.

Conclusion

The rapidity with which a mathematical school of high distinction has been built up in America is one of the most striking phenomena in the history of science. It should be borne in mind that the American Mathematical Society, which has been an important factor in this development, was not founded until 1888, and that only in isolated cases was research carried on before that date. Building on the splendid foundations already laid, great forward movements are possible if the spirit of cooperation now animating mathematicians is fostered. The challenge of the future inspires the mathematical fraternity to high endeavor.

Many of the statistics exhibited in this paper will be of interest to those who follow the development of teaching and research in the field of mathematics. Those concerned with the strategy of promoting mathematical thought and achievement in America can find in the assembled material several sign-posts for their further guidance. Other studies could be made from the data which have been collected and which are available to anyone; the investigations appear to be worth pursuing further.

DIMENSION THEORY AND DIMENSION MODELS

By DR. ALFRED BASCH of the Austrian Bundesamt für Eich- und Vermessungswesen and lecturer in the Technische Hochschule in Vienna

(Translated and adapted by JOSEPH JABLONOWER, Fieldston School, New York)

I. *Dimension as a Problem in Elementary Mathematics*

While for most students of elementary and even secondary schools the systematic study of dimension remains a closed book, the problem is present, at least implicitly, in elementary arithmetic. In the simple operation of division, where the pupil is concerned, for example, with partition of a given class of objects into a given number of small and numerically equal classes, the dividend is a denominate number of dimension one, the divisor is of dimension zero, and the quotient is of dimension one. On the other hand, if 840 pears are to be packed in baskets each of which hold 120 pears, we have the relation:

$$(1) \qquad \frac{840 \text{ pears}}{120 \text{ pears}} = 7.$$

It is not unusual, in elementary instruction, for the teacher to require the pupil to write the equation in the following incorrect form:

$$(1a) \qquad \frac{840 \text{ pears}}{120 \text{ pears}} = 7 \text{ baskets}.$$

Of course, to make the equation express the relationship correctly from the point of view of dimensionality, we should write it:

$$\frac{840 \text{ pears}}{120 \text{ pears/basket}} = 7 \text{ baskets}.$$

The equation written in this form has the quality of formal mathematical correctness. But it has additional significance. If we should pack the pears, in the instance chosen, into baskets whose capacity is one-third (in general, $1/\lambda$) times as large, the quotient will be three (in general, λ) times as large.

II. *Invariance with Respect to Scale in Ratios of Physical Quantities*

Mathematical physics is concerned with the problem of expressing discoverable relations among measurable magnitudes. To measure a physical magnitude, for example, length, time, velocity, mass, is to assign to various intensities which these quantities assume numbers in unequivocal and reversible order. A given intensity U is chosen arbitrarily, and is assigned the number 1, or, as we ordinarily say, it is represented by the number 1. It is possible then to assign to every intensity of this variable different from the chosen intensity, some other number; to intensity I_1 the number n_1, to intensity I_2, the number n_2, and so on. The ratio of the numerical measures will, of course, remain invariant with respect to the scale used in expressing the measures themselves.

We postulate a procedure whereby, in accordance with the unit chosen, any intensity of a quantity can be assigned an appropriate number. Physical quantities for which such procedure is available make advantageous fundamental quantities as, for example, length and time. We assume also that there exist quantities of another sort, namely, the sort whose numerical measures are derivable in terms of numerical measures of fundamental quantities. Velocity, for example, is of this class, because velocity is considered the greater when, in the same time interval, t, a greater distance, s, is covered, or when the same distance, s, is covered in a shorter time interval, t. This condition is expressed by the formula

$$v = C\frac{s}{t}. \tag{2}$$

The formula which yields the numerical measure x of any derived or secondary quantity X from the numerical measures $x_1, x_2, x_3, \cdots, x_n$ of the n fundamental or primary quantities $X_1, X_2, X_3, \cdots, X_n$,

$$x = f(x_1, x_2, x_3, \cdots, x_n) \tag{3}$$

cannot have any arbitrary form, if we impose the requirement that invariability of ratios of different intensities of secondary quantities be maintained despite variation in the units chosen for the measure of the fundamental quantities, that is, as we have already indicated, if we are to secure the invariance of ratios in respect to scale. It can be demonstrated that it is necessary and sufficient, to this end, that the formula have the form*

* Cf. P. W. Bridgman: *Dimensional Analysis*, 1931, Yale University Press.

$$x = Cx_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3} \cdots x_n^{\xi_n}. \tag{4}$$

C denotes here an arbitrary numerical constant. If the units of the fundamental quantities are reduced in the ratios $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, the numerical measures of the fundamental magnitudes are enlarged in the ratios $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, and the numerical measures of the secondary quantities in the ratios $\lambda_1^{\xi_1}, \lambda_2^{\xi_2}, \lambda_3^{\xi_3}, \cdots, \lambda_n^{\xi_n}$.

III. *The Dimension Model*

Let the numbers $\xi_1, \xi_2, \xi_3, \cdots, \xi_n$ be the dimension numbers of the derived quantity X in terms of the system of fundamental quantities $X_1, X_2, X_3, \cdots, X_n$. This is ordinarily expressed by means of the equation

$$[X] = [X_1^{\xi_1}X_2^{\xi_2}X_3^{\xi_3} \cdots X_n^{\xi_n}]. \tag{5}$$

In most of the physics, certainly in mechanics (including acoustics) and in the general theory of electricity, it is easily possible to manage with three fundamental magnitudes. Indeed, it is possible, although not ordinarily useful, to make the same fundamental magnitudes serve in the theories of heat and photometry. It is possible even to manage with only two fundamental magnitudes in the field of mechanics, but the device of employing two such magnitudes has been found valuable in so few instances that it is not often resorted to. The larger part, by far, of physical theory is based, therefore, on *three* fundamental magnitudes, in the sense that every physical magnitude has *numerical measures in three dimensions*. The dimensions are length, mass, and time. The units of measure for all other quantities are derivable in terms of length, mass, and time, according to methods first employed by K. F. Gauss, and W. Weber. The first to employ dimension theory in physics was J. B. Fourier.*

Because we employ three fundamental magnitudes we can express the dimensions of every physical magnitude in terms of three numbers (the dimension-numbers in respect to length, mass, time) and we are in a position to represent every magnitude by means of a point, the *dimension-point* in three-dimensional space, and this in the simplest and most obvious fashion by making the three dimension-numbers the coordinates of the dimension-point in a tri-rectangular system of coordinates.

This has been done by means of a spatial representation or model called the "Dimension Model," which the author constructed for the exposition held in 1932 in Vienna to celebrate "Sixty Years of the Metric System in Austria." In this model† the dimension-points are represented by wooden spheres which are connected with one another by means of rods (lattice-rods) laid out parallel to

* J. B. Fourier, Théorie analytique de la chaleur, Paris, 1822.

† Similar representations, have been proposed by H. Wüger (Graphic representation of measurement, Bulletin of the Electro Chemical Society, Zurich, 1931) and by H. Baudisch, (Universal coordinates of absolute and technical systems of measurement, Sparwirtschaft, Vienna, 1931). The model is now exhibited in the Technisches Museum fur Industrie und Gewerbe in Vienna.

the axes of coordinates. To the right of the origin of coordinates is the dimension in respect to length, ξ_1, backward is the dimension in respect to mass, ξ_2, upward in respect to time, ξ_3. In the chief vertical plane $\xi_1\xi_3$ are the dimension-points of kinematics such, for example, as velocity acceleration, angular acceleration, and so on. In the horizontal plane $\xi_1\xi_2$ are the dimension-points of the magnitudes of mass-geometry, as, for example, moment and moment of inertia.

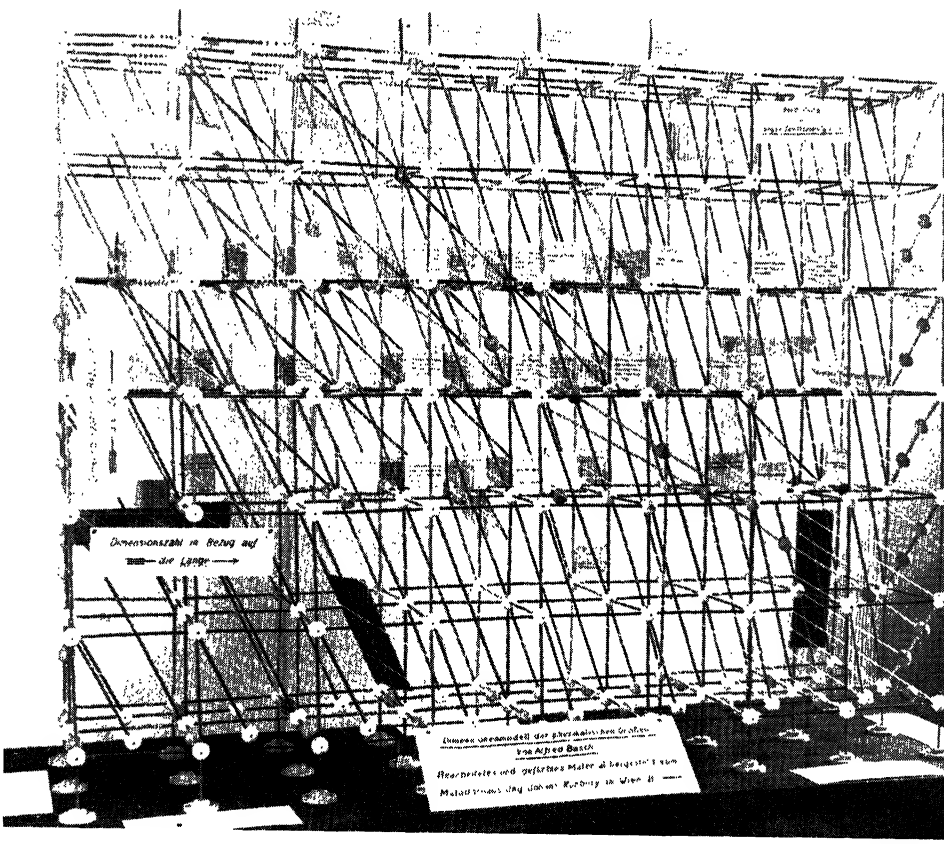


FIG. 1

Motions of translation from the non-dimension-point (origin) to dimension-points which represent physical magnitudes of one dimension we call *dimension-vectors* of these magnitudes. The dimension-vectors of the fundamental magnitudes we call fundamental vectors.

The dimension of any magnitude X can be given by the dimension equation:

$$[X] = [L^{\xi_1}M^{\xi_2}T^{\xi_3}] \tag{6}$$

in which ξ_1, ξ_2, ξ_3 are the dimension-numbers of the magnitude in respect to the fundamental system, length-mass-time. For the magnitudes of mechanics,

the dimension-numbers are in all cases integral. Dimension-points which have integral coordinates we call chief lattice-points. Dimension equation (6) corresponds to the vector equation

$$\vartheta = \xi_1\vartheta_L + \xi_2\vartheta_M + \xi_3\vartheta_T \tag{7}$$

which expresses the dimension-vector of the physical magnitude X in terms of the dimension-vectors of the three fundamental magnitudes (fundamental vectors). Thus, for example, the dimension-vector for acceleration is

$$\vartheta_A = \vartheta_L - 2\vartheta_T;$$

for force,

$$\vartheta_F = \vartheta_L + \vartheta_M - 2\vartheta_T;$$

for power,

$$\vartheta_P = 2\vartheta_L + \vartheta_M - 3\vartheta_T.$$

It is important to consider whether it is not possible to select, at least in part, magnitudes X_1, X_2, X_3 other than length, mass, and time, as fundamental. This can be done only when the dimension of every other magnitude, X, is representable in the form

$$[X] = [X_1^{\eta_1}X_2^{\eta_2}X_3^{\eta_3}]. \tag{8}$$

The magnitudes X_1, X_2, X_3, would have dimension-numbers obtainable through appropriate substitutions in the determinant

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \neq 0 \tag{9}$$

and, consequently, the dimension-vectors would satisfy the relation

$$\begin{aligned} \vartheta_1 &= \alpha_{11}\vartheta_L + \alpha_{12}\vartheta_M + \alpha_{13}\vartheta_T \\ \vartheta_2 &= \alpha_{21}\vartheta_L + \alpha_{22}\vartheta_M + \alpha_{23}\vartheta_T \\ \vartheta_3 &= \alpha_{31}\vartheta_L + \alpha_{32}\vartheta_M + \alpha_{33}\vartheta_T. \end{aligned}$$

The dimension-vector, ϑ, of any magnitude is expressible, then, in

$$\vartheta = \eta_1\vartheta_1 + \eta_2\vartheta_2 + \eta_3\vartheta_3 \tag{10}$$

when the scalar triple-product

$$D = \vartheta_1\vartheta_2\vartheta_3 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \neq 0. \tag{11}$$

Geometrically considered this means: We can resolve any vector (here the di-

mension-vector) into three components of prescribed directions that are non-coplanar, or when the parallelepiped constructed with these directions as edges has a volume other than zero. This condition is equivalent to the algebraic condition that the determinant, D, of the dimension-numbers shall not vanish. Vector equation (10) is equivalent to the three scalar equations

$$\begin{aligned} \alpha_{11}\eta_1 + \alpha_{21}\eta_2 + \alpha_{31}\eta_3 &= \xi_1 \\ \alpha_{12}\eta_1 + \alpha_{22}\eta_2 + \alpha_{32}\eta_3 &= \xi_2 \\ \alpha_{13}\eta_1 + \alpha_{23}\eta_2 + \alpha_{33}\eta_3 &= \xi_3. \end{aligned} \tag{10a}$$

The solution of this system of new dimension-numbers will be

$$\begin{aligned} \eta_1 &= \frac{1}{D}(A_{11}\xi_1 + A_{12}\xi_2 + A_{13}\xi_3) \\ \eta_2 &= \frac{1}{D}(A_{21}\xi_1 + A_{22}\xi_2 + A_{23}\xi_3) \\ \eta_3 &= \frac{1}{D}(A_{31}\xi_1 + A_{32}\xi_2 + A_{33}\xi_3). \end{aligned} \tag{12}$$

Here $A_{\rho\sigma}$ is the co-factor of $\alpha_{\rho\sigma}$ in the determinant D.

A system of magnitudes is obtained whose fundamental magnitudes are length, force, and time. The dimension equation of force is

$$[F] = [L^1M^1T^{-2}].$$

The dimension-vectors of the fundamental magnitudes (fundamental-vectors) are, accordingly, $\vartheta_L, \vartheta_F(=\vartheta_L - 2\vartheta_T), \vartheta_T$. These constitute a non-coplanar tripod; the determinant of the substitution matrix $D=+1\neq 0$. The fundamental system of magnitudes is therefore applicable, and the new dimension-numbers are

$$\eta_1 = \xi_1 - \xi_2, \quad \eta_2 = \xi_2, \quad \eta_3 = 2\xi_2 + \xi_3. \tag{13}$$

In the dimension model the dimension-point of force is treated as a basic point. The straight line determined by it and the origin of coordinates $L^0M^0T^0$ is the axis of dimension in respect to force. Rods are passed through lattice-points parallel to this axis of dimension in respect to force. Just as we were able to resolve any vector into appropriate length, mass, and time components, so we can now resolve any vector into appropriate length, force, and time components. We are able to measure these components with due regard to the sense of direction in the dimension-vectors of the fundamental magnitudes of the new length-force-time system. The numerical measures of the components are then given in terms of the dimension-numbers expressed in (13) in relation to the fundamental length-force-time system. The plane determined by the dimension-vectors of length and force contains the dimension-points of magnitude in theorems on statics.

IV. *Magnitudes in Electrical Theory*

The dimension-numbers of the magnitudes in mechanics are determined uniquely when we have once determined upon three fixed magnitudes of mechanics as fundamental magnitudes. In contrast with this situation is the one in electrical theory, where, employing the magnitudes of mechanics as fundamental, we obtain two sets of dimensions, according as we adopt as the basis the phenomena of electrostatics or the phenomena of electrodynamics. Thus, for example, for Coulomb's Law for the force exerted between two electric charges, we derive, in the electrostatic system, the dimension equation for the electric charge

$$[Q] = [L^{3/2}M^{1/2}T^{-1}]. \tag{14}$$

We derive a similar equation for the magnetic intensity from the law of the force exerted between two magnetic poles, and, from this, on the basis of the Biot-Savarts Law of the force between magnetic pole and current, for the strength of current, I, the dimension equation

$$[I] = [L^{1/2}M^{1/2}T^{-1}]. \tag{15}$$

For the dimension of the electric charge whose numerical measure is expressed by the equation $q=it$, we get the dimension equation in the electromagnetic system

$$[Q]=[L^{1/2}M^{1/2}T^{0}]. \tag{16}$$

Since it is possible to derive the dimensions of all other magnitudes in electrical theory from the dimensions of the electric charge, every magnitude in electrical theory is represented in the dimension model by two dimension-points: the one in the electrostatic, and the other in the electromagnetic system.

In addition, it is noteworthy that, by retaining the fundamental system of magnitudes, length-mass-time, we obtain for two of the three dimension-numbers, namely for those of length and mass, of some of the magnitudes in both electrostatic and electromagnetic theory, not integers but halves of uneven numbers. To this group, in addition to electric charge and strength of current already mentioned, belong electromotive force, intensity of magnetic field, magnetic moment. The dimension-number of these magnitudes in respect to mass is, in all cases, $\frac{1}{2}$. The dimension-points which represent these magnitudes on the model do not coincide with the dimension-points of the magnitudes in mechanics (the chief lattice-points). They lie, on the contrary, in the plane $M^{1/2}$, and, indeed, at the centers of the horizontal lattice squares, which might well be called *secondary* lattice-points. The quotient of the dimensions of such a magnitude in the two systems (electrostatic and electromagnetic), as is evident, for example, in the dimensions of electric charge (equations 14 and 16), is the dimension of velocity, L^1T^{-1}, or its reciprocal, $L^{-1}T^1$.

Other magnitudes in the theory of electricity whose numerical measures are

derived by division of the numerical measures expressed in the two systems and which are represented in the intermediate plane $M^{1/2}$ have, in all cases, integral dimension-numbers, as for example, electrical resistance. Their dimension-points are chief lattice-points and coincide for both electrostatic and electromagnetic systems with dimension-points of kinematics. Thus, electrical resistance in electrostatics has the dimension $L^{-1}M^0T^1$, in electromagnetics it has the dimension $L^1M^0T^{-1}$. The quotient of these dimensions $L^{-2}T^2$ is the dimension of the reciprocal of the square of the velocity, L^1T^{-1}. The quotient of the dimensions of those corresponding magnitudes in the two systems of electrical theory whose dimension-points lie in the plane M^0 (of dimension-points of kinematic magnitudes) is always L^2T^{-2}. Such magnitudes are, for example, the constants of self-induction, capacity, dielectric constants, magnetic permeability.

V. *The Heuristic Value of the Study of Dimension*

The postulate of Fourier that every quantity has its unique dimension is valid insofar as one proceeds from a fixed system of fundamental magnitude and confines himself to a fixed and restricted domain of phenomena. Only quantities of the same dimensions can be equated or combined additively. This has given a certain heuristic value to the study of dimension and has led to a method of investigation which in German literature is known as similitude or model theory, and in English and American literature as dimensional analysis. On both sides of an equation which expresses a relation in theoretic or technical physics must appear magnitudes of the same dimension. This means, in the case of the three fundamental magnitudes, three scalar equations or a vector equation in three-dimensional space which can be represented by a closed vector polygon. In this fashion it is possible to derive the mathematical laws of many phenomena except for the numerical constants, which are, of course, not thus determinable. We need only know, in advance, what magnitudes characterize a given phenomenon.

Thus, for example, the period of a mathematical pendulum depends—disregarding the negligible effect of amplitude—on the length, l, and the acceleration due to gravity, g. Therefore,

$$\tau = f(l, g),$$

and this unknown function must reduce, in dimensionality, to time, its dimension-vector must coincide in magnitude and direction with the dimension-vector ϑ_T. In Figure 2, a portion of the plane of kinematics is taken from the dimension model. The origin of coordinates is O, and L, T, V, G, Ω, $d\Omega/dt$ are respectively the dimension-points of length, time, velocity, acceleration, angular velocity, and angular acceleration. The dimension-vector, $\vartheta_T = \overrightarrow{OT}$, of time, is to be resolved into components in the directions of the dimension-vectors $\vartheta_L = \overrightarrow{OL}$, and $\vartheta_G = \overrightarrow{OG}$. It is obvious from the figure that

$$\vartheta_T = \tfrac{1}{2}\vartheta_L - \tfrac{1}{2}\vartheta_G \tag{17}$$

which vector-equation corresponds to the relation

$$\tau = c\sqrt{\frac{l}{g}} \tag{17a}$$

between the period of a pendulum, the length, and the acceleration due to gravity, in which the numerical constant, c, for small amplitudes, has the value, 2π.

From Figure 3, which represents a portion of the model, we can interpret, in addition to the law of the pendulum which we have just considered, laws of

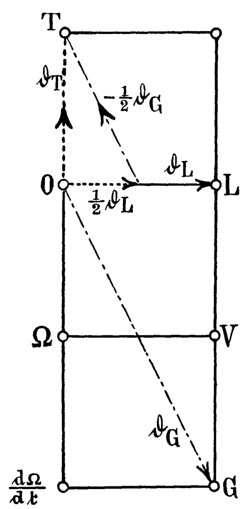


FIG. 2

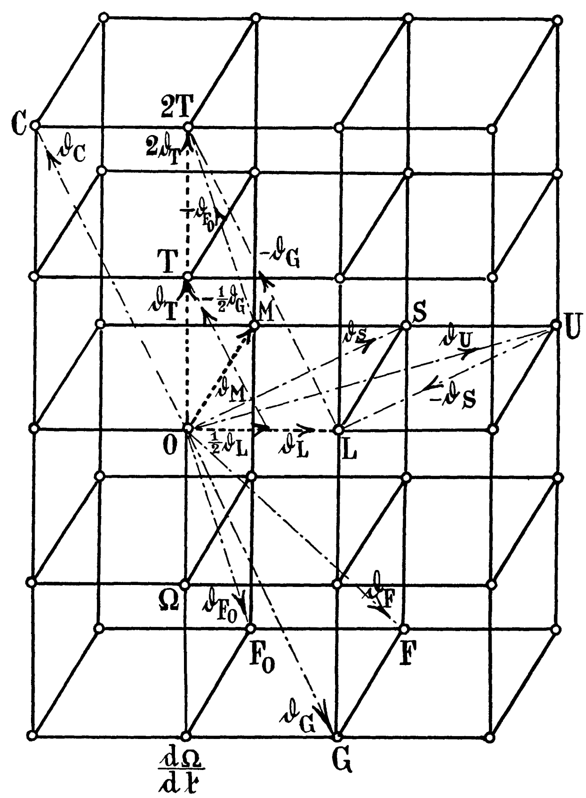


FIG. 3

other phenomena of vibration. The front plane of the figure is the plane of kinematics; the points indicated therein have the same meaning as in Figure 2; C represents the dimension-point of capacity in the electromagnetic system. The dimension-point, L, of length, is coincident with that of the constant of self-induction. In the plane to the rear of this one, M, S, U, F and F_0 represent respectively, the dimension-points of mass, static moment, moment of inertia, force, and the constant of elasticity of a spring. The points of the dimension-

vectors of the individual quantities are consistent with the labeling of the dimension-points. In the drawing are indicated, in addition to equation (17), the following vector equations by means of directed polygons:

$$\left.\begin{aligned} 2\vartheta_T &= \vartheta_U - \vartheta_S - \vartheta_G \\ 2\vartheta_T &= \vartheta_M - \vartheta_{F_0} \qquad (\vartheta_{F_0} = \vartheta_F - \vartheta_L) \\ 2\vartheta_T &= \vartheta_L + \vartheta_C \end{aligned}\right\} . \tag{18}$$

These equations express—except for the numerical constant 2π—in addition to the period of the mathematical pendulum, the following relations:

(a) the period of the compound pendulum,

$$\tau = 2\pi\sqrt{\frac{U}{Sg}}; \tag{18a}$$

(b) the period of the vibrating spring to which is attached a mass m and which has the elasticity constant $F_0 = F/l$,

$$\tau = 2\pi\sqrt{\frac{m}{F_0}} = 2\pi\sqrt{\frac{ml}{F}}; \tag{18b}$$

(c) and, finally, the period of oscillation of an electric circuit whose self-induction constant is L and whose capacity is C,

$$\tau = 2\pi\sqrt{LC}. \tag{18c}$$

In Figure 4 we see illustrated an important law of celestial mechanics. The revolution of the planets is a phenomenon of gravitation. The period τ of a planet is dependent upon the mass M of the central body, the gravitation constant G_0, and finally, its mean distance d from the central body. We need, therefore, a closed vector polygon composed of ϑ_L, ϑ_T, ϑ_M and ϑ_{G_0}. Every vector dependent on ϑ_L and ϑ_T lies in the kinematic plane, in which lies also the dimension-vector which is dependent on ϑ_M and ϑ_{G_0}.

$$\vartheta_K = \vartheta_M + \vartheta_{G_0} = 3\vartheta_L - 2\vartheta_T \tag{19}$$

from which vector equation follows the relation (except for the factor 4π)

$$4\pi\frac{d^3}{\tau^2} = MG_0. \tag{19a}$$

This relation expresses that fact that the ratio of the third power of the mean distance to the square of the period of revolution is constant. This is the familiar Third Law of Kepler. K is the dimension-point of this constant.

The method here developed whereby is determined analytically the mutual dependence of the numerical measure of three or four quantities fails when the number of quantities exceeds four. Among three quantities there is a relation

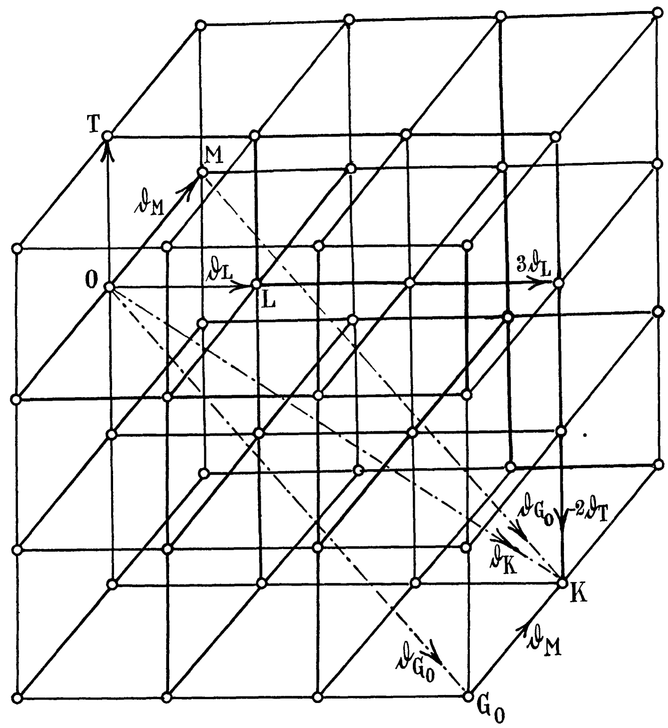


FIG. 4

when multiples of their dimension-vectors form a triangle, that is, when these vectors are coplanar. Among four magnitudes the mutual dependence is determinable when their dimension-vectors are non-coplanar. Dimensional analysis shows, to a certain extent, the form which must obtain for a relation of more than four quantities that determine a phenomenon.*

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

GRAEFFE'S METHOD AND COMPLEX ROOTS

By B. A. HAUSMANN, New Haven, Conn.

The excellent article by C. A. Hutchinson on *Graeffe's Method for the Numerical Solution of Algebraic Equations* which appeared in this MONTHLY for March, 1935, suggests a further problem. It concerns the application of Graeffe's

* E. Buckingham, Phys. Rev., vol. 4 (1914), p. 345; also P. W. Bridgman, *Dimensional Analysis.*

method to the case when there are more than two pairs of complex roots in a given equation. Textbooks on the subject dismiss the matter with the suggestion that methods similar to those employed for the case when there are two pairs of complex roots can be used.* As we shall see, analogous procedure presents difficulties which increase as the number of complex roots increases and makes the solution of such problems almost impossible. A simple modification of Graeffe's method however obviates all these difficulties.

Since the presence of real roots neither complicates nor simplifies the process of finding the complex roots, we shall confine our discussion to equations all of whose roots are complex.

When we apply Graeffe's method to an equation whose roots are complex, we get directly not the roots themselves but their absolute values. To determine the roots we must have recourse to the original equation and to the explicit expressions of the elementary symmetric functions of the roots of the equation. Thus suppose that the given equation has two pairs of complex roots, say $a_1 \pm b_1 i$ and $a_2 \pm b_2 i$. Let the equation be

$$f(x) = c_0 x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4 = 0. \tag{1}$$

If we rewrite this equation with explicit expressions for the elementary symmetric functions, we get

$$\begin{aligned} f(x) = x^4 - 2(a_1 + a_2)x^3 + (r_1^2 + r_2^2 + 4a_1a_2)x^2 \\ - 2(a_1 r_2^2 + a_2 r_1^2)x + r_1^2 r_2^2 = 0, \end{aligned} \tag{2}$$

where

$$r_1^2 = a_1^2 + b_1^2, \qquad r_2^2 = a_2^2 + b_2^2$$

are the squares of the absolute values of the complex roots. Since Graeffe's method gives us these values, we can easily determine a_1, a_2 from the coefficients of the terms x^3 and x in equations (1) and (2), for we have

$$\left.\begin{aligned} 2a_1 + 2a_2 &= -\frac{c_1}{c_0} \\ 2r_2^2 a_1 + 2r_1^2 a_2 &= -\frac{c_3}{c_0} \end{aligned}\right\}$$

Having found a_1 and a_2 we determine b_1 and b_2 from

$$\begin{aligned} r_1^2 &= a_1^2 + b_1^2 \\ r_2^2 &= a_2^2 + b_2^2 . \end{aligned}$$

The determination of the roots in this case is easy enough. Now let us see how this method applies when there are more than two pairs of complex roots present.

* Brodetsky and Smeal discuss the same problem and obtain another solution. Cf. Proc. Camb. Phil. Soc., vol. 22, (1924).

The general equation which has the three pairs of complex roots

$$a_j \pm b_j, \qquad j = 1, 2, 3$$

is

$$\begin{aligned} x^6 &- 2[a_1 + a_2 + a_3]x^5 + [r_1^2 + r_2^2 + r_3^2 + 4a_1a_2 + 4a_1a_3 + 4a_2a_3]x^4 \\ &- [2a_1(r_2^2 + r_3^2) + 2a_2(r_1^2 + r_3^2) + 2a_3(r_1^2 + r_2^2) + 8a_1a_2a_3]x^3 \\ &+ [r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 + 4(a_1a_2r_3^2 + a_1a_3r_2^2 + a_2a_3r_1^2)]x^2 \\ &- 2[r_2^2r_3^2a_1 + r_1^2r_3^2a_2 + r_1^2r_2^2a_3]x + r_1^2r_2^2r_3^2 = 0. \end{aligned}$$

Hence after we have found the absolute values of the roots, if we are to proceed as we did in the case when there were only two pairs of complex roots, we must solve for a_1, a_2, a_3 the simultaneous equations

$$\left.\begin{aligned} 2a_1 + 2a_2 + 2a_3 &= -\frac{c_1}{c_0} \\ 2(r_2^2 + r_3^2)a_1 + 2(r_1^2 + r_3^2)a_2 + 2(r_1^2 + r_2^2)a_3 + 8a_1a_2a_3 &= -\frac{c_3}{c_0} \\ 2r_2^2r_3^2a_1 + 2r_1^2r_3^2a_2 + 2r_1^2r_2^2a_3 &= -\frac{c_5}{c_0} \end{aligned}\right\}.$$

Here we note that one of the equations is not linear. Of course the equations have a solution and that solution can be found but it is admittedly not an easy task.

When an equation has four pairs of complex roots, the problem becomes even more complicated. Then we have to solve simultaneous equations, where the only unknowns are a_1, a_2, a_3, a_4, of the form

$$\left.\begin{aligned} a_1 + a_2 + a_3 + a_4 &= A \\ B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 + a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 &= B \\ C_1a_1 + C_2a_2 + C_3a_3 + C_4a_4 + C_5a_1a_2a_3 + C_6a_1a_2a_4 + C_7a_1a_3a_4 + C_8a_2a_3a_4 &= C \\ D_1a_1 + D_2a_2 + D_3a_3 + D_4a_4 &= D \end{aligned}\right\}.$$

Now we can obviate the necessity of solving simultaneous equations of the above type if we proceed as follows. Let the original equation be

$$f(z) = c_0z^n + c_1z^{n-1} + \cdots + c_n = 0.$$

Then setting $z = x + iy$ we have by Taylor's theorem

$$f(x + iy) = f(x) + f'(x)yi - f''(x)\frac{y^2}{2!} - f'''(x)\frac{y^3i}{3!} + f^{(\mathrm{IV})}(x)\frac{y^4}{4!} + \cdots = 0$$

and hence we may write

$$\left.\begin{aligned}(3)\qquad & f(x) - f''(x)\frac{y^2}{2!} + f^{(\mathrm{IV})}(x)\frac{y^4}{4!} - \cdots = 0\\ (4)\qquad & f'(x) - f'''(x)\frac{y^2}{3!} + f^{(\mathrm{V})}(x)\frac{y^4}{5!} - \cdots = 0.\end{aligned}\right\}$$

We might, of course, eliminate y from these last two equations and get an equation $F(x)=0$, the real roots of which would give us $a_1, a_2, \cdots$ but the labor involved is rather prohibitive.* Besides we would be wasting the valuable information which we already have obtained, namely knowledge of the absolute values of the complex roots. The better procedure then is to substitute

$$(5)\qquad y^2 = r_3^2 - x^2$$

in (4) and solve the resulting equation (again by Graeffe's method) for x. One of the real roots obtained will be a_3. If this equation has more than one real root, we calculate the corresponding values of y for each real x by means of (5) and substitute each pair in (3). The pair that satisfies (3) gives us the values we are seeking, namely a_3 and b_3.

If the original equation has more than three pairs of imaginary roots, we repeat the process with

$$y^2 = r_4^2 - x^2$$

to find a_4 and b_4. Obviously this procedure can be repeated as often as may be necessary. The values of a_1 and a_2 are best found from the symmetric functions of the roots by using the second and the second last terms of the original equation which are linear expressions in $a_1, a_2, \cdots$ and substituting in them the values of $a_3, a_4, \cdots$ which we have already found. Using Graeffe's method in this way we can always find all of the complex roots of an equation no matter how many there may be. The merit of the procedure lies in the fact that each step is clearly defined and all of them are of an elementary nature.

An illustrative problem (synthetic) may make the whole matter clearer. Consider the equation

$$f(z) = z^6 - 12z^5 + 72z^4 - 262z^3 + 601z^2 - 850z + 650 = 0.$$

Applying Graeffe's method we find that all the roots are complex with the squares of the absolute values $r_1^2=13$, $r_2^2=10$, $r_3^2=5$. Hence expanding by Taylor's formula we have

$$\left.\begin{aligned}(3)\qquad & f(x) - f''(x)\frac{y^2}{2!} + f^{(\mathrm{IV})}(x)\frac{y^4}{4!} - f^{(\mathrm{VI})}(x)\frac{y^6}{6!} = 0\\ (4)\qquad & f'(x) - f'''(x)\frac{y^2}{3!} + f^{(\mathrm{V})}(x)\frac{y^4}{5!} \qquad\qquad = 0\end{aligned}\right\}$$

* For a treatment of this method cf. a paper by Frazer and Duncan, *On the Numerical Solution of Equations with Complex Roots*, Proc. Royal Soc. of London, series A, vol. 125 (1929), p. 68.

where

$$
\begin{aligned}
f'(x) &= 6x^5 - 60x^4 + 288x^3 - 786x^2 + 1202x - 850 \\
f''(x) &= 30x^4 - 240x^3 + 864x^2 - 1572x + 1202 \\
f'''(x) &= 120x^3 - 720x^2 + 1728x - 1572 \\
f^{(\mathrm{IV})}(x) &= 360x^2 - 1440x + 1728 \\
f^{(\mathrm{V})}(x) &= 720x - 1440 \\
f^{(\mathrm{VI})}(x) &= 720.
\end{aligned}
$$

Now from $r_3^2 = 5 = x^2 + y^2$ we have $y^2 = 5 - x^2$. Substituting this value of y in (4) together with the values of $f'(x), f''(x)$, etc. we get the equation

$$E(x) = 32x^5 - 192x^4 + 416x^3 - 328x^2 - 88x + 160 = 0.$$

Solving this equation by Graeffe's method, we find that the only root which satisfies (3) is 1. Hence we have $a_3 = 1$. We find a_1 and a_2 from

$$
\left.
\begin{aligned}
2a_1 + \quad 2a_2 + \quad 2a_3 &= 12 \\
2r_2^2 r_3^2 a_1 + 2r_1^2 r_3^2 a_2 + 2r_1^2 r_2^2 a_3 &= 850
\end{aligned}
\right\}
$$

This gives $a_1 = 2$, $a_2 = 3$. From the relations $r_1^2 = a_1^2 + b_1^2$, $r_2^2 = a_2^2 + b_2^2$, $r_3^2 = a_3^2 + b_3^2$, we get $b_1 = 3$, $b_2 = 1$, $b_3 = 2$. Hence the roots of $f(z) = 0$ are

$$2 \pm 3i, \quad 3 \pm i, \quad 1 \pm 2i.$$

An Exponential Diophantine Equation

By C. J. Everett, Jr., Tulane University

Introduction. In his *Analysin Infinitorum*, L. Euler* gave a method for finding the rational solutions of the equation

$$x^y = y^x \tag{1}$$

and H. L. Slobin† has recently investigated the same problem using a novel logarithmic method. This note treats the corresponding Diophantine equation from the standpoint of elementary number theory, and contains a simple proof of the form solutions must have.

Types of solutions. The set of all integral solutions of (1) may be grouped in the following manner:

I. *Solutions such that* $x = y > 0$. Obviously every pair of values

$$x = y = i \qquad (i = 1, 2, \cdots)$$

is a solution of (1).

---

* L. Euler, *Introductio in Analysin Infinitorum* II, p. 294.

† H. L. Slobin, this Monthly, vol. 38 (1931), pp. 444–447.

II. *Solutions such that* $x>y>0$. We prove the

THEOREM: *In order that a solution be of type II it is necessary and sufficient that* $x=4$, $y=2$.

The sufficiency is evident by direct substitution in (1). For the necessity, let $x=y+R$ where R is a positive integer. Then (1) becomes

$$(y+R)^y = y^{y+R}.$$

Writing

$$y = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$$

we have

$$(y+R)^y = y^R (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s})^y.$$

Therefore $p_j^{\alpha_j y} \mid (y+R)^y$ and $p_j^{\alpha_j} \mid (y+R)$, $j=1, \cdots, s$. Hence $y \mid y+R$ and $y \mid R$, and we may now write $x=k\cdot y$ $(k \geqq 2)$. Then (1) becomes

$$(ky)^y = y^{ky} \text{ or } k = y^{k-1}. \tag{2}$$

We write

$$y^{k-1} = (1+t)^{k-1} = 1 + (k-1)t + \frac{(k-1)(k-2)t^2}{2!} + \cdots + t^{k-1}.$$

For $k=2$, $y=2$ is the only solution of (2).

Furthermore there exist no integers $k>2$, $y \geqq 2$ such that (2) is true; for then

$$y^{k-1} = (1+t)^{k-1} > 1 + (k-1)t \geqq k \text{ and } y^{k-1} \neq k.$$

Hence it is necessary that $y=2$ which implies that $x=4$.

III. *Solutions such that* $y>x>0$.

We merely state the analogous theorem: A necessary and sufficient conditon that a solution be of type III is that $x=2$, $y=4$.

IV. *Solutions such that* $x<0$, $y<0$.

Writing $x=-X$, $y=-Y$, we have

$$(-X)^{-Y} = (-Y)^{-X}$$

and in all cases which signs permit,

$$X^Y = Y^X.$$

It is an immediate consequence of the results in I, II, III, that, for a solution to be of type IV, it is necessary and sufficient that one of the following systems holds:

$$\begin{cases} x = -4 \\ y = -2 \end{cases} \qquad \begin{cases} x = -2 \\ y = -4 \end{cases} \qquad \begin{cases} x = -i \\ y = -i \end{cases} \qquad i = 1, 2, \cdots$$

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Modern Pure Solid Geometry. By N. Altshiller-Court. New York, The Macmillan Company, 1935. x+311 pages. $3.90.

Professor Coolidge once embarrassed the mathematical department of a certain New England college by asking them whether the altitudes of a tetrahedron are concurrent; not one knew whether they are or are not. Perhaps the reader can answer the question. Any subject which has disappeared so rapidly from the curriculum of both college and high school as solid geometry in the past generation must face the realization that it is no longer deemed essential. Does that mean it must die? Surely all will agree that the lack of growth or interest in any particular branch of mathematics will soon put it in a position where it cannot survive. The life of pure solid geometry has been prolonged by the interest of men like Neuberg, Servais, Thébault, Coolidge and Altshiller-Court. By the discovery of new theorems, by research published in various mathematical magazines—mostly not in English—a few persons still keep alive interest in this subject.

One of the most active stimulants in this direction should prove to be the publication of the text under review. It begins rightly in the most elementary phase of the subject and presupposes very little knowledge on the part of the reader. Here are presented in connected form, along lines similar to the author's treatment of plane geometry in his text *College Geometry*, the fundamental theorems and propositions dealing with the tetrahedron and the sphere. The synthetic method is used throughout, and the material "limited to the point, line, plane, circle and sphere. Conic sections are not dealt with, nor are their properties invoked in the proofs." As in his former book the author utilizes the idea of harmonic ratio, the only concept not strictly belonging to classical synthetic solid geometry.

In the three opening chapters is placed information regarding harmonic forms, homothetic figures, the trihedral angle, the skew quadrilateral and any other ideas which will make the succeeding chapters more understandable. The real substance of the book begins with a sixty-page chapter on the general tetrahedron, including also a treatment of the orthocentric, the trirectangular and the isosceles tetrahedron. Here the reader learns about bimedians and bialtitudes, that a general tetrahedron possesses a centroid, a circumcenter, an Euler line but no orthocenter. It possesses an inscribed and seven escribed spheres, and numerous relations between the radii of these spheres are given.

Then follow brief chapters on transversals and the oblique cone with a circular base. In the eighty pages of Chapter VI, devoted to spheres, are dis-

cussed orthogonal spheres, poles and polar planes with respect to a sphere, the "imaginary sphere"—an abbreviation for real geometric relations which are less clumsily stated by this device—centers, axes and planes of similitude, the power of a point with respect to a sphere, the radical plane of two spheres, coaxal pencils and coaxal nets of spheres. After a discussion of inversion the book ends with a sixty-five-page chapter on the recent geometry of the tetrahedron. This includes tetrahedral poles and polar planes, desmic systems, isogonal points, antiparallel sections, the sixteen point sphere, the twelve point sphere and the Apollonian spheres. We find here also the properties of the circumscriptible, the orthocentric, the isodynamic and the isogenic tetrahedron.

Some six hundred exercises placed after each section and at the end of each chapter, graded in difficulty, will furnish enough problems for the reader to test his skill and ability. Seven pages of biographical notes and an index are to be found at the end of the book.

This text should be of interest not only to the teacher of geometry in a high school or one preparing for such a career, but also to one desirous of doing research in solid geometry. Nowhere in such compact form can be found the material that is here presented. Professor Altshiller-Court deserves the heartiest commendation for the time and effort spent in giving us such a readable book.

J. R. Musselman

Solid Mensuration. By W. F. Kern and J. R. Bland. New York, John Wiley & Sons, 1934. Paper, viii+74 pages. $1.25.

This book is designed to furnish supplementary material in solid geometry, or to be used in a "practical" course in mensuration. The rules for computing areas and volumes of solids are given as formulas, with no explanation or proof except in the case of the prismatoid formula. The exercises number 235 in all, and are graded from immediate applications of the formulas to more puzzling types. There are no exercises in spherical geometry, but a brief syllabus of plane and solid geometry in the back summarizes the important theorems of spherical triangles. The inexcusable spelling *parallelopiped* is used. An appendix is devoted to logarithms and the slide rule; it would seem to the reviewer that in the answers to the slide rule exercises, the toleration of error is much larger than need be.

R. A. Johnson

Elements of Statistics with Application to Economic Data. By Harold T. Davis and W. F. C. Nelson. Bloomington, Indiana. The Principia Press, Inc., 1935. xi+424 pages. $4.00.

Chapters I and II contain among other things sections on collection, sources and classification of statistical data, frequency distributions, the binomial theorem, plotting of functions and empirical data, fitting straight lines, parabolas and exponential curves to data, the method of least squares and the skew-normal probability curve.

Class limits and class marks are not properly defined. Explanations of the rectangular coordinate system are inadequate as given in Chapter I. If the reader is unfamiliar with the rectangular coordinate system he is certainly not able to grasp plotting histograms as the first lesson in plotting. Too many fundamental ideas are presented in Chapter II for a beginner to grasp. The average beginner in statistics will be greatly bewildered while reading Chapters I and II.

The third chapter defines and explains how to find averages used in statistics, measures of dispersion, the coefficient of variability, and Sheppard adjustments of moments.

No method is given for finding the mean deviation without finding the deviations. The median and quartiles for grouped distributions are not presented correctly. No formulas are given for adjustments of moments of grouped discrete variates. The harmonic mean is well presented.

Chapters IV and V present index numbers, "ideal index" formulas, the secular trend, seasonal variations, link and chain relatives, corrections for seasonal variations and secular trends, correlation of time series and harmonic analysis.

After solving the problems listed one should have a good idea of how to find index numbers and analyze time series. The beginner will have many difficulties with harmonic analysis as presented.

The sixth chapter presents the ordinary elementary ideas concerning probability. Some space is given to more difficult problems.

Chapters VII and VIII introduce the binomial frequency distribution and its chief characteristics, Stirling's formula, the derivation of the skew-normal frequency function, the Gaussian function, graduation of data by use of ordinates and areas of standard curves, a neat derivation of the standard error of the mean under the assumption that the first sample average is approximately equal to the mean of the parent, measure of goodness of fit, the theory of least squares, and the use of standard errors.

Chapter IX contains ideas on fitting various curves to empirical data, together with tables for finding certain unknown coefficients. These tables greatly reduce the amount of labor. The standard error of prediction should have been introduced here to show how well the curves fit.

Chapters X and XI present linear, rank, non-linear, partial and multiple correlation together with regression curves.

The standard error of prediction is derived in Chapter X but is not used in any illustrative problem in that chapter.

Chapter XII treats of kurtosis of frequency distributions, Lexis ratio, the Charlier coefficient of disturbancy, Poisson and Lexis distributions and graduation.

The appendices contain biographic notes, logarithms and tables for the following: common logarithms, $e^{\pm x}$, x^2, $x^{1/2}$, x^{-1}, ordinates and areas for the normal curve, probability for goodness of fit, coefficients for fitting straight lines and parabolas to data.

The book contains many fine examples which should bring out the meaning of many ideas in elementary statistics. Each formula is explained by use of illustrative examples.

W. D. BATEN

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

CLUB TOPICS

1936 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By W. C. EELLS, Stanford University

In continuation of previously published lists (see this MONTHLY, vol. 42 (1935), pp. 171–73 for a list of 1935 centennial events and for references to previous volumes for corresponding lists from 1925 to 1934) of centennial dates in the history of mathematics, the following group of significant 1936 centennial dates is presented.

636 A.D. Death of Isidorus, bishop of Seville, author of *Origenes*, an encyclopedia of ancient learning in twenty volumes, one of which was devoted to the quadrivium—arithmetic, music, geometry, and astronomy.

836 A.D. Birth of Tabit ibn Korra, "one of the most brilliant and accomplished scholars produced by the Arabs" (Ball), translator of Greek works on mathematics and earliest writer outside of the Chinese to discuss magic squares.

1136 A.D. Publication of Gerard's translation of Ptolemy's *Almagest* in which it is believed that the Arabic numerals were used.

1336 A.D. Death of Richard of Wallingford, "one of the best known mathematicians of his time" (Smith), lecturer on the liberal arts at Oxford, and author of works on trigonometry and arithmetic.

1336 A.D. Introduction of a rule at the University of Paris that no student should take a degree without attending lectures on mathematics.

1436 A.D. Birth of Regiomontanus (Johannes Mueller) "one of the most prominent mathematicians of his generation" (Ball), German mathematician who wrote the first systematic treatise on trigonometry.

1636 A.D. Spiral of Fermat, the curve $\rho^2=\theta$, proposed by Fermat in a letter to Mersenne.

1736 A.D. Birth of Joseph Louis Lagrange, "the greatest mathematician of the eighteenth century" (Ball), and "one of the greatest mathematicians of all time" (Cajori).

1736 A.D. Publication of Euler's *Mechanica sive motus scientia analytice exposita*, which, according to Lagrange, was "the first great work in which analysis is applied to the science of movement."

1736 A.D. Publication of James Hodgson's *Doctrine of Fluxions* in London.

1836 A.D. Death of Ampére, French mathematician and physicist, after whom the ampere is named.

1836 A.D. Publication of first issue of Charles Gill's *Mathematical Miscellany*, a semi-annual mathematical periodical.

1836 A.D. Joseph Liouville, French mathematician, began his thirty-eighth term as editor of *Liouville's Journal*.

Club Activities

So far as this department knows, Hunter College of the City of New York has more mathematics clubs than any other institution of higher learning in the country. We quote from a recent letter:

"There are really three mathematics clubs in addition to Pi Mu Epsilon. One is for the Freshmen, one for the Sophomores and one for the Juniors and Seniors. Pi Mu Epsilon, the honorary mathematical fraternity, is the highest in rank. Pi Mu Epsilon has a program every month besides business and social meetings. The other clubs meet once a fortnight, or even once weekly for a formal program and have a party once a semester. The staff take turns in serving these clubs as faculty advisers."

Phi Chi Mu of Washington and Jefferson College is an honorary organization with a highly restricted eligibility list among juniors and seniors majoring in science. A letter from this school recently gave this interesting note. Of recent members of Phi Chi Mu, eight are this year working toward doctorates, one each in physics and mathematics at Yale University, two in mathematics at Johns Hopkins University, and one each in mathematics in Iowa State College, Princeton University, University of Michigan and University of Pennsylvania. From so restricted a group this is a noteworthy record.

Club Reports

All reports are of programs of 1934–35 and all officers are for that year unless otherwise specified.

Pi Mu Epsilon of Hunter College

The fact that this was the tenth year of the chapter's life gave opportunity for special social activities which culminated in a banquet attended by nearly half of all those initiated during the ten years. Professor Tomlinson Fort of Lehigh University was the principal speaker. It was under his inspiration that the chapter was organized while he was teaching at Hunter. Professor Lao G. Simons, Dr. Ziniti Patti, and Henrietta Chafets were the other speakers.

Following the regular custom of the chapter a book was presented to the student judged, each semester, to stand best in original preparation and exposition of a topic at a regular meeting, The winners were Sarah Forman and Bella Manel.

The officers were: Director, Professor L. S. Hill; Vice Director, Bella Manel; Corresponding Secretary, Gladys Lieder; Recording Secretary, Mabel Sander.

At the regular meetings the following subjects were discussed: "A novel treatment of geometry on a spherical surface"; "The icosahedral group"; "Groups of automorphic rotations of the regular tetrahedron and of the regular octahedron"; "The Goursat generalization of the prismatoid formula"; "Classification and analysis of quadric surfaces in ordinary space"; "Elementary corollaries of Cauchy's fundamental integration theorem"; "The tensors associated with the space-time manifold of Einstein"; "An elementary study of tensors."

Mathematics Club of Hunter College

President, Rose Bloom; Vice President, Jessie Schroeder; Secretary, Celia Levine; Treasurer, Lillian Nassau; Publicity Manager, Marie York.

The programs and speakers were as follows: "Intrinsic equations" by Mollie Orshansky; "Hyperbolic functions" by Marie York; "Theory of inversion" by Marion Crowe; "Women mathematicians" by Beatrice Falk; "Mathematical paradoxes" by Professor J. H. Bushey; "Mathematical economics" by Sarah Forman; "Mathematics and literature" by Edith Richter; "Finite geometry" by Bella Manel; "Mathematical recreations" by Ruth Bentien and Marie York; "Algebraic numbers" by Professor Mina S. Rees, "Approximate integration" by Jeanette Domoff; "Center of population" by Jessie Schroeder.

Social activities included afternoon in the gymnasium, theatre party, hike and assisting in the All-Department party.

Phi Chi Mu of Washington and Jefferson College

Officers for 1935–36 are: President, J. D. Maxwell; Secretary-Treasurer, J. L. Bryant. Fifteen students comprise the active membership. The heads of the departments of mathematics, physics and chemistry are honorary members, designated as Counsellors.

The list of papers presented was as follows: "Odds and ends in the history of chemistry" by R. G. Hay; "Borax beads" by S. C. Shaw; "Endocrines and their functions" by G. E. Hess; "Non-Euclidean geometry" by J. W. Popow; "Cathode rays" by C. M. Parshall; "Bacterial enzymes" by B. R. Mooney; "Inheritance of acquired characteristics" by P. C. Grana; "Colloids" by J. F. McPherson; "Pythagoras and his school" by R. C. Morrow; "Complex numbers" by J. I. Weinstein; "Mendelejeff and the periodic table" by H. S. Sakulsky.

At the affiliation banquet when new men were welcomed, Professor C. S. Atchison of the department of mathematics spoke on "History, development and purpose of the club," Professor T. S. Taylor of the physics department spoke on "Artificial radioactivity" and Dr. R. C. Hutchinson, President of the College, spoke on "Scientific mind in society":

The Mathematics Club of the University of British Columbia

President, G. Mossop; Vice-President, J. Wallace; Secretary-Treasurer, Phoebe Riddle. The speakers and subjects follow: "The three body problem" by Dr. D. Buchanan; "The modern mathematician" by F. Brand; "Mathematical fallacies" by I. Niven; "Some applications of vectors to mathematics" by G. Volkoff; "Graphical methods of solving various mechanical problems" by A. Frigidga; "Relativity" by G. Mossop; "Non-Euclidean geometry" by R. Christy; "Binary stars" by J. Parnall; "Theory of least squares" by D. Mitchell; "Comets" by R. Hunter; "Newton's calculus" by H. Clayton; "Spiral nebulae" by Phoebe Riddle; "Wave motions" by T. How.

The Mathematics Club of The George Washington University

President, T. E. Berry; Secretary-Treasurer, Mary A. Henderson; Faculty Adviser, Dr. Francis E. Johnston.

The programs were as follows: "Distribution of prime numbers" by T. E. Berry; "Transcendental numbers" by N. Goldman; "Mathematical logic" by W. Hoppman; "An elementary problem in geometry" by R. Been; "Statistical problem involving the Lexis theory of dispersion" by W.

Hendricks; "N-dimensional geometry" by W. Hoppman; "Curve tracing" by W. H. Gammon; "Mathematical phases of astronomy" by F. P. Scott; "Waring's problem" by Dr. F. E. Johnston; "Classification of cubic curves" by V. Schult; "Perfect numbers" by C. Lennahan.

The club held a Christmas party and a banquet and closed the year as guests of Professor and Mrs. J. H. Taylor at their summer home in Long Beach, Maryland.

Kappa Mu Epsilon of Kansas State Teachers College, Pittsburg

An outdoor picnic, entertaining the National Convention of Kappa Mu Epsilon, a Christmas party, and a mid-summer dinner were included in the chapter's activities.

The subjects and speakers at the regular meetings were as follows: "Early concepts of size and shape of the earth" by Colleen Hawkins; "Astronomical observations" by E. Blick; "Meteorites" by L. Eubanks; "The story of the stars" by Professor J. A. G. Shirk; "Extraneous solutions" by L. Smith; "Dependent and inconsistent equations" by P. Harbison; "Falacious proofs" by J. Theising and F. Gift; "Philosophy of mathematics" by Dr. R. G. Smith; "Symposium on mathematics," "In Art" L. Reeve; "In biology" by Lois Allison; "In chemistry" by H. Phillips; "In business and economics" by Josephine Schoofs; "In astronomy" by Ruby Fulton; "In physics" by V. Stevens; "As its own goal" by Professor R. W. Hart; "Organizations for mathematics teachers"; "State" by Kathryn Kelley; "National" by Florence Soskoff; "National organizations for research workers" by Edna Wasser; "International Mathematical Congress" by Henrietta Courtright; "Electrical demonstration" by V. Stevens and D. Dickinson; "Solution of the cubic equation by the use of the slide rule" by Professor L. E. Curfman; "Symposium on trends in mathematics"; "In junior high schools" by Ruth Kriegsman; "In senior high schools" by W. Guthridge; "In junior colleges" by Professor R. W. Hart.

The Mathematics Club of The University of Buffalo

Officers for 1935–36 are: President, Josephine W. Pound; Secretary-Treasurer, Harriet Luth.

The Wilfrid Sherk Memorial Prize, awarded annually by the club to the pupil writing the best original paper in mathematics, was won by H. Mackel.

The regular programs included the following papers: "Conformal mappings"; "Mathematical paradoxes"; "Game of nim"; "To construct a triangle given vertex, incenter, and orthocenter."

Pi Mu Epsilon of The Pennsylvania State College

Officers for 1935–36 are: President, C. M. Kearns; Secretary, S. H. Galiszewski; Treasurer, W. O. Gordon; Director, Dr. H. L. Krall.

The chapter is sponsoring a prize examination in differential calculus, open to all sophomores in the College.

One of the interesting papers in the autumn of 1935 was given by R. H. Van Horn on "16th and 17th century mathematics." He submits his bibliography as follows:

D. E. Smith, *Source Book of Mathematics*, 1929; F. Cajori, *A History of Mathematics*, 1919; and *History of Mathematical Notation*, 1928; W. R. R. Ball, *A Short history of Mathematics*, 1901; Sullivan, *A Short History of Mathematics.*

Pi Mu Epsilon of The University of California at Los Angeles

This chapter at the initiation of new members asked each to make a brief talk on subjects of less than serious importance. At another initiation the new members were required to solve trick problems.

At the regular meetings papers were presented on the following subjects: "The history of the development of trigonometry"; "The Fourier series"; "Hermite's process of integrating rational fractions"; "The theory of sampling"; "The anomalous dispersion discovered by Knudsen."

Officers for 1935–36 are, Director, J. LeR. Kelly; Vice Director, Olive Hoover; Secretary, Erma L. Schmalzried; Treasurer, Professor W. E. Mason; Librarian, Jean Lewis.

Mathematics Club of The University of California at Los Angeles

Anyone interested in mathematics is welcomed. Each meeting has a social half hour and several social events are held during the year. At one meeting a play, "The evolution of numbers," was presented and the best known athlete in the club, J. LuValle, gave a dialogue, "The mathematical coach." At another meeting was held a spelling bee, mathematical terms being spelled backward.

At the regular meetings the papers included: "Mathematical notation"; "Leonardo da Vinci"; "Problems with fallacious reasoning"; "The polar planimeter"; "In commemoration of Alcuin"; "The organization of tables."

Officers for 1935–36 are: President, W. H. Glenn, Jr.; Vice-President, Lucille Donovan; Secretary, Helen Yerian; Treasurer, Jean Lewis; Librarian, D. Gorman.

Pi Mu Epsilon of Brooklyn College

Director, Professor L. T. Moore; Vice Director, Marcia Brener; Secretary, Florence Plotkin; Treasurer, W. Forman; Librarian, J. Schlamberg.

Besides social and business meetings the chapter held seven regular meetings with papers as follows: "Irrational numbers" by B. Greenspan; "Boolean algebra" by Esther Block; "Numerical right triangles" by I. Richards; "Inscribed and escribed circles and spheres" by J. Schlamberg; "Empirical formulae and method of least squares" by A. Blumenfeld; "Higher plane curves of the fourth order" by Professor L. T. Moore; "Graeffe's method" by J. Lorell.

Euclidean Circle of The University of Indiana

The club held an initiation for new members, and a Christmas meeting as the guests of Dean Agnes E. Wells at which Brigette Raphael of Berlin spoke on "German Christmas customs" and Professor Cora B. Hennel discussed "Grafting Christmas spirit." An annual picnic completed the social program of the club.

At the regular meetings the following papers were read: "College education" by Dr. S. G. Hacker; "Mayan mathematics" by Professor D. A. Rothrock; "The shape of the molecule" and "Mathematical physics" by W. J. Kirkham, "Principles of applied resonance" by M. Wells.

Kappa Mu Epsilon of Mississippi Agricultural and Mechanical College

President, B. K. Buder; Secretary, R. L. Wilson; Treasurer, E. H. Bourquard; Corresponding Secretary and Faculty Adviser, Professor C. D. Smith. Two luncheons and a banquet were held, and the initiation of seventeen new members, as the social part of the chapter's activities.

Dr. Kathryn Wyant, National President of Kappa Mu Epsilon, spoke on "National programs in mathematics." A general discussion of "A century of progress in mathematics" was lead by S. A. Cooper and, at a joint meeting with the chapter at Mississippi State College for women, W. D. Chadwick spoke on "Types of correspondence" and a general discussion was held on "Trends in state school systems regarding emphasis on mathematics."

During the year more than sixty drawings were submitted for display on large placards. They were drawn in colors for use as a mathematical exhibit. This is proposed as an annual project.

Hypatia of Harris Teachers College

Besides parties and business meetings, this club held monthly meetings with the following papers: "History of astronomy" by C. R. Mutshnicky; "The earth and its movements" by Dr. N. Whitney; "A trip to the moon" by A. Stark; "Gauss, the mathematician" by Rev. J. H. Gauss, President of St. Louis Bible School; "Seeing the earth from other worlds" by A. Dolan; "Cosmic rays" by Dr. J. D. Whitney; "Hypatia" by Professor Jesse Osborn; "The earliest arithmetics" by H. Barr.

Professor Osborn writes, "Rev. Gauss, now eighty years old, is a classical scholar. He related

some of the traditions of the Gauss family and his grandfather. In speaking of the genius of Karl Friedrich Gauss, his grandson said 'There was no promise of it in ancestry and no trace of it in posterity'."

The Northeastern University Mathematics Society

Seven meetings were held with the following lectures: "Transformation of motion" by Professor G. A. Meserve; "The path of a projectile" by Professor J. Spear; "Non-Euclidean geometry" by Professor Norbert Wiener, Massachusetts Institute of Technology; "Spherical trigonometry" by E. E. Haskins; "Hyperbolic trigonometry" by Professor W. L. Smith; "Mathematics of the inexact" by Professor A. A. Bennett of Brown University; "Equivalence of equations" by Professor A. E. Whittaker; "$e^{\pi i}=-1$" by Professor J. Spear; "One to one correspondence" by E. E. Haskins.

Pi Mu Epsilon of The University of Washington

Officers: Director, W. Duthie; Vice Director, R. Johnson; Secretary-Librarian, Marigale Osborne.

Besides a banquet, a picnic with Zeta Mu Tau, the general mathematics club of the University, and an evening at the home of Professor and Mrs. J. P. Ballantine, the chapter held twelve regular meetings with the following programs: "Theory of definite integrals" by J. Carlson; "Transfinite numbers" by G. Whetstone; "Pendulums", by N. Mihailoff; "On foundations of arithmetic" by E. Rodenhouse; "Partial differentials" by R. Reed; "Number theory" by W. Duthie; "Polyhedrons" by Marigale Osborne; "Irrational numbers" by Donna Seaman; "Confocal concoids" by R. Johnson; "Matrices" by S. Rubens; "Statistical mechanics" by N. K. Olson; "Chinese mathematics" by Dr. K. Saltan; "Arccotangent triads" by Professor J. P. Ballantine; "Use of vectors for deriving the formulas of trigonometry" by M. Hostetter; "Types of classes" by H. Robinson; "History of numbers" by Florence Tongawa; "History of calculus" by Alberta Bierbaum.

Pi Mu Epsilon of Iowa State College

Director, Harriet Wilson; Vice Director, A. R. Brown; Treasurer, F. J. McCormick; Secretary, Winifred McBeath; Librarian, F. A. Brander; Faculty Adviser, Dr. D. L. Holl; Permanent Secretary, Professor E. R. Smith.

The winner of the Pi Mu Epsilon prize for the highest scholastic average in the first two years was D. J. Byers.

The chief activity of the chapter was the sponsoring of a mathematics club, with a large number of junior college students appearing on the program.

The chapter enjoyed an evening at the home of Professor and Mrs. E. R. Smith. There were business meetings and initiations. At the regular meetings the following programs were presented: "Six women go shopping" by E. Jebe; "A cat feasts on mice" by O. McCoy; "A duck swims a river" by Madelyn Kerr; "Mathematics in the Orient" by Professor Julia Colpitts; "i^i" by R. H. Cook; "Reduction of fractions by wrong method with correct results" by E. L. Anderson; "Use of the abacus" by Alice Churchill; "Mathematics in Germany" by Professor J. J. L. Hinrichsen; "Boomerang functions" by Phyllis Christy; "Divine kine" by J. Clendeninn; "Gay dice" by A. Christenson; "The differential analyzer" by H. W. Anderson; "Rational right triangles" by J. Egan; "Magic squares and cubes" by Laura Bliss; "Floating lead on H_2O" by A. T. Jaques; "An evening of mathematical paradoxes," "All triangles are isosceles" by L. Nichol, "Achilles and the tortoise" by T. E. McConnel, "The Petersburg problem" by J. Gustafson, "The paradox of Easter" by W. Dyer; "Cube root by division, without logarithms" by R. Boudinot; "How to make a perpetual motion machine" by G. C. Truedell; "Pandromic rings" by R. D. Jensen; "An obtuse angle equals a right angle" by R. W. Richards.

The year closed with a play, "Falling in love with mathematics," presented by the new initiates and by selections from the mathematics quartette.

Pi Mu Epsilon of The University of Nebraska

Of the annual prizes offered by this chapter in 1934–35, G. Petersen won the prize in calculus, and J. King and G. Wooters divided the prize in analytics.

Besides an initiation banquet and a picnic, the chapter held monthly meetings with the following programs: "The use of the complex variable in alternating current problems" by M. Halderson; "Solution of the cubic equation" by R. Doubt; "The harmonic analyzer" by J. Cowgill; "The slide rule" by M. Young; "Life and work of Isaac Newton and Wilhelm Leibniz" by Mary Swift; "Planetary orbits" by G. Brubaker; "Cramer's rule for the solution of simultaneous equations" by J. Marvin; "Trisection of an angle" by R. Pate; "Egyptian mathematics and the Rhind papyrus" by E. Johnson; "Old mathematics books" by Professor J. D. Fitzpatrick; "Discussion of a curve tracing machine" by Professor M. G. Gaba.

Pi Mu Epsilon of The Ohio State University

Director, W. A. Patterson; Vice Director, L. D. Rodabaugh; Treasurer, A. H. Bailey; Secretary, F. J. Krieger.

The chapter this year established the Charles L. Arnold Memorial Fund, in honor of its late founder. The initial gift was $100. The memorial is to consist of a collection of mathematical models representative of the mathematics taught at The Ohio State University.

The first steps were taken for an undergraudate club to reach the junior college students. One evening party and a dinner in honor of Professor E. P. Lane of the University of Chicago were held, and also the initiation banquet to which, as is the annual custom, the leading boy and girl freshman and sophomore students were invited guests. Professor L. LePaz acted as master of ceremonies at the banquet and Professor A. Avey, of the department of philosophy, spoke on "The relationship between mathematics and philosophy."

At the monthly meetings, the following papers were read: "Properties of the Simson lines of a set of triangles" by Professor J. H. Weaver; "Space curves belonging to a non-singular linear line complex" by Dr. C. E. Wylie; "The composition of forms" by Dr. E. D. Jenkins; "Mathematics by machinery" by Professor H. L. Hazen of the Massachusetts Institute of Technology; "The basis of a cubic field" by Professor C. C. MacDuffee; "The isoperimetric inequality" by Professor T. Radó; "Some new results in projective differential geometry" by Professor E. P. Lane; "Postulational development of quaternions" by Professor F. R. Bamforth; "The use of mathematics by the civil engineer" by Professor C. T. Morris of the department of civil engineering.

Pi Mu Epsilon of University of Georgia

Dr. D. F. Barrow, Director; Eleanor G. Adams, President; Mabel Stephens, Vice President; C. C. Bolger, Secretary; H. H. Walker, Treasurer.

Besides business meetings and an initiation banquet the chapter held eight meetings with programs of papers and discussions. The list of papers was as follows: "Number systems" by H. H. Walker; "Comprehensive mathematics tests for freshman in the university system of Georgia" by Professor H. M. Cox; "Symbolic logic and mathematics" by J. Gitler; "A Problem on rational roots of the general quadratic equation" by M. Stephens; "Interesting facts about Charles L. Dodgson" by Lucy Dillard; "The illustrations in Dodgson's works" by F. Foster; "Mnemonic devices for teachers of mathematics" by K. Thaxton; "The digits of π" by. Pope Hill; "The attained and the unattained in the teaching of mathematics" by Dr. F. A. Beers; "Revolving numbers" by Dr. D. F. Barrow; "Applying law of probability to newspaper article 'Who made that ladder' " by Professor P. R. Hill; "Nasik magic squares" by Dr. R. A. Hefner, Georgia School of Technology.

The Mathematics Club of Smith College

Nine meetings were held, including a Christmas party with games and grinds, a picnic with mathematical games and contests, and an amusement meeting in charge of Lee Bottome and Katharine Kester. A play, "The eternal triangle" was given at the first meeting of the year.

The other programs were as follows: "Science and culture" by Professor J. Murlen, read by Ann Huthsteiner; "Hindu-Arabic numbers" by Olive Oliver; "Greek mathematicians" by Alice L. Forbes; "Report on the meeting of the Mathematical Association of America" by Professor Susan M. Rambo; "Conformal mapping" by Miriam Ayer; "The mathematics major and her place in society" by Professor Suzan R. Benedict; "The magic number nine" by Margaret Plante.

Mathematics Club of Albion College

This club has introduced a feature, unique in club activities. At each meeting there is a roll call to be answered as announced at a previous meeting. Some of these roll calls were: Name of a mathematician, together with some outstanding event of his life; Formula from spherical trigonometry and its application; Formula from integral calculus with an application; A plane geometry theorem and an application.

Following is the list of papers presented at regular meetings: "Three schools of mathematics" by Kathryn Howes; "Original locus problem" by R. Fowler; "Advantages of different number bases" by R. Edick; "Mathematics and beauty" by J. Ludwig; "Mathematical theory of planimeters" by A. Larson; "Mathematics of finance" by Martha Sylvester; "Engineering problem, buckling of plates" by Dr. M. Miles; "Canterbury puzzles" by Harland Bristah; "Mathematics and chemistry" by W. Abbott.

Allan Larson's paper is expected to appear soon in School Science and Mathematics. He is a senior this year.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 205. *Proposed by Emma Lehmer, Lehigh University.*

Prove that

$$\frac{1}{k+1}\sum_{t=1}^{n}(-1)^{t+1}\frac{{}_nC_t}{{}_{t+k}C_{k+1}} = H_{n+k} - H_k,$$

where H_n is the sum of the reciprocals of the first n positive integers.

E 206. *Proposed by L. S. Johnston, University of Detroit.*

Show that for any plane quadrilateral (convex or otherwise) the intersection of the diagonals, the intersection of the medians, and the centroid are collinear, and that the distance from the intersection of the diagonals to the intersection of the medians is three times the distance from the intersection of the medians to the centroid. (A median of a quadrilateral is here understood to be a line joining the midpoints of opposite sides.)

E 207. *Proposed by V. Thébault, Le Mans, France.*

Find a number of six digits which is a perfect square both in the decimal system and in the system of enumeration based on five. Show that the solution is unique.

E 208. *Proposed by G. A. Whittemore, New York City.*

How may eight married couples play a seven-round bridge tournament, if each man plays one round as partner with each lady except his wife, and everybody plays against everyone else except their spouse? How many solutions exist? Can this be generalized for $4n$ couples playing $4n-1$ rounds? Does a solution exist for $4n+2$ couples playing $4n+1$ rounds?

E 209. *Proposed by J. M. Feld, New York City.*

Show that the equation $a \cos \theta + b \sin \theta = c$, $0 < c$, can be solved graphically by the following method: With center having the rectangular Cartesian coordinates $(a/2, b/2)$, describe the circle K passing through the origin O. With O as center and radius c, strike arcs cutting K at A and B. Then the angles XOA and XOB are the desired values of θ.

E 210. *Proposed by S. A. Anderson, New York City.*

In the well-known problem of the two ladders leaning across the street from opposite sides, it is customary to give the lengths of the ladders, a and b, and the height c of their intersection above the street. Assuming the touching sides of the two ladders to be in a vertical plane straight across the street, with each end of each ladder touching one of the two vertical walls bounding the street, it is usually required to find the width d of the street. In the present version of the problem however, it is required to find the set of positive integer values which may be consistently assigned to a, b, c and d, such that $b < a$, and such that no such set exists with a smaller a.

SOLUTIONS

E 176 [1935, 565]. *Proposed by V. Thébault, Le Mans, France.*

Find a number of five digits which are consecutive, though not in their natural order, and such that the square of this number contains the ten different digits. Show that there is just one solution.

Solution by E. P. Starke, Rutgers University.

Let N be the required number. Since the sum of the digits of N^2 is divisible by 9, the sum of the digits of N is divisible by 3. Let d be the smallest digit in N. Then $d+(d+1)+(d+2)+(d+3)+(d+4)$ is a multiple of 3, and d is 1 or 4. Therefore 12345 or 45678, in some order, are the digits of N.

Thus there are 240 possible values of N to be tested. From a table of the squares of the numbers from 1 to 1000 can be read the first three and the last three digits of N^2. Only these few in which no duplication of digits has thus appeared, need be tested by actual multiplication. In this manner it appears

that $12543^2 = 0157326849$ and $46587^2 = 2170348569$ are the only values of N^2. If an initial zero is not acceptable, there remains the unique solution, $N = 46587$.

Also solved by W. E. Buker, J. E. Burnam, Mary L. Constable, C. W. Trigg, Simon Vatriquant and the proposer.

E 177 [1935, 566]. *Proposed by F. A. Lewis, University of Alabama.*

If the cosines of the half-angles of a triangle be divided by the lengths of the corresponding bisectors, the sum of the three ratios thus formed equals the sum of the reciprocals of the sides of the triangle.

Solution by W. R. Hardman, West Lafayette, Indiana.

Let AD, BE and CF be the respective bisectors of the angles of the triangle ABC. Then the area of the triangle equals

$$\tfrac{1}{2}CA \cdot AB \sin A = \tfrac{1}{2}AD \cdot AB \sin \tfrac{1}{2}A + \tfrac{1}{2}CA \cdot AD \sin \tfrac{1}{2}A$$

or

$$CA \cdot AB \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A = \tfrac{1}{2}AD \sin \tfrac{1}{2}A(AB + CA).$$

From this we get

$$\frac{\cos \frac{1}{2}A}{AD} = \frac{AB + CA}{2CA \cdot AB} = \frac{1}{2}\left(\frac{1}{CA} + \frac{1}{AB}\right)$$

Adding the two symmetrical equations obtainable from this by cyclic permutation of A, B and C, and also D, E and F, we obtain the desired relation,

$$\frac{\cos \frac{1}{2}A}{AD} + \frac{\cos \frac{1}{2}B}{BE} + \frac{\cos \frac{1}{2}C}{CF} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}.$$

Also solved by M. W. Aylor, W. E. Buker, Mannis Charosh, Brent Clark, W. B. Clarke, Wm. Douglas, O. E. Eggert, E. W. Franz, H. E. H. Greenleaf, L. M. Kelly, Leon Recht, Augustus Sisk, E. P. Starke, J. E. Thompson, C. W. Trigg, Simon Vatriquant and the proposer.

E 178 [1935, 566]. *Proposed by C. H. Forsyth, Dartmouth College.*

A, B and C play nine holes of golf, counting 5 points for the lowest score on a hole, 3 points for the next lowest, and 1 point for the highest. Their final total scores are; A 29, B 27, and C 25, although A makes the least number of low scores and C the greatest.

If there are no ties on any hole, determine the scores on each of the nine holes, without respect to the order of the holes. Show that the result is unique.

Solution by E. E. Heimann, East Central State Teachers College, Ada, Okla.

Since C makes the greatest number of low scores, we find that he must make four scores of five each, since he cannot score only three scores of five each and still score the greatest number, nor can he score more than four such scores and still score on each of the remaining holes. Similarly, since A makes the least

number of low scores, he must make low score twice and B thrice. That leaves the score card with these points scored (independent of the order of the holes):

A	55x	xxx	xxx	29
B	xx5	55x	xxx	27
C	xxx	xx5	555	25 .

We then find that the remaining five scores that C makes must be worth one point each. When we enter these on C's score card, we also know what the scores of A and B on these holes must be. At this stage the score card looks like this:

A	553	33x	xxx	29
B	335	55x	xxx	27
C	111	115	555	25 .

We now find that A has a total of nineteen points on five holes, and must therefore score ten points on the remaining four holes, by an appropriate number of 3's and 1's. Obviously, A must score three more 3's and one 1. We can now finish A's score card, and then B's, since he got the scores not otherwise accounted for. The final score card reads:

A	553	333	331	29
B	335	551	113	27
C	111	115	555	25 .

Since the order of the holes is not to be considered in the problem, the solution is shown to be unique from its manner of derivation.

Also solved by L. J. Adams, Frank Ayres, Jr., W. E. Buker, J. E. Burnam, Mannis Charosh, W. B. Clarke, E. W. Franz, H. E. H. Greenleaf, K. W. Johnson, M. W. Keller, O. M. Rogers, E. P. Starke, C. W. Trigg, M. J. Turner and the proposer.

E 179 [1935, 566]. *Proposed by J. Rosenbaum, Hartford Federal College.*

In a triangle ABC the points D, E and F trisect the sides such that $BC=3BD$, $CA=3CE$, and $AB=3AF$. Similarly the points G, H and I trisect the sides of triangle DEF such that $EF=3EG$, $FD=3FH$, and $DE=3DI$. Prove that the sides of triangle GHI are parallel to the sides of triangle ABC and that each side of the smaller triangle is one-third as long as its parallel side in the larger triangle.

Solution by T. C. Esty, Amherst College.

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the position vectors of the vertices, with a similar notation for the vectors of the other points mentioned in the problem. Then $\mathbf{D}=(\mathbf{C}+2\mathbf{B})/3$, $\mathbf{E}=(\mathbf{A}+2\mathbf{C})/3$, $\mathbf{F}=(\mathbf{B}+2\mathbf{A})/3$, $\mathbf{G}=(\mathbf{F}+2\mathbf{E})/3$, $\mathbf{H}=(\mathbf{D}+2\mathbf{F})/3$, $\mathbf{I}=(\mathbf{E}+2\mathbf{D})/3$, and $\mathbf{H}-\mathbf{G}=(\mathbf{D}-2\mathbf{E}+\mathbf{F})/3=(\mathbf{C}+2\mathbf{B}-2\mathbf{A}-4\mathbf{C}+\mathbf{B}+2\mathbf{A})/9$ $=(\mathbf{B}-\mathbf{C})/3$. Similarly, or by cyclic permutation, $\mathbf{I}-\mathbf{H}=(\mathbf{C}-\mathbf{A})/3$ and $\mathbf{G}-\mathbf{I}$ $=(\mathbf{A}-\mathbf{B})/3$. This completes the proof that each side of the inner triangle is parallel to, and equal to one-third of, the corresponding side of the outer triangle.

Also solved by M. W. Aylor, W. E. Buker, Mannis Charosh, W. B. Clarke, H. E. H. Greenleaf, W. R. Hardman, M. W. Keller, Leon Recht, E. P. Starke, Virginia Sweeney, C. W. Trigg, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3778. *Proposed by L. J. Adams, High School, Beverly Hills, Calif.*

Solve

$$\frac{dy}{dx}=\frac{1}{x}+\frac{1}{y}\cdot$$

3779. *Proposed by W. H. Roever, Washington University.*

It is well known that there exist linear transformations which carry a quadratic form in n variables into itself (See *On the linear transformations of a quadratic form into itself*, Percey F. Smith, Trans. Amer. Math. Soc., vol. 6, 1905, pp. 1–16).

A simple proof is desired, especially for the cases $n=2, 3$.

3780. *Proposed by J. M. Feld, New York City.*

In triangle $A_1A_2A_3$ the transversal A_iD_i divides A_jA_k in the ratio $A_jD_i:D_iA_k=p_i:q_i$, where ijk is a cyclic permutation of 123. Find the value of the cross ratio

$$\frac{P_3P_2}{P_2A_1}\bigg/\frac{P_3D_1}{D_1A_1}$$

in terms of the p's and q's. Show that Ceva's theorem is a special case.

3781. *Proposed by Otto Dunkel, Washington University.*

Evaluate

$$\sum_{k=1}^{n}\frac{\cos m(2k-1)\pi/n}{\sin^2 (2k-1)\pi/2n}$$

in terms of the integers m and n, $0\leqq m\leqq n$: also the sum where the sine is in the numerator instead of the cosine.

3782. *Proposed by Otto Dunkel, Washington University.*

Given the developable surface whose edge of regression is a circular helix, find the form of a section of the surface by a plane normal to the edge of regression.

SOLUTIONS

3705 [1934, 581]. *Proposed by Raphael Robinson, University of California at Berkeley.*

Show that when the quadratic form

$$\sum_1^n |i-j|\, x_i x_j,\ n > 1,$$

is reduced to the sum of squares by a real linear transformation, one of the terms will be positive, the other $n-1$, negative. The determinant of the form has been proposed for evaluation in problem 3667 [1934, 193].

Solutions and Discussion by G. Szegö, Washington University.

Solution I. Denote by D_n the determinant of the $(n+1)$th order of the quadratic form of the problem in that number of variables, and by Z_ν the νth row of the determinant. Replace the Z_0 row by $Z_0+Z_2-2Z_1$ and we obtain a determinant in which the first row is

$$0,\ 2,\ 0,\ 0,\ \cdots,\ 0,$$

and the remaining elements are unchanged. The same operation for the first column gives for its elements $-4,\ 2,\ 0,\ 0,\ \cdots,\ 0$, so that the development of this last determinant in terms of its first row gives

$$D_n = -4D_{n-1} - 4D_{n-2}, \qquad n = 2, 3, 4, \cdots,$$

where $D_0=0$ and $D_1=-1$. Mathematical induction gives

$$D_n = (-1)^n 2^{n-1} n.$$

Since $D_n \neq 0$, $n \geqq 1$, the rank is $n+1$. Furthermore, a known criterion (cf. Frobenius, *Journal f. Math.* vol. 114, 1895, p. 188) gives for the signature of the form the value

$$\sum_{\nu=0}^{n} \operatorname{sign}(D_{\nu-1}D_\nu) = -(n-1), \qquad D_{-1} = 1;$$

and this proves the theorem of the problem.

Solution II. The theorem can be easily obtained also from an interpolation formula derived in my paper *Über einen Satz des Herrn Serge Bernstein*, Schriften der Königsberger Gel. Ges. 1928, p. 65. Let $f(\phi)$ be a trigonometric polynomial with real coefficients and $g(\phi)$ its conjugate polynomial:

$$\begin{aligned} f(\phi) &= a_0 + a_1\cos\phi + b_1\sin\phi + \cdots + a_n\cos n\phi + b_n\sin n\phi, \\ g(\phi) &= \quad\; - b_1\cos\phi + a_1\sin\phi + \cdots - b_n\cos n\phi + a_n\sin n\phi. \end{aligned} \tag{1}$$

Then we have

$$g'(\phi) = \frac{1}{2}nf(\phi) - \frac{1}{n}\sum_{l=1}^{n}\frac{f[\phi + (2l-1)\pi/n]}{1 - \cos(2l-1)\pi/n}, \tag{2}$$

so that for $\phi = 0$

$$a_1 + 2a_2 + \cdots + na_n = \frac{1}{2}nf(0) - \frac{1}{n}\sum_{l=1}^{n}\frac{f[(2l-1)\pi/n]}{1-\cos(2l-1)\pi/n}. \tag{3}$$

Setting

$$\begin{aligned} f(\phi) &= |x_0 + x_1e^{i\phi} + x_2e^{2i\phi} + \cdots + x_ne^{ni\phi}|^2 = \sum_{p,q=0,1,\cdots,n} x_p\bar{x}_qe^{i(p-q)\phi} \\ &= \sum x_p\bar{x}_q\cos(p-q)\phi + i\sum x_p\bar{x}_q\sin(p-q)\phi, \end{aligned} \tag{4}$$

where $\bar{x}_q$ means the conjugate of x_q, we obtain the identity

$$\begin{aligned} \sum_{p,q=0,1,\cdots,n} |p-q|\,x_p\bar{x}_q &= \frac{1}{2}n|x_0 + x_1 + \cdots + x_n|^2 \\ &\quad - \frac{1}{n}\sum_{l=1}^{n}\frac{|x_0 + x_1z_l + \cdots + x_nz_l^n|^2}{1-\cos(2l-1)\pi/n}, \\ z_l &= e^{i(2l-1)\pi/n}, \end{aligned} \tag{5}$$

where the vertical bars mean the absolute value of the expression between them. This last result contains the theorem of the problem, since the $n+1$ linear forms

$$x_0 + x_1 + \cdots + x_n, \quad x_0 + x_1z_l + \cdots + x_nz_l^n, \quad l = 1, 2, \cdots, n, \tag{6}$$

are linearly independent.

The representation just given yields easily the determinant D_n in the first solution. For, if we denote by Δ the determinant of the system (6), we have

$$D_n = |\Delta^2|\frac{n}{2}\prod_{l=1}^{n}\frac{1}{n}\,\frac{-1}{1-\cos(2l-1)\pi/n}. \tag{7}$$

A reduction similar to that used for the Vandermonde determinant gives

$$\Delta = (-1)^n\prod_{l=1}^{n}(e^{i(2l-1)\pi/n} - 1)\begin{vmatrix} \epsilon & \epsilon^2 & \cdots & \epsilon^n \\ \epsilon^3 & \epsilon^6 & \cdots & \epsilon^{3n} \\ \cdot & \cdot & \cdots & \cdot \\ \epsilon^{2n-1} & \epsilon^{2(2n-1)} & \cdots & \epsilon^{n(2n-1)} \end{vmatrix}, \qquad \epsilon = e^{i\pi/n}.$$

The multiplication theorem for determinants then yields

$$(-1)^nD_n = \left(\frac{2}{n}\right)^{n-1}\left\{\operatorname{abs}\begin{vmatrix} \epsilon & \epsilon^2 & \cdots & \epsilon^n \\ \epsilon^3 & \epsilon^6 & \cdots & \epsilon^{3n} \\ \cdot & \cdot & \cdots & \cdot \\ \epsilon^{2n-1} & \epsilon^{2(2n-1)} & \cdots & \epsilon^{n(2n-1)} \end{vmatrix}\right\}^2$$

$$= \left(\frac{2}{n}\right)^{n-1} \begin{vmatrix} n & 0 & \cdots & 0 \\ 0 & n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & n \end{vmatrix} = 2^{n-1}n.$$

Discussion. 1. The quadratic form

$$(1) \qquad \sum_{p,q=0,1,\cdots,n} |p-q|\, x_p x_q$$

can be treated from the more general point of view of the "striped" or Toeplitz' forms. Let $f(\phi)$ denote a real periodic function with the period 2π integrable say in Riemann's sense. Introducing its Fourier development

$$(2) \qquad f(\phi) \sim a_0 + 2\sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

and writing $c_n = a_n + ib_n$ $(n=0, 1, 2, \cdots, b_0=0)$, $c_{-n} = \bar{c}_n$, we have for arbitrary integral values of n:

$$(3) \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)e^{in\phi}d\phi = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)z^n d\phi, \qquad z = e^{i\phi}.$$

The Hermitian forms

$$(4) \qquad \begin{aligned} H_n = H_n(f) &= \sum_{p,q=0,1,\cdots,n} c_{p-q}x_p\bar{x}_q \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)\,|x_0 + x_1 z + x_2 z^2 + \cdots + x_n z^n|^2 d\phi \\ &\qquad\qquad (z = e^{i\phi};\ n = 0, 1, 2, \cdots) \end{aligned}$$

are called the Toeplitz' forms corresponding to the function $f(\phi)$. They have numerous remarkable properties and close connections with the function $f(\phi)$. The determinant

$$(5) \qquad D_n = [c_{p-q}], \qquad p, q = 0, 1, \cdots, n,$$

is "striped," i.e. it contains constant elements in all lines parallel to the main diagonal.*

2. In the special case of a trigonometric polynomial of the first order: $f(\phi) = a_0 + 2(a_1 \cos \phi + b_1 \sin \phi)$, the determinants (5) can be calculated easily. We find that†

$$(6) \qquad D_n = (a_1^2 + b_1^2)^{(n+1)/2} \frac{\sin (n+2)\alpha}{\sin \alpha} \text{ with } \cos \alpha = \frac{a_0}{2(a_1^2 + b_1^2)^{1/2}}.$$

* Cf. my papers: *Beiträge zur Theorie der Toeplitz'schen Formen.* Mathematische Zeitschr., vol. 6, 1920, pp. 167–202; vol. 9, 1921, pp. 167–190.—Here a list of the previous literature is given.

† Cf. e.g. Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, pp. 319–320.—Of course, $\sin (n+2)\alpha/\sin \alpha$ is a polynomial in $\cos \alpha$ of the degree $n+1$. Thus (6) is valid also for $|\cos \alpha| > 1$ i.e. for imaginary α.

Taking $f(\phi)-\lambda$ instead of $f(\phi)$, we obtain the "characteristic values" of $H_n(f)$:

(7) $$\lambda_{\nu n} = a_0 - 2(a_1^2 + b_1^2)^{1/2} \cos \frac{\nu+1}{n+2}\pi, \qquad \nu = 0, 1, 2, \cdots, n,$$

which are, by the way, identical with the values of $f(\phi)$ at some equidistant places.* Using (7) there is no difficulty in calculating the rank and signature of the form $H_n(f)$ in this special case.

3. The set (1) of quadratic forms (or of the corresponding Hermitian forms) cannot be interpreted as a special case of (4) because

$$2\sum_{n=1}^{\infty} n \cos n\phi$$

is not a Fourier series. However a slight modification of (1), namely

(8) $$\sum_{p,q=0,1,\cdots,n} |p-q|\, r^{|p-q|-1} x_p x_q, \qquad -1 < r < +1,$$

satisfies this condition. We prove now for $-1 \leqq r \leqq +1$ the following theorem:

Let the real angle α be defined by the conditions $\cos\alpha = -2r/(1+r^2)$, $0 \leqq \alpha \leqq \pi$. The determinant of the form (8) *is given by the identity in r*

(9) $$\left[\,|p-q|\, r^{|p-q|-1}\right]_0^n = -(1+r^2)^{n-1}\frac{\sin n\alpha}{\sin \alpha}.$$

The following table gives an account of the number of the positive, negative and vanishing squares in the canonical representation of (8) $(n \geqq 2)$:

	+	−	0
$k\pi/n < \alpha < (k+1)\pi/n$ $k = 0, 1, \cdots, n-1$	$n-k$	$k+1$	
$\alpha = 0$	n	1	
$\alpha = k\pi/n$, $k = 1, 2, \cdots, n-1$	$n-k$	k	1
$\alpha = \pi$	1	n	

* This property holds generally in an asymptotical sense. Cf. the papers quoted in (1).

For $r=1$, i.e. $\alpha=\pi$, we obtain the form (1). The expression (9) becomes then $(-1)^n 2^{n-1} n$ which, together with the last row of the table, furnishes the answer to the special problems 3667 and 3705.

4. *Proof.* The formula (9) is obvious for $n=0$ and $n=1$. We may therefore assume $n \geqq 2$. We consider for $-1<r<+1$ the function

$$f(\phi) = 2\sum_{n=1}^{\infty} nr^{n-1}\cos n\phi = \frac{\partial}{\partial r}\frac{1-r^2}{1-2r\cos\phi+r^2} = \frac{-4r+2(1+r^2)\cos\phi}{(1-2r\cos\phi+r^2)^2}.$$

The corresponding Toeplitz' form of $n+1$ variables will be

$$\sum_{p,q=0,1,\cdots,n} |p-q|\, r^{|p-q|-1} x_p \bar{x}_q$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)\,|x_0+x_1z+x_2z^2+\cdots+x_nz^n|^2 d\phi, \qquad z=e^{i\phi}.$$

The identity in z

$$x_0+x_1z+\cdots+x_nz^n = y_0+y_1z+(z-r)^2(y_2+y_3z+\cdots+y_nz^{n-2})$$

defines a linear transformation $\{x_\nu\}\to\{y_\nu\}$ with the determinant 1. Consequently the quantities in question remain unchanged in going over to the form

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)\,|y_0+y_1z+(z-r)^2(y_2+y_3z+\cdots+y_nz^{n-2})|^2 d\phi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)\{|y_0+y_1z|^2+2R(\bar{y}_0+\bar{y}_1z^{-1})(z-r)^2(y_2+y_3z+\cdots+y_nz^{n-2})$$
$$+|z-r|^4|y_2+y_3z+\cdots+y_nz^{n-2}|^2\}d\phi, \qquad z=e^{i\phi},$$

R denoting the real part. We have, however,

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)d\phi = 0, \qquad \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\phi)e^{i\phi}d\phi = 1$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)(\bar{y}_0+\bar{y}_1z^{-1})(z-r)^2(y_2+y_3z+\cdots+y_nz^{n-2})d\phi$$

$$(10)\qquad = \frac{1}{2\pi i}\oint \frac{-4r+(1+r^2)(z+z^{-1})}{(z-r)^2(z^{-1}-r)^2}(\bar{y}_0+\bar{y}_1z^{-1})(z-r)^2 (y_2+y_3z+\cdots+y_nz^{n-2})\frac{dz}{z},$$

where the integration has to be extended over the unit circle $|z|=1$, $z=e^{i\phi}$. Here the factor of $z^{-1}dz$ can be written in the form

$$\frac{-4rz+(1+r^2)(z^2+1)}{(1-rz)^2}(\bar{y}_0z+\bar{y}_1)(y_2+y_3z+\cdots+y_nz^{n-2})$$

which is regular for $|z|\leqq 1$. Thus we obtain its value in $z=0$, i.e. $(1+r^2)\bar{y}_1y_2$ for the integral (10) so that the form in question will be:

$$\begin{aligned}
&2R(\bar{y}_0y_1)+2(1+r^2)R(\bar{y}_1y_2)\\
&\quad+\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(\phi)\,|z-r|^4|y_2+y_3z+\cdots+y_nz^{n-2}|^2d\phi\\
&=\tfrac{1}{2}|y_0+y_1+(1+r^2)y_2|^2-\tfrac{1}{2}|y_0-y_1+(1+r^2)y_2|^2\\
&\quad+\frac{1}{2\pi}\int_{-\pi}^{+\pi}\{-4r+2(1+r^2)\cos\phi\}\,|y_2+y_3z+\cdots+y_nz^{n-2}|^2d\phi.
\end{aligned}$$

This is valid also for $r=\pm 1$. The latter integral is a Toeplitz' form Y in $n-1$ variables belonging to the trigonometric polynomial $-4r+2(1+r^2)\cos\phi$. We consider now a transformation of Y into a sum of squares by means of a linearly independent set of $n-1$ linear forms of the variables $y_2, y_3, \cdots, y_n$. By adjunction of the two forms

$$2^{-1/2}(y_0\pm y_1+(1+r^2)y_2)$$

we obtain a set of $n+1$ forms of the variables $y_0, y_1, y_2, \cdots, y_n$, which is obviously linearly independent again. Therefore we have only to calculate the number of the positive, negative and vanishing squares in Y and to increase each of the first two numbers by one unit. Moreover the determinant in question is

$$\begin{vmatrix}2^{-1/2} & 2^{-1/2}\\ 2^{-1/2} & -2^{-1/2}\end{vmatrix}\Delta=-\Delta,$$

Δ being the determinant of Y.

Now according to (7) the characteristic values of Y are

$$-4r-2(1+r^2)\cos\frac{\nu+1}{n}\pi=2(1+r^2)\left(\cos\alpha-\cos\frac{\nu+1}{n}\pi\right),$$

$$\nu=0,1,\cdots,n-2,$$

whence the items of the table follow easily. On account of (6) we have

$$\Delta=(1+r^2)^{n-1}\frac{\sin n\alpha}{\sin\alpha}$$

whence (9) follows.

Editorial Note. Professor Szegö was requested to add the above discussion in order to give readers information regarding the Toeplitz forms which generalize

the one of the problem. It was unnecessary for him to state that the reduction of the forms used as illustrations can be obtained by the less interesting ordinary methods, some of which will be mentioned. The proposer's solution made use of the criterion in I above; and he referred to exercises 4 and 5 on page 147 of Bôcher's *Higher Algebra*, remarking that the matrix of the quadratic form is here regularly arranged, i.e., the sequence of the n principal minor determinants in the upper left hand corner has no two consecutive zero determinants and the determinant of the matrix is not zero. The terms signature and signum (sgn) are defined in exercise 6. Kronecker's reduction is explained in §49 of the text cited. We observe also that exercise 1 on page 133 is the special case of the problem for $n=5$. The solution by Williamson appears in his solution of 3667 [1935, 400]. In solution II it is easily shown that

$$D_n = \frac{(-1)^n}{4}\left(\frac{2}{n}\right)^{n-1} |\,\Delta\,|^2, \qquad |\,\Delta\,|^2 = \Delta\bar{\Delta} = 4n^n.$$

However, this reduction does not simplify the work of the solution, since the solution is arranged so that a complicated product cancels in the evaluation of D_n.

The consideration of quadratic forms is simplified by the elementary theory of symmetric matrices, and in this theory there are two very elementary transformations which are useful theoretically as well as practically. If the column C_j of the matrix a is replaced by a linear combination of its columns in which the coefficient of C_j is unity, the determinant of a is unaltered in value. This is equivalent to multiplying a on the right by a matrix of the same order with the determinant unity. This matrix is easily written in terms of the coefficients, and it gives immediately the linear transformation of the form which is mentioned later. If we now carry out the same transformation of the row R_j, this is equivalent to the multiplication of a on the left by a matrix of the same order which is the conjugate of the right hand factor matrix. These two transformations together are equivalent to a linear transformation with the determinant unity of the quadratic form. If we interchange two columns the sign of the determinant of a is altered, but if we make the conjugate transformation, i.e., the interchange of the corresponding two rows, the sign is restored. The result is a linear transformation with the determinant -1 of the form. This latter transformation is useful in making a regular. None of these transformations alter the rank of a.

Suppose that a is a symmetric matrix of real elements of order n and rank r, $0<r\leqq n$; and that it is regularly arranged with respect to the r principal minors $M_1, M_2, \cdots, M_r$ in the upper left hand corner, where D_r, the determinant of M_r, is not zero. The minor $M_{r+1}^{(k)}$ formed by M_r and the proper elements of the kth column and the kth row has the determinant $D_{r+1}^{(k)}=0$, $k>r$, but the cofactor D_r of a_{kk} is not zero. Denote by e_{kj} the quotient of the cofactor of a_{kj} in $M_{r+1}^{(k)}$ divided by D_r; and let e_{ik} have a similar meaning. The transformation

$$
\begin{aligned}
&C_k \to e_{k1}C_1 + e_{k2}C_2 + \cdots + C_k,\\
&R_k \to e_{1k}R_1 + e_{2k}R_2 + \cdots + R_k,
\end{aligned} \tag{1}
$$

reduces to zeros the elements of a in the kth column and row. Hence the transformations (1) in turn for $k=n, n-1, \cdots, r+1$ reduce to zeros all the elements of a outside of M_r, while its elements are unaltered. If the determinant D_{r-1} of M_{r-1} is not zero, we carry out the transformation similar to (1) for $k=r$, where the divisor D_r is now replaced by D_{r-1}. The result is that all elements but one of the rth column and row are reduced to zero, while a_{rr} is replaced by D_r/D_{r-1}. If $D_{r-1}=0$, then $D_{r-2}\neq 0$ by hypothesis; and we carry out (1) for $k=r-1$, where the divisor is now D_{r-2}. The result is that the elements in the column and row bordering M_{r-2} are all zero, and the elements of the rth column and row are unchanged except for the change to $\bar{a}_{r-1,r}$ and $\bar{a}_{r,r-1}$ which are equal since a is symmetric. Moreover, $D_r = -D_{r-2}\bar{a}_{r-1,\,r}\bar{a}_{r,r-1}$, and hence neither of these transformed elements is zero. Thus

$$
D_rD_{r-2} < 0, \quad \text{if } D_{r-1} = 0. \tag{2}
$$

We now carry out the transformation

$$
\begin{aligned}
&C_j \to C_j - a_{rj}C_{r-1}/\bar{a}_{r,r-1}, \qquad && R_j \to R_j - a_{jr}R_{r-1}/\bar{a}_{r-1,r},\\
&C_r \to C_r - a_{rr}C_{r-1}/2\bar{a}_{r,r-1}, && R_r \to R_r - a_{rr}R_{r-1}/2\bar{a}_{r-1,r},\\
&\qquad j = 1, 2, \cdots, r-2.
\end{aligned} \tag{3}
$$

Continued use of these transformations in the form required by the two possible cases reduces the matrix of M_r to a form such that the elements of the jth column and row are all zero except at their intersection where we have D_j/D_{j-1}; or all elements of two consecutive columns and rows, say the jth and $(j+1)$th, are zero except $\bar{a}_{j,j+1}\bar{a}_{j+1,j}$. This last case results from $D_j=0$, and we then have

$$
-\bar{a}_{j,j+1}\bar{a}_{j+1,j} = D_{j+1}/D_{j-1} < 0, \qquad D_0 = 1. \tag{4}
$$

It is easily seen how to replace such pairs of elements by diagonal elements having opposite signs. The combined transformations give a real linear transformation with the determinant unity which transforms the original form into square terms whose coefficients appear in a simple manner in terms of the determinants D_j. The number of positive and negative terms is determined by a rule similar to Descartes rule of signs for equations whose roots are all real (see the solution of 3678 [1935, 577]. Thus in the sequence

$$
1 = D_0, D_1, D_2, \cdots, D_r \neq 0, \tag{5}
$$

if $D_i=0$, $0<i<r$, we replace it by $+1$ (or -1); and the number of positive terms is equal to the number of permanences of signs in the sequence, the number of negative terms is the number of variations of sign.

If a is not regularly arranged, a slight modification of the above process with the transformations of exchanging a pair of columns and the corresponding rows shows how to rearrange it regularly; in fact, it would not be difficult to combine the rearranging and reduction. The matrix of a Hermitian form may be treated in the same manner. The problem is a simple case where the matrix reduces by this process to

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \tag{6}$$

in the upper left hand corner, and all other elements are zero except the diagonal elements $D_i/D_{i-1} = -2(i-1)/(i-2)$, $i=3, 4, \cdots, n$.

In special cases, such as that of the problem, it may be possible to simplify the transformations without using the values of the determinants. The following general transformation exhibits the peculiar simplicity of the matrix of the problem. In any matrix a subtract from each column the following one, beginning with the first. This is the right hand operation. The left hand, or conjugate, operation is the same process with respect to the rows. The combined change may be indicated by

$$a_{ij} \to a_{i+1,j+1} + a_{ij} - a_{i,j+1} - a_{i+1,j} = \overset{2}{\Delta}_{ij} a_{ij}, \tag{7}$$

where on the right $a_{ij}=0$, if i or j exceeds n, the order of a; and Δ_{ij}^2 denotes the second partial difference. Thus

$$| a_{ij} | \to | \overset{2}{\Delta}_{ij} a_{ij} |, \tag{8}$$

by means of a real linear transformation whose determinant is unity. In the problem $a_{ij} = |i-j|$; and it suffices to examine in turn the cases $i=n$, $i=j<n$, and $n>i>j\geqq 1$. It will then be found that

$$\begin{aligned} \overset{2}{\Delta}_{ij} a_{ij} &= 0, \quad && \text{if } i = j = n, \\ &= -2, && \text{if } i = j < n, \\ &= 1, && \text{if } i = n, j < n, \text{ or if } j = n, i < n, \\ &= 0 && \text{in all other cases.} \end{aligned}$$

In the resulting matrix add one half of each of the other columns to the last, and perform the conjugate. This reduces the original form in x to

$$-2(x_1''^2 + x_2''^2 + \cdots + x_{n-1}''^2) + \frac{n-1}{2} x_n''^2, \tag{9}$$

and also evaluates the determinant of the form. This is Williamson's result. The linear transformations are respectively

$$
(10)\qquad
\begin{array}{lll}
x_1 = x_1', & x_i' = x_i'' + \tfrac{1}{2}x_n'', & x_1 = x_1'' + \tfrac{1}{2}x_n'',\\
x_i = -x_{i-1}' + x_i'; & x_n' = x_n''; & x_i = -x_{i-1}'' + x_i'',\\
 & & x_n = -x_{n-1}'' + \tfrac{1}{2}x_n'',\\
2 \leqq i \leqq n, & 1 \leqq i \leqq n-1, & 2 \leqq i \leqq n-1.
\end{array}
$$

In order to evaluate the determinant of the problem, the solution I applies the transformation of the first column and row

$$a_{i1} \rightarrow \Delta_i^2 a_{i1}, \qquad a_{1j} \rightarrow \Delta_j^2 a_{1j}.$$

Thus a_{11} has been subjected to the transformation

$$(11)\qquad a_{ij} \rightarrow \overset{4}{\Delta}_{iijj} a_{ij}.$$

This transformation shows that the determinant D_n of the $(n+1)$th order satisfies the difference equation

$$(12)\qquad (U+2)^2 D_n = 0, \qquad D_0 = 0, \qquad D_1 = -1,$$

where $Uf(n) = f(n+1)$. From the form of (12) we see that the general solution in terms of two arbitrary constants is

$$D_n = A(-2)^n + Bn(-2)^{n-1}.$$

The initial values determine the constants with the result above. The equation (12) may be simplified further by a general method for repeated factors. If we set

$$D_n = (-1)^n 2^{n-1} P(n),$$

we have

$$\Delta^2 P(n) = 0, \qquad P(0) = 0, \qquad P(1) = 1,$$

and the solution satisfying the initial conditions is obviously $P(n) = n$.

Returning to the original difference equation in the form

$$(13)\qquad D_{n+2} + 4D_{n+1} + 4D_n = 0,$$

it is obvious that we can set up a simple determinant solution by taking all the elements of the principal diagonal equal to -4; each element of the adjacent parallel above equal to any non-zero number $\bar{c}$; each element of the adjacent lower parallel equal to c, where $c\bar{c} = 4$; and all other elements equal to zero. But, since this solution does not satisfy the given initial conditions, we prolong each of its rows and columns at their ends by two zero elements, and place in the remaining lower right hand corner the minor

$$\begin{matrix} 0 & \bar{b} \\ b & 0 \end{matrix}, \qquad b\bar{b} = 1.$$

In order to obtain a real symmetric matrix we take $c=\bar{c}=2$, $b=\bar{b}=1$. It is obvious that we can pass from the original quadratic form to the corresponding new form by a real linear transformation with a determinant whose square is unity. Such a transformation with unity for the value of its determinant may be obtained by first repeating (7), (8). This gives the transformation

$$|a_{ij}| \to |\overset{4}{\Delta}_{iijj}a_{ij}|, \tag{14}$$

where as before there is a similar remark as to the subscripts. We then subtract twice the last column from the one two units before it; add twice the last column to the one just before it; and carry out the same transformations for the rows.

The determinant D_n of order $n+1$ of the Hermitian form of the discussion corresponding to the trigonometric polynomial of the first order is similar to the one already mentioned which did not satisfy the required initial conditions. The difference equation is obtained immediately by development of the first column or row

$$D_{n+2} - 2a\rho D_{n+1} + \rho^2 D_n = 0, \quad \rho^2 = a_1^2 + b_1^2 > 0, \quad a_0 = 2a\rho, \tag{15}$$

where a_0, a_1, b_1 are real. If we set

$$D_n = \rho^{n+1}P(n+2),$$

we obtain

$$P(n+2) - 2aP(n+1) + P(n) = 0, \quad P(0) = 0, \quad P(1) = 1. \tag{16}$$

The solution of this last equation with the given initial conditions is the expression for x_n in (6) of the Note to the solution of 3677 [1935, 575]. Hence

$$\begin{aligned} P(n) &= \sin n\theta/\sin\theta, \ a^2 \leqq 1, \ \cos\theta = a, \ 0 \leqq \theta \leqq \pi; \\ &= \sinh n\theta/\sinh\theta, \ \cosh\theta = a > 1; \\ &= (-1)^{n-1}\sinh n\theta/\sinh\theta, \ \cosh\theta = -a > 1. \end{aligned} \tag{17}$$

The solution $P(n)$ of (16) may be written as a striped determinant of order $n-1$ with $P(2)=2a$ in the principal diagonal; any number $d\neq 0$ in the lower adjacent parallel; d^{-1} in the upper adjacent parallel; and all other elements zero. It is clear that the matrices in all these cases are regularly arranged; for, if two consecutive solutions of these difference equations of the second order are zero, then all the solutions are zero, contrary to the initial conditions. Also if $P(n)=0$, $P(n+1)P(n-1)<0$. The corresponding forms for different values of d are equivalent by means of real linear transformations which, however, are of the type where the transformation for x_i may not be the same as that for x_j in $a_{ij}x_ix_j$. In other words we distinguish the two variables as in the Hermitian forms, but the two transformations are simply related in terms of d. No $P(n)$ vanishes for $n>0$ except in the cases for which $a^2<1$; this fact is utilized when we consider the characteristic roots. With these results we can count the number of positive and negative terms when the matrix is reduced to the diagonal form

from the signs of $P(n)$. A reduction is simple in this case. We carry out in turn

$$(18)\quad C_j \to C_j - d^{-1}\frac{P(j-1)}{P(j)}C_{j-1}, \qquad R_j \to R_j - d\frac{P(j-1)}{P(j)}R_{j-1},$$
$$j = 2, 3, 4, \cdots$$

until we reach the first P which is zero, say $P(k)=0$. Since the difference equation may be written

$$(19)\qquad P(2) - \frac{P(j-1)}{P(j)} = \frac{P(j+1)}{P(j)}, \qquad P(j) \neq 0,$$

we easily show that we obtain in the principal diagonal $P(j+1)/P(j)$, $j=1, 2, \cdots, k-1$, with all other elements zero in the first $k-1$ columns and rows. We then carry out the transformations in turn

$$(20)\qquad \begin{aligned} &C_k \to C_k - \tfrac{1}{2}d^{-1}P(2)C_{k-1}, \qquad && C_{k+1} \to C_{k+1} - d^{-2}C_{k-1},\\ &R_k \to R_k - \tfrac{1}{2}dP(2)R_{k-1}; && R_{k+1} \to R_{k+1} - d^2R_{k-1}. \end{aligned}$$

After this we recommence the same process, so that we obtain along the principal diagonal the periodic arrangement

$$(21)\qquad \frac{P(2)}{P(1)}, \frac{P(3)}{P(2)}, \cdots, \frac{P(k-1)}{P(k-2)}, \begin{vmatrix} 0 & d^{-1} \\ d & 0 \end{vmatrix}, \frac{P(2)}{P(1)}, \frac{P(3)}{P(2)}, \cdots,$$

where the first vanishing is for $P(k)=0$, and then $P(k-1)=-P(k+1)$, $k \geqq 2$. The vanishing occurs only for $a^2<1$. If $P(2)=0$, we have merely a sequence of the second order determinants, and the sequence ends in a zero if n is even. In the discussion is a neat derivation of the characteristic roots. Here the characteristic equation has $P(2)-\lambda$ in the principal diagonal, and, as remarked above, the characteristic determinant can vanish only if this diagonal term is less than two in absolute value. Hence we get the roots by setting $P(2)-\lambda = 2\cos\beta$, $0<\beta<\pi$. Thus for the determinant $P(n+2)$ of order $n+1$ we have for the roots

$$(22)\qquad 2\left(a - \cos\frac{\mu\pi}{n+2}\right), \qquad \mu = 1, 2, \cdots, n+1.$$

The determinant for $r^{|i-j|-1}|i-j|$, r real, may be evaluated by a method analogous to that of solution I, that is we carry out the transformation of the first column and row

$$(23)\qquad C_1 \to C_1 - 2rC_2 + r^2C_3, \qquad R_1 \to R_1 - 2rR_2 + r^2R_3.$$

Then development by the first column gives

$$(24)\qquad D_{n+2} + 4rD_{n+1} + (1+r^2)^2D_n = 0, \quad D_0 = 0, \quad D_1 = -1.$$

If we set

$$(25)\qquad D_n = -(1+r^2)^{n-1}P(n), \qquad -4r/(1+r^2) = 2a,$$

we obtain the difference equation and initial conditions (16); and, since $a^2 \leqq 1$, the solution is in the first line of (17). As before the difference equation shows that the matrix is regularly arranged, and the number of the positive and negative square terms in the diagonal form may be determined from the values of D_n. A solution of (24) similar to that given for (13) may be obtained by setting $c\bar{c} = (1+r^2)^2$ and replacing the former -4 by $-4r$ in the principal diagonal. As before if one or both of the sets $(c, \bar{c})$, $(b, \bar{b})$ are unequal, the matrix is not symmetric and we should have to distinguish the transformations for x_i and x_j. For simplicity we consider the case $c = \bar{c} = 1+r^2$, $b = \bar{b} = 1$; it must be possible by a real linear transformation with a determinant whose square is unity to pass from the original form to the corresponding new one. Such a transformation with a unit determinant will be indicated for D_{n-1} of order n. We first carry out

$$C_j \to (rU_j - 1)^2 C_j = r^2 C_{j+2} - 2rC_{j+1} + C_j,$$
$$R_i \to (rU_i - 1)^2 R_i,$$
$$a_{ij} \to (rU_i - 1)^2 (rU_j - 1)^2 a_{ij}, \qquad i, j = 1, 2, \cdots, n.$$

To obtain the result it is necessary to examine only the principal diagonal and the elements below it. For $n-2 \geqq i \geqq j+2$ it is easily seen that all elements in this triangle reduce to zeros. This triangle of zeros is then easily seen to extend to the last two rows, and it then suffices to examine the minor of the 3rd order in the lower right hand corner. It will then be seen that all elements but one of the principal diagonal are $-4r$, and the last to the right is zero. The two adjacent parallels end to the right in unity and all their other elements are $1+r^2$. All other elements are zero. Thus in the lower right hand corner we have

$$\begin{matrix} -4r & 1+r^2 & 0 \\ 1+r^2 & -4r & 1 \\ 0 & 1 & 0 \end{matrix}$$

We now carry out

$$C_{n-2} \to C_{n-2} - (1+r^2)C_n, \qquad R_{n-2} \to R_{n-2} - (1+r^2)R_n,$$
$$C_{n-1} \to C_{n-1} + 2rC_n, \qquad R_{n-1} \to R_{n-1} + 2rR_n,$$

and we have the desired result. The successive transformations are

$$x_1 = x_1', \qquad x_i' = x_i'', \quad 1 \leqq i \leqq n-1,$$
$$x_2 = -2rx_1' + x_2', \qquad x_n' = -(1+r^2)x_{n-2}'' + 2rx_{n-1}'' + x_n'';$$
$$x_i = r^2 x_{i-2}' - 2rx_{i-1}' + x_i',$$
$$3 \leqq i \leqq n;$$

$$
\begin{aligned}
x_1 &= x_1'', \\
x_2 &= -\,2rx_1'' + x_2'', \\
x_i &= r^2x_{i-2}'' - 2rx_{i-1}'' + x_i'', \qquad 3 \leqq i \leqq n-1, \\
x_n &= -\,x_{n-2}'' + x_n''.
\end{aligned}
$$

A slight modification gives a result conforming to that of the discussion. The variables are subjected to the linear transformation

$$x_p \to (rU_p - 1)^2 x_p, \qquad p = 0, 1, \cdots, n,$$

where here as elsewhere a term of the result is zero if its subscript lies outside the range designated. Then the form undergoes the transformation

$$
\begin{aligned}
\sum a_{pq}x_p\bar{x}_q \to \sum a_{pq}(rU_p - 1)^2(rU_q - 1)^2 x_p\bar{x}_q \\
= \sum x_p\bar{x}_q(rU_p^{-1} - 1)^2(rU_q^{-1} - 1)^2 a_{pq}.
\end{aligned}
$$

The result of the last operation for $a_{pq} = r^{|p-q|-1}|p \dot{-} q|$ is easily found as before. All elements are zero except those in the principal diagonal and the two adjacent parallels. The two parallels begin at the left with unity and each of the rest is equal to $1+r^2$; the principal diagonal begins at the left with zero and each of its other elements is $-4r$. Then the transformation on this result

$$x_0 \to x_0 + 2rx_1 - (1 + r^2)x_2$$

gives

$$x_0\bar{x}_1 + \bar{x}_0x_1 + \sum_{j=2}^{n-1}[-\,4rx_j\bar{x}_j + (1 + r^2)(x_j\bar{x}_{j+1} + \bar{x}_jx_{j+1})] - 4rx_n\bar{x}_n.$$

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Mathematical Association of America, Inc., has formed a Placement Bureau under the direction of Professor E. J. Moulton, Northwestern University, Evanston, Illinois. The Bureau is undertaking to assist men and women holding the doctorate in mathematics to secure suitable positions. About sixty persons have registered with the Bureau. Prospective employers may secure information concerning candidates by addressing Professor Moulton.

The Placement Bureau may also be of service in arranging exchange professorships. At the present time an associate professor of mathematics in an Eastern college desires to exchange positions for the year 1936–37 with a man in a similar position in the West or Southwest; the only prescribed course in his schedule would be one in advanced algebra and modern geometry for students majoring in mathematics.

The Second International Congress for the Unity of Science will take place in Copenhagen, June 21–26, 1936. The central topic will be the relation of physics and biology (including psychology), with especial emphasis upon the concept of causality. A small number of invited speakers will provide the context for a general discussion. Niels Bohr will participate in the Congress.

Inquiries and notices of intended attendance at the Congress may be directed to the Secretary, Dr. Otto Neurath, Mundaneum Institute, 267 Obrechtstraat, The Hague, The Netherlands. The subscription for active membership is 100 francs.

Some time ago Professor David Eugene Smith presented to Columbia University his library on the history of mathematics, and he has now also presented his collection of mathematical instruments to the same institution. Professor Smith, as reported in Science, makes the following statement relative to this gift:

"A few years ago I gave to the University my library on the history of mathematics, consisting of upwards of 20,000 items—bound volumes, manuscripts, portraits of mathematicians (about 2,700 engravings and 160 medals), catalogued monographs (2000) and 4000 letters, including such leaders as Newton, Descartes, Leibniz and the Bernoullis. For use in my lectures I had, over a period of more than forty years, collected more than 275 instruments of early and medieval times for purposes of calculating, measuring (lengths, areas, volumes, weight and time) and astronomy (navigation, astrology and the calendar). It, therefore, seemed appropriate that this mathematical material should be placed alongside the books, many of which describe and illustrate the instruments themselves."

A new chapter of Pi Mu Epsilon, honorary mathematical fraternity, was installed at the University of Toledo on February 7. Professor J. H. Weaver acted as installing officer. There are now thirty-four chapters of this fraternity distributed over twenty states.

J. Hobart Bushey of Hunter College has been promoted to an associate professorship.

The following promotions have been made in the department of mathematics of Brooklyn College: to associate professorships, Edward Fleisher, O. W. Irvin, L. T. Moore; to assistant professorships, F. E. Smith, L. S. Kennison; to an instructorship, Mary I. Draper.

The following doctorate should be included in the list published in the August-September, 1935, issue of this MONTHLY:

Sister Mary Cleophas Garvin, St. Louis University, June, minor in physics. *A generalized Lambert series.*

Dr. Henry Taber, professor of mathematics at Clark University during the period 1888–1921 and professor emeritus since 1921, died January 6, aged 75.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Summer Meeting, Harvard University, Aug. 31, 1936.

Twenty-first Annual Meeting, Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, March.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov. 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S.C., April 17–18.
SOUTHERN CALIFORNIA, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, May 2.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 5, MAY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section was held at the University of Maryland, College Park, Maryland, on Saturday, December 7th, 1935.

Professor T. H. Taliaferro of the University of Maryland officially welcomed the members and their guests. The chairman, Professor G. T. Whyburn of the University of Virginia, presided over both sessions, morning and afternoon. At the invitation of the Section, Professor E. J. McShane of the University of Virginia delivered a lecture on "Recent developments in the problem of Plateau."

The attendance was sixty-eight, including the following forty-three members of the Association: O. S. Adams, N. H. Ball, Archie Blake, C. C. Bramble, Randolph Church, G. R. Clements, Abraham Cohen, John Curtiss, Alexander Dillingham, J. A. Duerksen, R. E. Gaines, Michael Goldberg, Harry Gwinner, Isabel Harris, E. K. Haviland, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, Gillie A. Larew, J. A. Larrivee, C. M. Lennahan, Florence P. Lewis, S. B. Littauer, J. J. Luck, E. J. McShane, Florence M. Mears, T. W. Moore, W. K. Morrill, F. D. Murnaghan, J. L. Nagle, O. J. Ramler, C. H. Rawlins, J. N. Rice, A. W. Richeson, R. E. Root, T. H. Taliaferro, John Tyler, C. H. Wheeler, G. T. Whyburn, John Williamson, E. W. Woolard, R. T. Zoch.

The spring meeting will be held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 9th, 1936.

The following seven papers were read:

1. "On the postulate of the arithmetic mean" by R. T. Zoch, United States Weather Bureau.

2. "The life and work of Simon Newcomb" by Dr. E. W. Woolard, United States Weather Bureau.

3. "The existence of alphabets having no interval repetitions" by Dr. A. Sinkov, Washington, introduced by Michael Goldberg.

4. "Surveying texts before 1800" by Professor W. F. Shenton, American University, introduced by the Secretary.

5. "Magic squares" by Professor H. M. Robert, United States Naval Academy, introduced by Professor C. H. Rawlins.

6. "An elementary envelope problem" by Professor R. E. Gaines, University of Richmond.

7. "Recent developments in the problem of Plateau" by Professor E. J. McShane, University of Virginia.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Suppose that n observations have been made of an unknown quantity. It is desired to know the most probable value of the unknown. When Gauss

gave his development of the so-called Normal Law of Error, he *assumed* that the Arithmetic Mean of the n observations is the most probable value. The question arises: Can this postulate be justified? Coolidge, in his book *An Introduction to Mathematical Probability* (Oxford, 1925) and Whittaker and Robinson in *Calculus of Observations* (London, 1929) give proofs each of which purports to deduce the postulate of the Arithmetic Mean from assumptions of a more elementary nature. These proofs are not correct. Mr. Zoch reviewed the essentials of these proofs and showed where they were lacking in rigor. His paper has appeared in the *Annals of Mathematical Statistics*, December, 1935.

2. Dr. Woolard gave a brief outline of the life of Simon Newcomb before proceeding to discuss his contributions to science, mainly in the field of astronomy. Since Simon Newcomb was born in 1835 this paper of Dr. Woolard's was very timely and appropriate.

3. The problem discussed by Dr. Sinkov arose in an attempt to obtain two permutations of the twenty-six letters of the alphabet having the property that no two letters be separated by the same interval in both sequences. He showed, by considering an alphabet of n letters and reducing the problem to a problem in Number Theory, that no such permutations are possible when n is even, but that they are always possible when n is odd.

4. The extremely rich collection of early surveying texts in English found in the Artemas Martin Mathematical Library prompted Professor Shenton to make a study from them of the development of the science of surveying and of surveying instruments previous to the nineteenth century. This study covered the texts available in the Library of Congress and the Library of the Coast and Geodetic Survey—in all about forty texts were examined—from the quaint early English of Richard Benese (1562) to the more ambitious texts of Zechariah Jess and Solomon Dewey which appeared in America in 1799. The paper was profusely illustrated with lantern slides of many interesting title pages, and figures showing early instruments and methods of surveying.

5. Professor Robert showed how to construct ordinary magic squares both of even and odd order. He also discussed methods for forming hyper-magic squares including bordered, compound and perfect magic squares.

6. Professor Gaines discussed the problem of finding the envelope of a variable chord of a conic which subtends a constant angle at a focus of the conic. He showed that the envelope is a conic with the same focus and directrix as the original one, and how, by repeating the process with the new conic, a whole system of confocal conics could be obtained.

7. The problem of Plateau, that is the problem of proving that every non-self-intersecting closed curve bounds a minimal surface, and the closely related problem of finding the surface of least area bounded by a given curve, have been studied by mathematicians over many decades. From time to time advances have been made toward their solution; geometry, the theory of differential equations, function theory, group theory and the calculus of variations all being pressed into service. But with the last few years the advance has been almost

discontinuously rapid, and these problems may now be considered to be completely solved. Professor McShane first discussed these modern methods of attack and the results yielded by them, and then proceeded to consider some recent work in extensions of the Plateau problem and related questions.

JOHN WILLIAMSON, *Secretary*

THE THIRD ANNUAL MEETING OF THE OKLAHOMA SECTION

The third regular meeting of the Oklahoma Section of the Mathematical Association of America was held at the First Methodist Church in Oklahoma City, Oklahoma, on Friday morning, February 7, 1936, Dr. E. F. Allen, presiding.

The number in attendance was one hundred and forty-two, including the following eighteen members of the Association: E. F. Allen, J. C. Brixey, N. A. Court, Mildred Dolezal, E. P. R. Duval, W. V. N. Garretson, H. L. Hall, Claire A. Harrison, J. O. Hassler, L. W. Johnson, J. E. LaFon, J. J. Miller, U. G. Mitchell, W. T. Short, C. E. Springer, S. B. Townes, E. B. Wedel, J. H. Zant.

The following officers were elected for the next year: Chairman, L. W. Johnson, Central State Teachers College; Vice-Chairman, H. L. Hall, Northwestern State Teachers College; Secretary, C. E. Springer, University of Oklahoma. The next meeting will be held in February 1937, at Oklahoma City, in connection with the Oklahoma Education Association.

The members of the Association joined with the members of the Mathematics Division of the Oklahoma Education Association in the basement of the church for a luncheon. Later, in the same room, two lectures were given; the first, by Dr. E. J. Ortman, School of Education, University of Oklahoma, on "Problems or personalities," and the second (illustrated), by Dr. U. G. Mitchell, Head of the Department of Mathematics, University of Kansas, on "Some glimpses into the history of mathematics."

The following papers were read at the morning assembly.

1. "The mathematics of geophysics" by B. E. Skipper, Tulsa, introduced by Professor Allen.
2. "On the inscribed circle of a triangle" by Professor L. W. Johnson, Central State Teachers College.
3. "On some theorems in the theory of numbers" by Professor J. C. Brixey, University of Oklahoma.
4. "Some notes on the Heaviside operational calculus" by Professor G. B. Drummond, Oklahoma A. and M. College, introduced by Professor Allen.
5. "Some considerations relating to irrational numbers" by Professor U. G. Mitchell, University of Kansas, by invitation.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Mr. Skipper presented a film entitled "A day with the seismograph" which depicted a résumé of a day's work by a seismograph crew. Animated drawings showing the reflection and refraction of waves set up in the ground by the detonation of a charge of dynamite made use of certain formulas significant in geophysical work.

2. Professor Johnson proved the following theorem. The second of the internal tangents to the incircles of the triangles into which a triangle ABC is divided by a variable line through A, passes through the point of contact (X) of the incircle of ABC with the side BC. The second of the internal tangents to the excircles of these triangles, relative to the vertices at A passes through the point X_1 on BC, analogous to the point X. This theorem can be used to solve the problem of constructing a transversal cutting the sides BC, CA, and AB of the triangle ABC at L, M, and N, respectively, given the ratio of LM to LN, such that the segment MN should subtend a right angle at an arbitrary point of BC. The above theorem and problem, in turn, can be used to solve the problem of drawing a line through the vertex of a triangle such that the inradii of the triangles thus formed should have a given ratio. Special cases of this problem arise for consideration.

3. Professor Brixey illustrated the application of mathematical induction to problems in number theory, considering in particular the proofs of the following theorems:

(a) $ax^2+by^2\equiv kp \pmod{p^n}$ has solutions x, y prime to an odd prime p when $(-ab/p)=1$.

(b) $ax^2+by^2\equiv 0 \pmod{2^n}$ has solutions x, y odd if $a+b\equiv 0 \pmod 8$, ab odd.

(c) $ax^2+by^2+cz^2\equiv 0 \pmod{2^n}$ is solvable, with xyz odd, if $a+b+c\equiv 0 \pmod 8$ and ab odd.

4. In the first part of this paper Professor Drummond gave a résumé of the ideas of the operational calculus of Heaviside, explaining the "unit function," the operators p, $p^2 \cdots$, p^{-1}, $p^{-2} \cdots$, and the method of "algebrizing" to obtain a solution of a linear differential equation with constant coefficients. The "expansion theorem" of Heaviside is useful in considering differential equations, for which the solution is a divergent series or a series convergent only in a certain region. The past decade has seen an increase in the application of the methods of Heaviside, particular attention being given to the physical interpretation of the algebraic solution.

5. Professor Mitchell reviewed briefly the historical development of the concept of irrational numbers, noting that the irrationality involved in finding the area of the ellipse is represented by the number π and that involved in finding the area of the hyperbola by the number e. He raised some questions which might enter into a classification of irrational numbers. Are there other plane curves whose properties involve irrational numbers not expressible in terms of e and π? Rational numbers are expressible in terms of the zero power of e or π. In view of Liouville's proof that e^x is not rational for x any other rational number than zero, what irrational numbers are expressible in terms of the powers

of e and π? It seems probable that great impetus would be given to the development of the theory of differential equations if a more usable notation and classification of irrational numbers could be developed.

C. E. SPRINGER, *Secretary*

GEOMETRIC CONSTRUCTIONS WITHOUT THE CLASSICAL RESTRICTION TO RULER AND COMPASSES*

By W. H. BUSSEY, University of Minnesota

Once when Mark Twain was about to give a lecture to the students of Barnard College, he was asked what was to be the subject of his talk. He said "I don't know, but I have some good illustrations." He finally decided to talk about "The eccentricities of memory" because, as he said, "anything would illustrate it."

W. W. R. Ball, when he was about to publish a new edition of his book on "Mathematical Recreations" changed its name to "Mathematical Recreations and Essays" so that he would feel free to put into the book anything he had to say about mathematics without violating the contract implied by the title.

The subject announced for this talk of mine is "Geometric constructions without the classical restrictions to ruler and compasses." The title which I gave to the program committee was slightly different. I used the words "Geometric constructions without the classical ruler and compass restriction." The committee changed the wording, I assume, to avoid the use of the word "compass" in the singular. Although that word is usually used in the plural, there is good precedent for its use in the singular form. I shall use it that way when the use of the plural seems awkward.

I have some things to say, mostly about geometric constructions, but I do not want to be too much restricted by a title. So perhaps I should say that the subject of my talk is this: "Some remarks about geometric constructions without the classical restrictions, and some other remarks." That title is not very binding, and besides it does not give the impression that I am going to give you an exhaustive account of what has been done by other means than just ruler and compasses. There are two good reasons why I am not going to try to give you any such complete exposition. One is that it would take more time than I have at my disposal; the other is that I do not know enough about the subject.

When I speak of geometric constructions without the classical restrictions, I have in mind, primarily, constructions by means of one or more of these simple instruments:

1. The ruler, by which I mean an unmarked straightedge.
2. Compasses.

* Presented to the Association at the meeting in Ann Arbor, Mich., September 9, 1935.

3. The graded ruler, which is what the word "ruler" means to most people and which is what you will get if you go to a store to buy a "ruler". By the words "graded ruler" I also mean a straight edge on which marks are made by the user. I have in mind particularly a paper or cardboard straightedge on which the user makes marks at his pleasure. It becomes a graded ruler when he makes two marks on it. Some of the German books use the word "Papierstreifen" in this connection.

4. A parallel ruler, which has two parallel straight edges like the long arm of a draftsman's T-square.

5. Draftsman's triangles, which are usually made of celluloid. The two in common use are right triangles with acute angles (45°, 45°) and (30°, 60°), respectively.

6. The draftsman's T-square.

7. The L-shaped or carpenter's square.

The straight edges on the parallel-ruler, draftsman's triangles, T-square, and L-shaped square may be graded or not graded.

These instruments are all in common use, chiefly for making shorter and easier those constructions which can be made with ruler and compasses alone. The fact that they can be used to solve construction problems of higher degree than is possible with ruler and compasses may or may not be known to the people who use them.

When I speak of geometric constructions without the classical restrictions, I do not have in mind complicated instruments for drawing such curves as the conchoid, quadratrix, etc. Nor do I have in mind, except incidentally, constructions by means of auxiliary curves, the drawing of which involves either some complicated instrument or the free-hand drawing of a smooth curve through points which have been located individually by ruler and compasses or other means.

The classical restrictions are those imposed by these three of Euclid's postulates:—(1) that a straight line can be drawn from any point to any point; (2) that a finite straight line can be produced continuously in a straight line; (3) that a circle can be drawn with any center and any radius.

Constructions made in accordance with those postulates are called "ruler and compass" constructions. They are made by drawing circles with compasses and straight lines with an unmarked straight edge called a "ruler." As the postulates have been interpreted, they impose a restriction which is not in these days usually implied by the phrase "ruler and compasses." I refer to the fact that they make no provision for the use of compasses to carry a line segment from one position to another. Those who wish to be fully bound by the classical tradition should, as De Morgan once said, "insist upon all the restrictions imposed by the postulates, which do not allow a circle to be drawn with a compass-carried distance." De Morgan's interpretation was that "we should suppose the compasses to close of themselves the moment they cease to touch the paper."

Most of the people who now speak of ruler and compass constructions have given up that restriction. They feel free to use compasses as "segment-carriers." Other segment carriers in common use are the ordinary graded ruler and the paper or cardboard straightedge on which the user makes marks.

If we use compasses to carry segments, we merely abbreviate constructions which can be made with all the classical restrictions. Indeed Euclid's second proposition shows how to construct at any given point a line segment equal to a given line segment. That proposition enables one to do, somewhat awkwardly, that which is usually done by means of a segment carrier. Incidentally let me say that by means of a ruler and a segment carrier, without any drawing of circles, it is possible to make some but not all of the constructions that can be made with ruler and compasses. There is a chapter on that subject in Hilbert's *Foundations of Geometry* which is available in an English translation by E. J. Townsend.

A good example of segment carrying is to be found in the construction of a regular decagon as it is usually given in the high school geometry textbooks. A radius of a circle is divided in what used to be called, and I think still is called, "extreme and mean ratio"; then the longer of the two segments, which is equal in length to one side of the regular inscribed decagon, is carried to a new position.

Greek geometers came early to the conclusion that some of the problems in which they were interested required for their solution something more in the way of equipment then just ruler and compasses. I refer especially to the three so-called "famous geometric problems of antiquity": (1) the trisection of an angle; (2) the duplication of a cube; (3) the squaring of a circle. They, the Greek geometers, proceeded successfully to find other means of solving those problems. Let us consider the first two. (The third one involves the transcendental number π and is much more difficult.) If they cannot be solved by means of ruler and compasses, just how much more instrumental equipment is necessary? The answer is that only a very little more is needed. That is evident from the following constructions by compasses and graded ruler.

A Construction for Trisecting an Angle

To trisect an acute angle ABC, mark a point D on BA at any convenient distance from B; and through D draw a line l perpendicular to BC and a line m parallel to BC. Then place a line segment EF, whose length is twice BD, in such a position that E will be on l; F will be on m; and FE produced will pass through B. (This can easily be done by marking the two points E and F on a cardboard straightedge and then moving that straightedge into the required position by a motion which is a combination of a rotation about B and a sliding through B.) That will complete the construction. The angle FBC, marked θ in the diagram, will be equal to one-third of the given angle ABC. To prove it, mark the middle point, G, of EF and draw DG. Denote the length of BD by k. By construction, $EF=2k$, and therefore $GF=k$; and $DG=k$ because the middle point of the

hypotenuse of a right triangle is equidistant from the vertices. The angle DFB equals θ because DF is parallel to BC; and the angle FDG equals θ because the triangle DGF is isosceles. The angle DGB equals 2θ because as an exterior angle of the triangle DGF it equals the sum of the two opposite interior angles; and the angle DBG equals 2θ because the triangle DBG is isoceles. Therefore the angle $ABC = \theta + 2\theta$; and θ is one-third of the given angle.

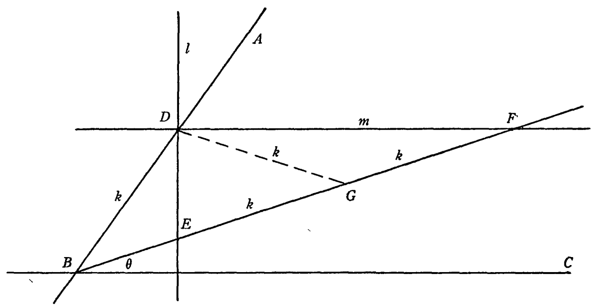


FIG. 1

This construction is very old. The essential idea involved in it is due to Nicomedes, who lived about 180 B.C. All but the placing of the line segment EF in the required position can be done with ruler and compasses. The use of the cardboard straightedge with two marks on it, which is a simple form of graded ruler, is the only departure from the classical restrictions. If one desires to use the common graded ruler which is marked off in inches, he can choose the point D so that the distance BD will be equal to that between two marks on the ruler. For a drawing of ordinary size, a good choice would be $BD = 2$ inches.

A Construction for Duplicating a Cube

Hippocrates of Chios, who lived about 460 B.C., showed that the problem of duplicating a cube involves the finding of two mean proportionals between two given line segments. The Greek geometers found those mean proportionals in more ways than one. But they had to use something more than ruler and compasses. I shall give a solution which I found in the works of Vieta.* It is related to a solution made by Apollonius.†

Two line segments, α and δ, being given, the problem is to construct line segments β and γ so that

$$\alpha:\beta = \beta:\gamma = \gamma:\delta.$$

These proportions lead to the equations

$$\beta = \sqrt[3]{\alpha^2\delta} \quad \text{and} \quad \gamma = \sqrt[3]{\alpha\delta^2}.$$

If α be taken as unity, β will be equal to $\sqrt[3]{\delta}$; and the construction which follows

---

* *Opera Mathematica*, p. 242.

† See Theodor Vahlen's *Konstruktionen und Approximationen* (B. G. Teubner, 1911), p. 87.

may be thought of as that of finding a line segment whose length is equal to the cube root of the length of a given line segment. If then δ be taken as a line of length 2, the construction will be that of finding the edge of a cube whose volume is twice that of a given cube whose edge is chosen as the unit of length.

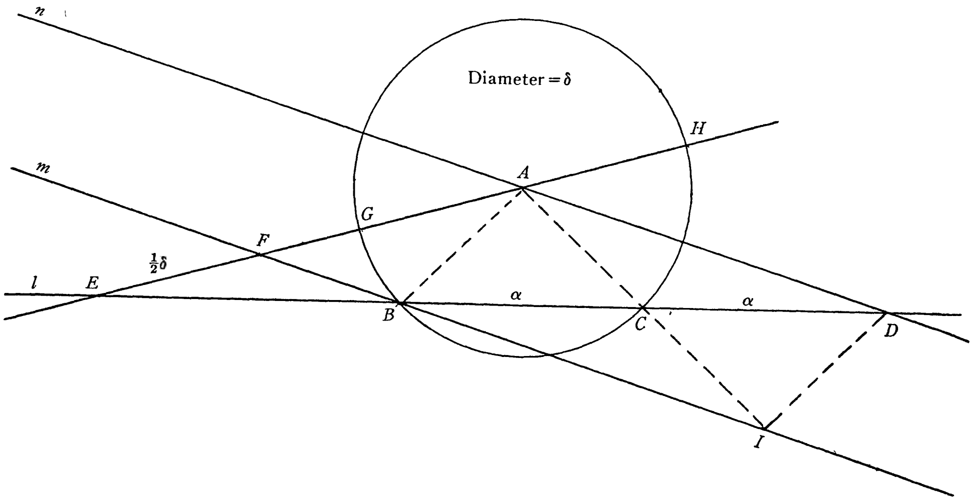


FIG. 2

In a circle whose center is A and whose diameter is equal to δ, draw a chord BC whose length is α. Produce BC to D so that $BC = CD = \alpha$. Denote the line BCD by the name l; draw a line through D and A, and call it n; and draw a line m through B parallel to n. Then place a line segment EF, of length $\delta/2$ (that is, equal to the radius of the circle), in such a position that (1) the point E will be on l; (2) the point F will be on m; (3) the line EF produced will pass through A. Mark the points G and H in which it meets the circle. That will complete the construction. (The point I, obtained by drawing DI parallel to AB, and the lines AB, DI, ACI are included in the diagram, not because they are necessary for our purpose here but because there will be a reference later on to the parallelogram $ABID$.) The line segments EG and EB will be two mean proportionals between α and δ. To prove this statement, we denote EG by β and EB by γ, and use two theorems of elementary geometry as follows: Since the line FB is parallel to the line AD, it divides the other two sides of the triangle EAD proportionally. That is

$$EF : EB = FA : BD.$$

But $EF = \delta/2$; $EB = \gamma$; $FA = (\delta/2) + FG = EG = \beta$; $BD = 2\alpha$. Therefore

$$(1) \qquad (\delta/2) : \gamma = \beta : 2\alpha; \quad \text{whence} \quad \beta\gamma = \alpha\delta.$$

From the two secants through E and the theorem that the product of a whole secant to a circle and its external segment is the same for all secants

through a point, it follows that

$$EG \cdot EH = EB \cdot EC \quad \text{or} \quad \beta(\beta + \delta) = \gamma(\gamma + \alpha). \tag{2}$$

From (1) and (2), it follows, by some elementary algebra, that

$$\alpha:\beta = \beta:\gamma = \gamma:\delta,$$

which was to be proved.

The essential idea in each of these constructions is very old. In fact the Greeks had a word for it—the word "νευσις" ("neusis," in Roman letters) which has been translated by the word "verging," a participle used as a noun. The Greek word means, with reference to straight lines, "the inclination or verging of lines toward a point." The corresponding Greek verb "νευω" means "to incline and meet so as to form an angle." By a "νευσις" problem the Greek geometers meant the placing of a straight line l with respect to a fixed point P and two

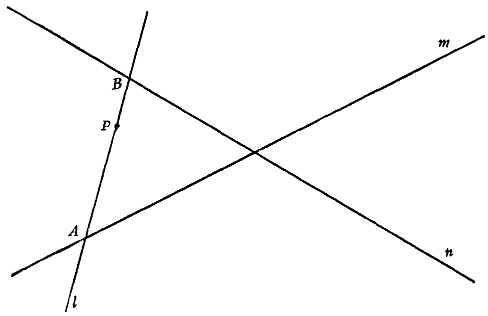


FIG. 3

fixed straight lines m and n so that the line l will pass through P and the part AB of l intercepted between m and n will have a given length. In the diagram (Fig. 3) the segment AB has the given length. Another way to state the problem, with more emphasis on the given length, is this: To insert a line segment AB, of given length, between two given lines m and n so that the end points A and B will be on m and n, respectively, and so that the straight line l which is determined by A and B will go through a given point P.

The word "verging" was used because the line was to "verge" or "incline" toward P. It does not call attention to the other conditions of the problem, which are that A must be on m and B must be on n. The Germans have used the word "Einschiebung" to express the fact that the line segment AB is to be "shoved in" or "inserted" between m and n. There are three conditions on the line l to be expressed: (1) It must go through P; (2) one end of a segment of l (of given length) must be on m; (3) the other end must be on n. No one word seems to express all three conditions. But I think that the word "Einschiebung" is more expressive of the general idea than the word "verging." I shall use the word "inserting" as a translation of "Einschiebung" and shall speak of inserting

a line segment between two given lines in such a way that the line segment, produced if necessary, will go through a given point.

One of these "νευσις" or "Einschiebung" or "inserting" problems has usually four solutions as may be seen from the following diagram. Each of the segments A_1B_1, A_2B_2, A_3B_3, A_4B_4 has the given length. If that given length is too short, two of the solutions will be imaginary.

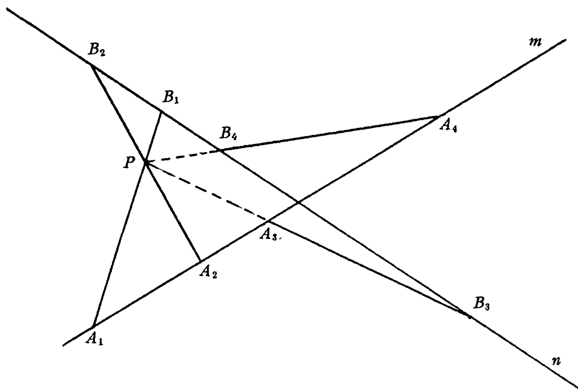


FIG. 4

If one makes an analytic geometry formulation of the problem and solves for the slope (S) of the line to be constructed, the equation for S will be of the fourth degree.

In the construction which I have given for trisecting an angle, if B be taken as the origin and BC as the positive x-axis; and if the tangent of the given angle be denoted by μ; the quartic equation for the slope (S) of the trisecting line BF will be

$$S^4 - 2\mu S^3 - 3(1 + \mu^2)S^2 - 2\mu S + \mu^2 = 0.$$

One factor of this is $S+\mu$. It gives one solution of the problem to insert a line segment of given length between the lines l and m; but it does not correspond to any solution of the trisection problem. The solutions which come from the cubic factor equated to zero,

$$S^3 - 3\mu S^2 - 3S + \mu = 0,$$

are

$$\tan \tfrac{1}{3}\phi; \quad \tan \tfrac{1}{3}(\phi + 180^\circ); \quad \tan \tfrac{1}{3}(\phi + 360^\circ).$$

In the Vieta construction which I have given, if $\alpha = 1$ and $\delta = 2$, the line m will be tangent to the circle at B. The diagram, in which I have put the extra line AJ, will be as shown in Fig. 5. By taking A as the origin of coordinates and the line AJ as the negative y-axis, the following quartic equation for the slope (S) of the line EF can be derived:

$$33S^4 + 30\sqrt{3}S^3 + 6\sqrt{3}S - 9 = 0.$$

All that is needed in the derivation is some elementary algebra and analytic geometry. One factor of the quartic is $(S+\sqrt{3})$, which gives the solution $S=-\sqrt{3}$. That this furnishes one solution of the "νευσις" problem is evident from the diagram. It gives CI as the line segment of given length (unity in this case) to be inserted between the lines l and m. But obviously it does not furnish a solution of the problem of finding the two mean proportionals. The solution of that problem comes from the cubic factor; that is from the equation

$$33S^3 - 3\sqrt{3}S^2 + 9S - 3\sqrt{3} = 0.$$

This equation has one real and two imaginary roots. The real root corresponds to that position of the inserted line segment EF for which $EG=\sqrt[3]{2}$.

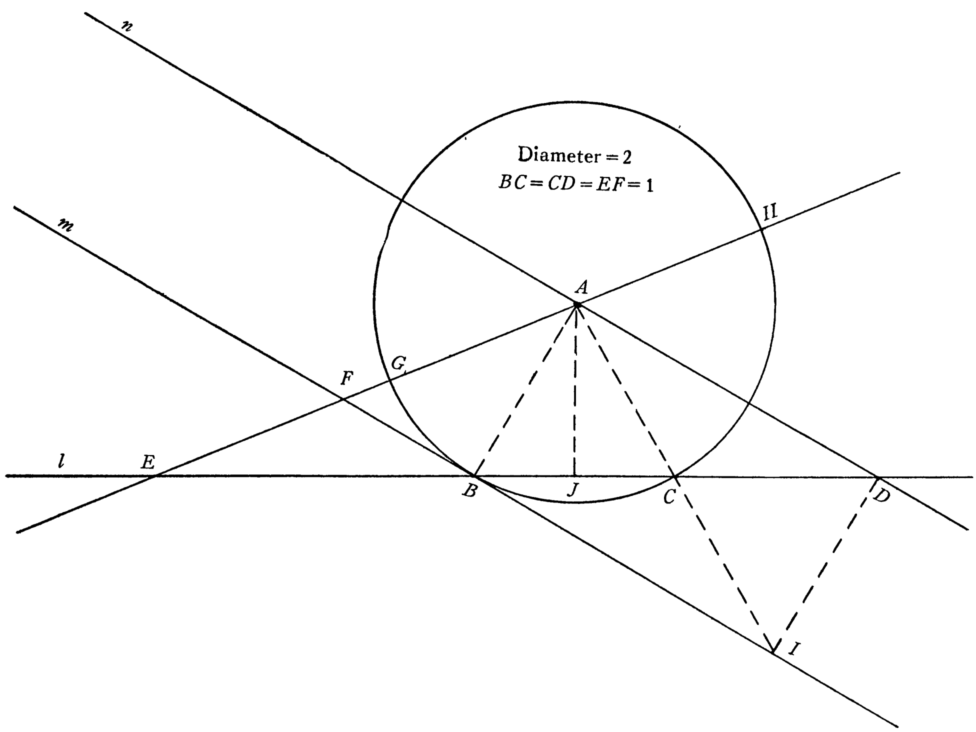


FIG. 5

The essential idea of the construction which I have given for trisecting an angle is due to the Greek geometer Nicomedes. He reduced the problem to a "νευσις" problem and invented the conchoid curve to solve that. He even devised a machine for constructing such a conchoid. He located the point corresponding to the F of my diagram as the intersection of a conchoid and a straight line corresponding to my m.

To me it seems that the use of a conchoid, or other auxiliary curve, whether it is made by the free-hand drawing of the curve through points individually located or by a mechanical device, is much more mechanical than the use of

a graded ruler. It is only a little more difficult to place a graded ruler in the proper position for the solution of a "νευσις" problem than it is to place a straightedge so that it will go through two given points. One has to watch three things instead of only two, namely: the fixed point through which the required line must pass and two points, marked on the ruler, which must be on fixed lines. Perhaps if graded rulers and cardboard straightedges had been as common in the second century B.C. as they are now, Nicomedes would have put his solution in the more modern form in which I have exhibited it. That form is not so modern, by the way. Pappus (about 300 A.D.), in commenting on the solution of Nicomedes, said that "the conchoid was not always actually drawn; that some people, for greater convenience, moved a ruler about a fixed point until by trial the intercept was found to be equal to the given length."

These examples of solution by compasses and graded ruler clearly show that much power to make geometric constructions is added by the substitution of a graded ruler for the ruler which is just an unmarked straightedge. It is evident that by such means it is possible to solve some of the construction problems whose analytic geometry formulations lead to equations of the third and fourth degrees. As a matter of fact all cubic and quartic equations whose coefficients are real numbers which represent the lengths of given geometric line-segments can be solved geometrically by compasses and graded ruler. The solution of a quartic can be reduced to that of its resolvent cubic; the general cubic can be transformed into the reduced cubic for which the Cardan formulas give the solution; if the reduced cubic has three real roots of which at least two are equal, it can be solved by ruler and compasses; if the three real roots are all different, it can be solved by ruler and compasses together with a properly chosen construction for angle trisection; and if the cubic has only one real root it can be solved by ruler and compasses together with a properly chosen Vieta construction for finding a cube root. To prove all these statements in detail would take more time than I have at my disposal; and the proofs* are not necessary for the purpose which I have in mind.

If the substitution of graded ruler for ruler adds so much power to the usual ruler and compass equipment, how much might we reasonably expect to be able to do with the whole lot of simple instruments which I have mentioned? I do not know the answer to that question. Partial answers are to be found in books to which I shall make reference later.

Geometric constructions are of great interest to amateur mathematicians. That has been true for centuries and is still true. Let me say here that there is a much greater amateur interest in mathematics than many professional mathematicians suspect. During my five years service as editor-in-chief of THE AMERICAN MATHEMATICAL MONTHLY I had letters from amateurs in many walks of life—from doctors, lawyers, paper-hangers, retired business men, etc. Some were

* Those who desire proofs should read §20 of August Adler's *Theorie der geometrischen Konstruktionen* (G. J. Göschen, 1906) together with §20 of chapter VII of the second volume of Federigo Enriques's *Fragen der Elementargeometrie* (B. G. Teubner, 1923).

interested in the honor and glory which they thought might come to them for solving problems which had baffled professional mathematicians; some were interested mainly in fancied financial rewards; but most of them seemed to be playing with mathematics for the fun of the thing.

I had a letter from a man in Iowa who claimed that he had proved Fermat's last theorem. He would not send it to me unless I would put up a bond to protect him against the theft of his ideas.

A man in the State of Washington wrote me that he had "a construction for trisecting angles by ruler and compasses, a thing which mathematicians for centuries had tried in vain to accomplish." He asked me to take the initiative in getting several of the larger universities to set aside a percentage of football receipts to build up a fund of $15,000 or more to pay him for his fifteen years of work on the problem. He said that all he was asking was the wages of an ordinary school teacher for fifteen years.

Aside from those two, all the amateurs who wrote to me sent me their stuff without any requests for financial reward. They wanted me to evaluate their work and to publish it if I thought it worth publishing.

The large amateur interest to which I have referred manifests itself mainly in these fields:

1. Mathematical recreations, especially puzzles of a mathematical nature.
2. Elementary theory of numbers.
3. Geometric constructions, especially angle trisection and the squaring of the circle. The duplication of the cube has not aroused so much interest.)

These are fields in which amateurs can play even if they have had no courses in mathematics beyond those which they took in high school.

Of the amateurs interested in angle trisection, some know what is meant by the classical ruler and compass restrictions and some do not. Of those who do not know, some make good approximations (with or without fallacious proofs that they are exact); and some make constructions by means of auxiliary curves, thinking that they are making what mathematicians call ruler and compass constructions in the strict sense. One of these came from Mr. Joseph Trost, of Lakota, North Dakota, to the chief justice of the state of Minnesota, to the president of the University of Minnesota, to Professor Dunham Jackson. The man used an auxiliary curve which, as I found out later, was invented for angle trisection and described by a man named P. Delanges in a paper published in Verona in 1783 under the title *La trisegante nuova curva*. The same curve was invented independently (about 100 years later) by W. H. Hillhouse who gave a different description of the curve. He published his work in an article *On a new curve for the trisection of an angle* in the Analyst for 1882 (vol. 9, p. 181). In that article he described a machine for drawing the curve.

What I have said should not be taken as meaning that the man from North Dakota contributed nothing new. As a matter of fact he gave a different and more simple definition of the curve by means of which its use for angle trisection can be explained easily to any high school student of geometry.

When I was examining this trisection, and the proof of it, there was something vaguely familiar about it, although I was sure that I had never had anything to do with that particular auxiliary curve. (It was not until later that I found a reference to the trisecting curve of Delanges and Hillhouse*.) At last it occurred to me that it was related to a trisection by means of a carpenter's square which, as editor, I had published for Mr. H. T. Scudder, of Marinette, Wis., in May, 1928.† He used a carpenter's square—no compasses. He thought he had made a wonderful achievement; that he had succeeded in solving a problem which had defied mathematicians for centuries. He had been told, perhaps

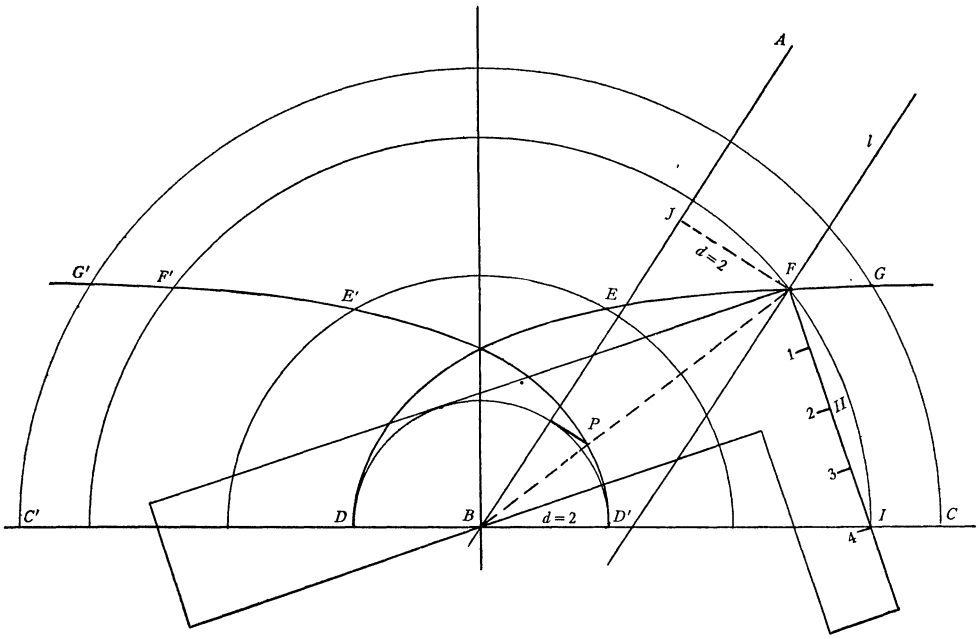


FIG. 6

in a high school geometry class, that mathematicians could not trisect angles with ruler and compasses. He did not understand just what was meant by the classical restrictions and did not know in what sense a trisection by ruler and compasses was impossible. I told him about two of the centuries-old constructions by compasses and graded ruler and thus got him over the notion that he had made an achievement of great magnitude. Then I complimented him on his solution and told him that I would publish it. It was an ingenious one which I could not find anywhere in the literature. Since that time I have found that J. W. Nicholson used what is essentially a carpenter's square in an article on

* Gino Loria, *Spezielle Algebraische und Transzendente Ebene Kurven* (B. G. Teubner, 1910), vol. 1, pp. 231–233.

† This MONTHLY, vol. 35 (1928), p. 250.

The Multisection of Angles in the Analyst for 1883 (vol. 10, p. 41). But Nicholson used the L-shaped square to draw a curve called the polyode which he used as an auxiliary curve for angle trisection.

The essential features of the carpenter's square are these: (1) it has two arms at right angles with their outer edges graded in inches; (2) the longer arm has parallel edges two inches apart. The way in which Mr. Scudder used the square is indicated in the diagram (Fig. 6). His construction resembles the trisection by compasses and graded ruler in that it involves a three-point contact of an instrument with a drawing. I have put in the diagram the trisecting curve of Delanges and some semi-circles in terms of which Mr. Trost defined the curve.

Let ABC be the angle to be trisected. By means of the parallel edges of the long arm of the square, draw the line l parallel to BA and at a distance $d=2$ from BA. Then put the square in such a position that (1) the point B will be on the inner edge of the long arm; (2) the outer corner will be on the line l (at F); (3) the mark for 4 ($=2d$) on the short arm will be on the line BC (at I). Mark H, the middle point of FI, and remove the square. The lines BF and BH will trisect the given angle ABC, because if J is the foot of the perpendicular from F to BA, the triangles BHI, BHF, BJF will be congruent right triangles with equal angles at B.

The curved line $DEFG$ is a part of the trisecting curve of Delanges. Another branch is the curved line $D'E'F'G'$. The whole curve may be obtained from those two parts by reflection in the line $C'BC$. The polar equation of the locus of F as the angle ABC changes in size is $d=\rho \sin(\theta/2)$. Delanges used the equivalent form $d=\rho\cos(\theta/2)$. The point (ρ_1, θ_1) of the first of these loci is identical with the point $(\rho_1, \theta_1+180°)$ of the second.

Mr. Trost defined the curve $DEFG$, in terms of the concentric circles with centers at B, as the locus of the end points of chords of fixed length, the other ends of the chords being on the line BC. He obtained the locus by the free-hand drawing of a smooth curve through individually plotted points; and then located the point F as the intersection of the curve with the line l. For the angle ABC, Delanges used the branch $D'E'F'G'$ and located the point P (which is on the trisecting line BF) as the intersection of the curve with the tangent to the circle of radius d at the point where BA meets that circle. He used the equation $d=\rho\cos(\theta/2)$ to make the proof.* Hillhouse† gave a mechanical description of the curve in terms of a jointed parallelogram.

The construction of Mr. Scudder's is a good example of what can be done in the way of geometric construction with a simple instrument. We know one problem of the third degree can be solved by means of it. Just what other problems of higher degree than the second can be solved by its use I do not know.

To get back to the man from North Dakota—it was while I was examining

* See Loria, loc. cit.

† Loc. cit.

his solution that I realized that the carpenter's square as Mr. Scudder used it, located a point on the trisecting curve of Delanges, just the one point needed. Mr. Scudder did not mention any such curve. Indeed he was not aware that there was one. Neither was I, at the time, although I might have suspected it. In just such a manner the graded ruler may be used to locate a point, just the one point needed, on the conchoid curve for trisecting an angle by the method of Nicomedes.

Such a locating of a point on an auxiliary curve really dispenses with the use of the curve as an auxiliary and eliminates the drawing of a curve through plotted points. Such free-hand drawing is, to my mind, a defect in any geometric construction. That it has been so regarded is evidenced by the fact that so many mechanical devices have been made for the drawing of such curves by continuous motion.

Perhaps I have said enough to make it clear that the addition of very simple instruments to the traditional ruler and compass equipment makes it possible to solve many old problems in a more simple way and to solve some new problems. There are many interesting things that I might say about geometric constructions which involve some but not all of the instruments which I have mentioned. I shall mention only a few of the topics about which you can read in the books to which I shall refer you.

I mention first the chapter on geometric constructions in L. E. Dickson's *First Course in the Theory of Equations*. The author made a real contribution to the education of our undergraduate students of mathematics when he wrote that chapter. It tells about what constructions are possible by ruler and compasses; it shows that constructions for the trisection of an angle, the duplication of the cube, the regular polygons of seven and nine sides are not thus possible; and it exhibits a ruler and compass construction for the regular polygon of seventeen sides. The chapter does not have anything to say about constructions by means of the graded ruler and other simple instruments to be used in connection with ruler and compasses.

The three books which I most highly recommend for an account of what has been done in the way of geometric constructions without the classical restrictions are these:

Theorie der Geometrischen Konstruktionen, by August Adler;

Konstruktionen und Approximationen, by Theodor Vahlen;

Fragen der Elementargeometrie, by Federigo Enriques.

These books contain a wealth of material on the subject. There are no books in English which give such complete expositions. But there are books in English which contain material of the kind I am discussing. I have in mind particularly: Hilda P. Hudson's *Ruler and Compasses*; Hilbert's *Foundations of Geometry*, to which I have made reference; the books on the history of mathematics by W. W. R. Ball, Florian Cajori, Vera Sanford, and D. E. Smith; and books on Greek mathematics by Thomas L. Heath and James Gow.

I shall now indicate briefly some of the topics discussed in those books:

What constructions can be made

(1) with ruler alone?

(2) with parallel ruler alone?

(3) with compasses alone (Mascheroni constructions)?

(4) with ruler and one fixed geometric figure—square, parallelogram, circle, etc. (Steiner constructions)?

(5) with one or two squares (where the word "square" means an instrument for drawing right angles, such as the draftsman's triangles, the T-square, the L-shaped square)?

(6) with ruler and segment carrier?

(7) by means of auxiliary curves?

(8) by means of various combinations of the instruments mentioned?

Also discussed in these books are approximate geometric constructions.

I now come to what is the main purpose of this talk, if it has any purpose other than that of occupying a place on the program with the hope that it will interest some of you. That purpose is to influence those members of this Association who are teachers—and most of them are—to see to it that their students get, somewhere in their college course, some knowledge of geometric constructions without the classical restrictions. Students in high school and college have been told a lot about what can and cannot be done with ruler and compasses. A lot of publicity has been given to the fact that with those instruments alone it is not possible to trisect angles and duplicate cubes. But students have, in general, not been told how easy it is to solve two of those famous problems by other simple means. The constructions which students are usually asked to make when studying elementary geometry can all be made with ruler and compasses; and high school students may well be told that those are the traditional instruments; that the ancient Greek geometers, and many others since their day, have considered the making of geometric constructions a better game if the only instruments used are ruler and compasses; that if you please it may be more sportsmanlike to limit one's self to those instruments (just as it may be more sportsmanlike to catch brook trout with artificial flies than to follow the country boy's more easy method of using angle worms and grasshoppers); but they should not be told or given the impression that there is anything un-mathematical about using graded rulers, T-squares, and other simple instruments. On the contrary they should be told that many of the constructions which can be made with ruler and compasses can be shortened and made easier by means of such instruments; and that some constructions which cannot be made with the classical restrictions in effect can be very easily made if just a little is added to the ruler and compass equipment. High school students may well be told about the famous geometric problems of antiquity and that they cannot be solved with ruler and compasses alone. But they should not be told that and nothing more. That has been going on for years and is responsible for the prevalent idea that mathematicians have been unable to solve such a simple problem as that of dividing an angle into three equal parts. Students ought to be

told something about what can be done and not exclusively what cannot be done with simple geometric instruments.

A few years ago a certain university president got a lot of publicity for his claim that he had succeeded in solving a problem that had defied the efforts of mathematicians for many centuries. He claimed that he had a construction for trisecting angles by ruler and compasses according to the strict meaning of that phrase. He was visiting in the neighborhood of Minneapolis and managed to get a long feature story in the *Minneapolis Tribune.* From there the story went all over the United States, and to foreign countries for all I know to the contrary. He did not explain his construction but gave a few hints; and he said that he might publish it later, perhaps in a textbook on geometry. The story was commented on in the press for days. Because he occupied a relatively high educational position, the newspaper men took his word for it that he had made a wonderful achievement. They thought it made a good story.

When a story like that gets such wide publicity, it puts mathematicians in a bad light. It must seem strange to the general public that mathematicians cannot solve such a simple problem. That which they have been told and which they vaguely remember has left them with the impression that mathematicians have no theoretically exact construction for dividing an angle into three equal parts. That is the result of the fact that both high school and college teachers have said so much about what cannot be done and so little about what can be done with problems of higher degree than the second.

When one is a witness in court he is sworn to tell the truth, the whole truth, and nothing but the truth. Now I am not advocating that we try to tell the whole truth about what can or cannot be done with simple drawing instruments. We do not know the whole truth; and it is impossible for us to tell as much of it as we do know to high school and less advanced college students. But it is easy to explain to high school students how two of the famous problems can be solved by compasses and graded ruler or by ruler, compasses, and an auxiliary curve. The proofs involve only easy geometric theorems with which they are familiar.

Students in college can be told more of the whole truth. I am not suggesting that colleges offer a special course in the subject (although such a special course would make a good addition to the summer offerings of any college department of mathematics). I am suggesting rather that some of the subject matter which I have in mind be worked into other courses.

The chapter on geometric constructions in L. E. Dickson's *First Course in the Theory of Equations*, to which I have referred, can appropriately be supplemented by the teacher to include constructions by graded ruler and other instruments whose analytic geometry formulation leads to equations of third and fourth degree. (Let me say, by the way, that there is one problem in that chapter, leading to a reciprocal quartic equation, which is really a "*νευσις*" problem similar to the one for angle trisection by graded ruler and compasses. I refer to Example 16 on page 40.)

Courses in "The History of Mathematics" and courses in "College Geometry," which are more commonly taught now than they were some years ago, may appropriately include material on constructions without the classical restrictions. The courses in "College Geometry" have been introduced in colleges in recent years because of a feeling that the more traditional college courses give to prospective high school teachers much less adequate preparation for teaching geometry than for teaching algebra. The textbooks commonly used in courses in "College Geometry" have a wealth of very interesting subject matter but not much, if any, material on geometric constructions by other means than the traditional ruler and compasses. For that reason the teacher who wants to follow my suggestion will have to gather material from other sources.

I have mentioned three college courses into which I think can be put something about geometric constructions without the classical limitations; and it may be that teachers of analytic geometry and calculus can appropriately work some such material into those courses. Some of the auxiliary curves which have been used for trisecting angles and duplicating cubes are to be found in textbooks on analytic geometry and differential calculus; and the use of the quadratrix for the squaring of the circle can be explained to students who have studied the chapter on the evaluation of indeterminate forms.

"Well," as one of the more prominent radio speakers would say, "I see that my time is up and I've got to buzz off." So I shall put on a concluding paragraph and call the whole thing a lecture. Here it is:—If college teachers do not give their students more of the material about which I have been talking than they have in the past, the next generation of high school teachers will be as fully bound by the classical tradition as their predecessors; and they will have missed a lot of interesting subject matter that is very definitely related to their job as teachers of elementary geometry.

THE TRANSFORMATION OF TWO ARBITRARY FUNCTIONS INTO LINEAR FUNCTIONS

By ALBERT WERTHEIMER, U.S. Navy Department, Washington, D.C.

1. *Introduction.* In a paper† dealing with graphical methods, it is shown how to transform graphically by approximations any two curves satisfying certain mild conditions into straight lines. Analytically the problem may be stated as follows: Given two curves with equations $y=F(x)$, and $y=G(x)$, does there exist a continuous transformation of the form $y^*=y^*(y)$; $x^*=x^*(x)$, such that the equations of the curves are transformed into $y^*=a_1+b_1x^*$, and $y^*=a_2+b_2x^*$. In this paper we obtain such a transformation when both curves are either increasing or decreasing, are continuous, and have one and only one point in common.

2. *Auxiliary Functions.* Without loss of generality we may take the common point as the origin. Let the equations of the curves C_1 and C_2 be respectively

† A. Wertheimer, Journal Franklin Inst., vol. 219 (1935), p. 343.

$y=F(x)$, and $y=G(x)$ defined in the interval $0>x_1' \leqq x \leqq x_1>0$.† The functions F and G as well as their inverses $\overline{F}$ and $\overline{G}$ are assumed to be continuous and single valued in the intervals considered. For definiteness, we assume that both F and G are increasing functions and that $|F(x)|>|G(x)|$ for all $x\neq 0$.

Let the function ψ be defined by $\psi \equiv F(\overline{G})$, then its inverse $\overline{\psi}\equiv G(\overline{F})$; from the conditions imposed on F and G it follows that ψ and $\overline{\psi}$ are continuous, single valued, increasing functions. Let $F(x_1)=y_0$, and $F(x_1')=y_0'$, then define the following sequence of numbers:

$$y_1 = \overline{\psi}(y_0);\ y_2 = \overline{\psi}(y_1) = \overline{\psi}^2(y_0);\ \cdots\ y_n = \overline{\psi}(y_{n-1}) = \overline{\psi}^n(y_0)\ \cdots$$

and a similar sequence for the y-primes.

From the assumptions and definitions it follows that

(a) $\overline{\psi}(y) > 0$ for all $y > 0$; (b) $\overline{\psi}(0) = 0$.

Also since $|G(\overline{F})| < |F(\overline{F})|$ for all $\overline{F}\neq 0$, we have

(c) $|\overline{\psi}(y)| < |(y)|$ for all $y \neq 0$.

From (a) and (c) it follows that the sequence y_r is a positive monotonically decreasing sequence and therefore has a limit L, thus

$$\lim_{n\to\infty} y_n = L.$$

Now since $\overline{\psi}$ is continuous everywhere in the interval defined, we have

$$\lim_{n\to\infty} \overline{\psi}(y_n) = \overline{\psi}(L),$$

so that

$$\lim_{n\to\infty} y_{n+1} \equiv \lim_{n\to\infty} \overline{\psi}(y_n) = \overline{\psi}(L) = L,$$

since L is assumed to be the limit of the sequence; hence from (b) and (c) it follows that $L=0$. A similar proof holds for the sequence of the y-primes.

Let a function ϕ be defined by $\phi\equiv\overline{F}(G)$, and define the following sequence of numbers:

$$x_2 = \phi(x_1);\ x_3 = \phi(x_2) = \phi^2(x_1);\ \cdots\ x_n = \phi(x_{n-1}) = \phi^{n-1}(x_1);\ \cdots$$

and a similar sequence for the x-primes. Again it can be shown that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n' = 0.$$

3. *The transformations for x and y.* Let $\xi(y)$ be a continuous monotonically increasing function in the interval $0\leqq y\leqq y_0$, such that $\xi(y)>0$ for all $y>0$.‡

† The restriction to a finite interval is not essential. The method of proof given below can readily be extended to an infinite interval as well.

‡ One such function is simply $\xi(y)=y$.

The transformation of y is then defined by y^* where y^* takes on the following forms in the different intervals:

$$\begin{array}{ll} (y_0 y_1) & y^* = \xi(y) \\ (y_1 y_2) & y^* = k\xi[\psi(y)] \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (y_n y_{n+1}) & y^* = k^n \xi[\psi^n(y)]\dagger \end{array} \qquad \left\{ k \equiv \frac{\xi(y_1)}{\xi(y_0)} < 1 \right\}.$$

Let $\eta(y)$ be a continuous monotonically increasing function in the interval $y' \leqq y \leqq 0$, such that $\eta(y) < 0$, for all $y < 0$, and $\eta(y_1') = k\eta(y_0')$.‡ For the y-primes then the transformation y^* takes on the following forms:

$$\begin{array}{ll} (y_0' y_1') & y^* = \eta(y) \\ (y_1' y_2') & y^* = k\eta[\psi(y)] \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (y_n' y_{n+1}') & y^* = k^n \eta[\psi^n(y)] \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

From the definitions it follows that

$$k\xi[\psi^r(y_r)] = \xi[\psi^{r-1}(y_r)]$$

and

$$k\eta[\psi^r(y_r)] = \eta[\psi^{r-1}(y_r)],$$

hence y^* as well as its inverse are continuous and single valued. Also since $k < 1$, we have that

$$\lim_{n\to\infty} y^*(y_n) = \lim_{n\to\infty} y^*(y_n') = 0.$$

The transformation x^* takes on the following forms in the different intervals:

$$\begin{array}{ll} (x_1 x_2) & x^* = m\xi[F(x)] \quad (m \text{ is any constant } \neq 0) \\ (x_2 x_3) & x^* = mk\xi\{\psi[F(x)]\} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (x_{n+1} x_{n+2}) & x^* = mk^n\xi\{\psi^n[F(x)]\} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

For the x-primes, the forms are the same except that ξ is replaced by η. Here again it can be shown that x^* and its inverse are continuous, single valued, and that

---

† k^n is an ordinary power while ψ^n is the result of repeated functional operation.

‡ One such function is $\eta(y) = ky_0'^{\,r} - y_1'^{\,r} + (1-k)y^r$, where r is any positive odd integer such that $y_1'^{\,r}/y_0'^{\,r} \leqq k$.

$$\lim_{n\to\infty} x^*(x_n) = \lim_{n\to\infty} x^*(x_n') = 0.$$

4. *The transformation of the equations of the curves.* From the definitions of the sequences of x and y it follows directly that $F(x_r)=y_{r-1}$, and $G(x_r)=y_r$, so that to the interval $(x_r x_{r+1})$ there corresponds on C_1 the interval $(y_{r-1}y_r)$, and on C_2 the interval $(y_r y_{r+1})$, and similarly for the x-primes and y-primes. Substituting in the equations $y=F(x)$, and $y=G(x)$, the corresponding values of y^* and x^* we get in each of the intervals the following equations:

$$C_1^*:\quad y^* = x^*/m, \qquad\qquad C_2^*:\quad y^* = kx^*/m.$$

The two curves are thus transformed into straight lines with arbitrary slopes. The case when F and G are both decreasing functions is exactly the same after interchanging x and y with x' and y' respectively.

THE DECOMPOSITION OF $4(x^p-1)/(x-1)$

By CORNELIUS GOUWENS, Iowa State College

If $X=(x^p-1)/(x-1)$ where p is an odd prime, then

$$4X = Y^2 - (-1)^{(p-1)/2}pZ^2$$

where Y and Z are polynomials in x. For $p\leqq 37$ we find the decomposition cited in *Recherches sur la theorie des nombres* by M. Kraitchik (1924), p. 126. For $37\leqq p\leqq 61$ the decomposition is given by Pocklington in Nature vol. 107 (1921), pp. 456 and 587. For all decompositions Y is a polynomial of degree $(p-1)/2$ while Z is a polynomial of degree $(p-3)/2$. We list herewith the decomposition for $67\leqq p\leqq 97$, giving only the coefficients in each case first for Y, then for Z.

$p=67$.

2, 1, −16, 9, 33, −44, −18, 79, −39, −48, 75, −35, −14, 69, −89, 10, 106, −106, −10, 89, −69, 14, 35, −75, 48, 39, −79, 18, 44, −33, −9, 16, −1, −2.

1, 0, −3, 3, 4, −8, 1, 9, −8, −1, 7, −8, 5, 5, −14, 8, 8, −14, 5, 5, −8, 7, −1, −8, 9, 1, −8, 4, 3, −3, 0, 1.

$p=71$.

2, 1, −17, −26, −5, 31, 58, 64, 60, 33, −29, −89, −106, −91, −70, −44, −16, −3, 3, 16, 44, 70, 91, 106, 89, 29, −33, −60, −64, −58, −31, 5, 26, 17, −1, −2.

1, 1, −2, −5, −5, −2, 2, 6, 11, 13, 9, 2, −3, −6, −8, −8, −7, −7, −8, −8, −6, −3, 2, 9, 13, 11, 6, 2, −2, −5, −5, −2, 1, 1.

$p=73$.

2, 1, 19, 28, 61, 106, 158, 251, 322, 442, 544, 652, 783, 868, 983, 1050, 1113, 1164, 1156, 1164, 1113, 1050, 983, 868, 783, 652, 544, 442, 322, 251, 158, 106, 61, 28, 19, 1, 2.

1, 1, 4, 7, 12, 20, 27, 40, 50, 64, 78, 89, 104, 113, 124, 131, 134, 138, 134, 131, 124, 113, 104, 89, 78, 64, 50, 40, 27, 20, 12, 7, 4, 1, 1.

$p=79$.

2, 1, −19, −29, 24, 69, 4, −64, 19, 75, −67, −125, 81, 166, −60, −137, 87, 110, −152, −148, 148, 152, −110, −87, 137, 60, −166, −81, 125, 67, −75, −19, 64, −4, −69, −24, 29, 19, −1, −2.

1, 1, −3, −6, 1, 8, 0, −7, 5, 11, −9, −17, 8, 18, −7, −13, 14, 14, −18, −18, 14, 14, −13, −7, 18, 8, −17, −9, 11, 5, −7, 0, 8, 1, −6, −3, 1, 1.

$p=83$.

2, 1, −20, 11, 20, −65, 58, 42, −137, 115, 47, −223, 184, 62, −290, 273, 44, −343, 319, 7, −348, 348, −7, −319, 343, −44, −273, 290, −62, −184, 223, −47, −115, 137, −42, −58, 65, −20, −11, 20, −1, −2.

1, 0, −3, 4, −1, −8, 12, −3, −13, 21, −7, −20, 30, −12, −23, 40, −17, −23, 43, −22, −22, 43, −23, −17, 40, −23, −12, 30, −20, −7, 21, −13, −3, 12, −8, −1, 4, −3, 0, 1.

$p=89$.

2, 1, 23, 34, 52, 118, 123, 146, 237, 209, 203, 303, 257, 230, 368, 332, 266, 410, 364, 242, 365, 333, 198, 333, 365, 242, 364, 410, 266, 332, 368, 230, 257, 303, 203, 209, 237, 146, 123, 118, 52, 34, 23, 1, 2.

1, 1, 4, 8, 8, 15, 20, 17, 25, 29, 21, 29, 36, 26, 34, 44, 30, 34, 44, 27, 27, 40, 27, 27, 44, 34, 30, 44, 34, 26, 36, 29, 21, 29, 25, 17, 20, 15, 8, 8, 4, 1, 1.

$p=97$.

2, 1, 25, 37, 93, 171, 278, 466, 643, 936, 1219, 1556, 1960, 2295, 2750, 3111, 3520, 3930, 4225, 4632, 4846, 5114, 5302, 5347, 5468, 5347, 5302, 5114, 4846, 4632, 4225, 3930, 3520, 3111, 2750, 2295, 1960, 1556, 1219, 936, 643, 446, 278, 171, 93, 37, 25, 1, 2.

1, 1, 5, 9, 17, 30, 44, 69, 92, 125, 160, 194, 239, 274, 319, 358, 394, 437, 462, 498, 518, 534, 551, 546, 551, 534, 518, 498, 462, 437, 394, 358, 319, 274, 239, 194, 160, 125, 92, 69, 44, 30, 17, 9, 5, 1, 1.

These coefficients have been carefully checked and I think they are correct.

ON MONGE'S DIFFERENTIAL EQUATION

By J. W. LASLEY, JR., University of North Carolina

Monge's differential equation*

$$9q^2t - 45qrs + 40r^3 = 0,$$

where $dy/dx=p$, $dp/dx=q$, $dq/dx=r$, $dr/dx=s$, $ds/dx=t$, appeared first in 1810. In 1859 Boole† attributed the equation to Monge, adding the often quoted words, "But here our powers of geometric interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms."

---

* Monge, *Sur les équations différentielles des courbes du second degré.* Correspondence sur l'École Impériale Polytechnique, Paris, vol. 2, no. 2, Jan. 1810: reprinted in Société Philomathique de Paris, Bulletin, vol. 2 (1810), pp. 87–88.

† Boole, *A Treatise on Differential Equations*, Cambridge, 1859, p. 19 and 20.

Sylvester stated* in 1885 that he could find no justification for attaching Monge's name to this equation. Since, however, the left member of this equation furnished him with the first instance on record of a multinomial projective reciprocant,† Sylvester was particularly anxious to ascertain whether Boole was right in ascribing the equation to Monge. To this end Sylvester engaged the assistance not only of the mathematicians of London, Cambridge and Paris, but also of the "Universal Knowledge and Information Office." He wrote a letter about the matter to an eminent geometer in Paris, who replied that the equation could not be in the writings of Monge, else Halphen, who had written on the differential equation of the conic,‡ would have mentioned the fact. He then wrote to Halphen, but Halphen was unable to supply the reference. Search was then made in the works of Monge and in his manuscripts. Sylvester continued the search both because, as he says,† he knew "how scrupulous and painstaking Boole was, and the least likely of all men to make a quotation at random," and because* this differential form involved in the equation "may be regarded as the starting-point in the Differential Invariant Theory, as the Schwarzian is of the deeper-lying and more comprehensive Reciprocant Theory." The matter rested here until April 1886 when W. W. Beman, of Michigan, supplied the desired references which had been checked for him at the Yale College library. A few hours later his report was corroborated by the Universal Knowledge and Information Office. The lost was found. Apropos of the finding Sylvester said in his flamboyant way, "to think that a task found impossible in London and Paris should have been accomplished in the most satisfactory manner at Yale and Michigan."

The purpose of this note is to call attention to a method of integrating Monge's differential equation by applying to it the theory of linear dependence.

We may write the equation in the determinantal form

$$\begin{vmatrix} 3q & 0 & r \\ 4r & 3q & s \\ 5s & 10r & t \end{vmatrix} = 0.$$

If q is identically zero, the integral curves of the differential equation are straight lines. Let us suppose q is not identically zero. Let us multiply the elements of the second column by q. Then let us add to the elements of the second column the corresponding elements of the first column each multiplied by p; add also to the elements of the second column the corresponding elements of the third column each multiplied by y; and finally add to the elements of the first column the corresponding elements of the third column each multiplied by x. We obtain in this way our differential equation in the form

* Nature, vol. 33 (1886), pp. 224–225.

† Nature, vol. 33 (1886), p. 581.

‡ Société Mathém. de France, Bulletin, vol. 4 (1876), p. 62; vol. 7 (1879), p. 84. See also *Oeuvres de G.-H. Halphen*, vol. 1 (1916), p. 395 and vol. 2 (1918), p. 290.

$$\begin{vmatrix} 3q + xr, & 3pq + yr, & r \\ 4r + xs, & 4pr + 3q^2 + ys, & s \\ 5s + xt, & 5ps + 10qr + yt, & t \end{vmatrix} = 0.$$

This determinantal form of Monge's equation is, upon multiplying the elements of the second column by 2 and rearranging columns, equivalent to the elegant form in which Halphen obtained the differential equation of a conic, namely:

$$\begin{vmatrix} y''' & (xy)''' & (y^2)''' \\ y^{\mathrm{IV}} & (xy)^{\mathrm{IV}} & (y^2)^{\mathrm{IV}} \\ y^{\mathrm{V}} & (xy)^{\mathrm{V}} & (y^2)^{\mathrm{V}} \end{vmatrix} = 0,$$

the primes denoting derivatives with respect to x.

We observe that either in Halphen's form or the one from which we have derived it, the elements of each row are the derivatives of the corresponding elements of the preceding row. It thus appears that our differential equation is of the form

$$W\{y''', (xy)''', (y^2)'''\} = 0,$$

where W is the Wronskian of the three functions y''', $(xy)'''$, $(y^2)'''$. But the vanishing* of the Wronskian is known to be a necessary and sufficient condition for the three functions to be linearly dependent. We have then

$$2fy''' + 2h(xy)''' + b(y^2)''' = 0,$$

where f, h and b are constants, not all zero.

Upon integrating three times with respect to x we obtain

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, g and c are constants of integration.

Thus the integral curves of the Monge differential equation are seen to be the totality of conics in the plane.

ON THE MAPPING OF CERTAIN PLANAR QUADRATIC INVOLUTIONS

By MARY J. DEAN, Oakland High School, Susquehanna, Pa.

Suppose a (2, 1) point transformation is given between two planes π and π', such that a given P_1 in π has one and only one corresponding point P' in π', but P' has two corresponding points, P_1 and P_2, in π. Since either point P_1, P_2 uniquely determines the other, there is an involutorial transformation be-

* Goursat, *A Course in Mathematical Analysis*, translated by Hedrick and Dunkel, Boston, 1917, vol. 2, part 2, p. 103.

tween them in the plane π. The plane π' is then said to be a map of this involution, each point P' representing a pair of conjugate points P_1, P_2.

The plane π' may then be replaced by any convenient rational surface F when a (1, 1) correspondence between the points of π' and those of F is known.

In the case of the quadratic involutorial transformation I in the plane π, Godeaux* mapped I on the points of a certain cubic surface. In these papers two cases were considered: (a) when the three fundamental points are distinct, (b) when two of the three fundamental points coincide. The same process can be applied to a third case, (c) when all three fundamental points are coincident. The present paper will consider this third case.

By a proper choice of coordinates, any transformation of type (c) can be expressed in the form†

$$I:\quad \begin{aligned} \rho x_1' &= x_1^2 \\ \rho x_2' &= -\ x_1x_2 \\ \rho x_3' &= 2x_2^2 - x_1x_3. \end{aligned}$$

This involution has one fundamental point $O_3(0, 0, 1)$, and one fixed point $O_1(1, 0, 0)$. Any point on the line $x_1=0$ corresponds to the fundamental point $(0, 0, 1)$. The image of any straight line not through O_3.

$$a_1x_1' + a_2x_2' + a_3x_3' = 0, \qquad a_3 \neq 0,$$

is a conic

$$a_1x_1^2 + a_2(-\ x_1x_2) + a_3(2x_2^2 - x_1x_3) = 0, \tag{1}$$

through O_3. The common tangent to all these conics at O_3 is the line $x_1=0$, and the conics have three point contact with each other at this point. Consider two lines (neither through O_3) intersecting at a point. Since the two image conics have three point contact at O_3, the fourth point of intersection must correspond to the point of intersection of the two given lines.

If the pairs P_1, P_2 of points of the involution I are to be mapped on the points of a cubic surface F, a plane section of F should correspond to a curve in the plane π invariant under I. It is therefore required to find a linear ∞^3 system of cubic curves, each invariant under I. A cubic curve, to be invariant under I, must pass through $O_3(0, 0, 1)$, have the line $x_1=0$ for tangent, and have three point contact at that point with the conics (1). Otherwise the transformed curve would be of order higher than three.

A cubic curve through the point (0, 0, 1) is represented by an equation of the form

$$u_1x_3^2 + u_2x_3 + u_3 = 0, \tag{2}$$

* L. Godeaux: *Sur l'inversion et sur une surface cubique à quatre points doubles;* Mathesis, vol. 36 (1922), p. 19; and *Sur une transformation quadratique involutive;* Mathesis, vol. 40 (1926), p. 352.

† See Charlotte Scott, *Modern Analytical Geometry* (1894), p. 222.

in which u_i is a binary form of order 1 in x_1, x_2, as yet with arbitrary coefficients. The tangent to this cubic (2) at the point (0, 0, 1) is $u_1=0$. Hence we must put $u_1\equiv\lambda_1x_1$. It is then readily verified that this cubic has three point contact with the conics (1) at the point (0, 0, 1), and is invariant under I, if and only if

$$u_2 \equiv -\lambda_4x_1x_2 - 2\lambda_1x_2^2, \qquad u_3 \equiv \lambda_2x_1^3 + \lambda_3x_1x_2^2 + \lambda_4x_2^3,$$

where λ_1, λ_2, λ_3, λ_4 are arbitrary constants. The required system of cubics can therefore be written:

$$\lambda_1x_3(x_1x_3 - 2x_2^2) + \lambda_2x_1^3 + \lambda_3x_1x_2^2 + \lambda_4x_2(x_2^2 - x_1x_3) = 0. \tag{3}$$

Let the cubics of this system be associated with the planes of space by the equations

$$\begin{aligned} \rho X_1 &= x_3(x_1x_3 - 2x_2^2) \\ \rho X_2 &= x_1^3 \\ \rho X_3 &= x_1x_2^2 \\ \rho X_4 &= x_2(x_2^2 - x_1x_3). \end{aligned} \tag{4}$$

The result obtained by eliminating x_1, x_2, x_3, x_4 from these equations is the cubic surface

$$X_3^3 - X_2X_4^2 + X_1X_2X_3 = 0. \tag{5}$$

Therefore by equations (4) the involution I is mapped on the cubic surface F given by equation (5), and the cubic curves of the linear system (3) correspond to the plane sections of the cubic surface. A point $(0, a_2, a_3)$, $a_2\neq 0$, on the line $x_1=0$ in the plane π corresponds to the point $(-2a_3, 0, 0, a_2)$ on the line $X_2=X_3=0$ lying on the surface F; and conversely, to a point $(c_1, 0, 0, c_4)$, $c_4\neq 0$, on the line $X_2=X_3=0$, corresponds in the plane π the point $(0, -2c_4, c_1)$. As a limiting case, we may say that the fundamental point $O_3(0, 0, 1)$ in the plane π corresponds to the point $O_3'(1, 0, 0, 0)$ on F. Since the first partial derivatives of (5) with respect to X_1, X_2, X_3, X_4 all vanish at O_3' this point is a double point of the surface. The tangent cone at this point degenerates, being given by the equation $X_2X_3=0$, and hence the point is a biplanar point.

To the fixed point $O_1(1, 0, 0)$ in the plane π corresponds the point $O_1'(0, 1, 0, 0)$ on the surface. At this point also the first partial derivatives vanish, making it a double point of the surface F. The tangent cone at this double point is given by the equation $X_1X_3-X_4^2=0$.

CONTACT TRANSFORMATIONS IN SOLVED FORM

By C. K. ROBBINS, Purdue University

It is well known that the most general contact transformation in two dimensions is defined by

$$F(x, y, x', y') = 0, \qquad F_x + pF_y = 0, \qquad F'_x + p'F'_y = 0,$$

where $p = dy/dx$ and $p' = dy'/dx'$.

Solving this set gives

$$(1) \qquad \begin{aligned} x' &= A(x, y, p) \\ y' &= B(x, y, p) \\ p' &= P(x, y, p). \end{aligned}$$

This is the "solved form" mentioned in the title.

The object of this investigation is to obtain some general types of contact transformations in form (1). Therefore the equivalent definition is used, namely equations (1) together with the condition that $dy' - p'dx' = 0$ is a consequence of $dy - pdx = 0$.

Only reversible transformations will be considered, that is, those in which the Jacobian J of the functions involved does not vanish identically.

Although this study is mainly functional in character, geometrical terms are used to some extent. The phrases line element, union of elements and so on have the usual significance.

§1. The contact transformation defined above carries a union of line elements into a similar union. Applying the analytic condition gives

$$(B_x - PA_x)dx + (B_y - PA_y)dy + (B_p - PA_p)dp = 0$$

by means of $dy - pdx = 0$.

Eliminating dy and remembering that x, y and p are independent, we have

$$(2) \qquad B_x - PA_x + (B_y - PA_y)p = 0$$

$$(3) \qquad B_p - PA_p = 0.$$

We seek to solve these two equations.

§2. Any non-singular point transformation, that is one in which A and B are functions of x and y only, is a contact transformation, for (3) is satisfied and P is obtained from (2).

§3. If P is a function of x and y only, (3) gives at once

$$B = AP + \omega(x, y),$$

ω being arbitrary. Then (2) becomes

$$(P_x + pP_y)A + \omega_x + p\omega_y = 0,$$

so that the following general type may be written down

$$\begin{aligned} x' &= -\frac{\omega_x + p\omega_y}{P_x + pP_y} \\ y' &= AP + \omega \\ p' &= P \end{aligned}$$

in which P and ω are arbitrary functions of x and y only.

§4. If P is a function of p only, equation (2) may be solved treating p and therefore P as constants. If we let $u=B-PA$, (2) becomes $u_x+pu_y=0$, the solution of which is $u=\omega(y-px, p)$ where ω, as before, indicates an arbitrary function of the arguments. Then $B=PA+\omega$. This must satisfy (3). Substituting, we have $AP_p-x\omega_t+\omega_p=0$ where $t=y-px$.

Thus we have the following general type

$$x' = \frac{x\omega_t - \omega_p}{P_p}$$
$$y' = PA + \omega$$
$$p' = P$$

The dilation in which

$$x' = x \mp \frac{ap}{\sqrt{1+p^2}}$$
$$y' = y \pm \frac{a}{\sqrt{1+p^2}}$$
$$p' = p$$

is seen to be a special case of the above type by choosing

$$\omega = y - px \pm a\sqrt{1+p^2}.$$

§5. To attack the problem in a more general way, eliminate P between (2) and (3) getting

$$B_pA_x + pB_pA_y - (B_x + pB_y)A_p = 0. \tag{4}$$

This may be regarded as a partial differential equation in A. The solution A = a function of B cannot be used because then J would vanish. If $B_x+pB_y+MB_p=0$, where M is a function of x and p only, (4) becomes

$$A_x + pA_y + MA_p = 0. \tag{5}$$

The related system is $dx=dy/p=dp/M$, one solution of which is obtained from $Mdx-dp=0$ and may be written $\theta(x, p)=k$, k being an arbitrary constant. Thus $M=-\theta_x/\theta_p$.

To obtain another solution, solve $\theta(x, p)=k$ for p. Then $p=\omega(x, k)$ and using $dx=dy/p$ we have $\omega(x, k)\,dx-dy=0$, the solution of which is $y-\mu=$ constant where $\mu=\int\omega(x, k)dx$, the integration to be performed and k replaced by $\theta(x, p)$.

Still another solution, but not independent of the two above, may be obtained by solving $\theta(x, p)=k$ for x, say $x=\sigma(p, k)$. Then, using $dy/p=dp/M$ we have $dy=pdp/M(\sigma, p)$, the solution of which may be written $y-\nu=$ constant where

$$\nu = \int \frac{pdp}{M(\sigma, p)},$$

the integration to be performed and k be replaced by $\theta(x, p)$.

Since the equation in B is the same as that in A the solutions for B are the same as those for A namely θ, $y-\mu$ and $y-\nu$. Then the following type may be given.

$$x' = \alpha(s, t)$$
$$y' = \beta(u, v)$$
$$p' = \frac{\beta_p}{\alpha_p},$$

where α and β are arbitrary functions, s and t are any two different solutions of (5) and u and v are any two different solutions of (4).

To illustrate the method of writing down some types under the above general head, choose $\omega = k+3x^2$. Then $p = k+3x^2$ and $k = p-3x^2 = \theta$. Also $\mu = \int(k+3x^2)dx = kx+x^3 = px-2x^3$. Thus

$$x' = \alpha(y - px + 2x^3, p - 3x^2)$$
$$y' = \beta(y - px + 2x^3, p - 3x^2)$$
$$p' = \frac{\beta_t - \beta_s}{\alpha_t - \alpha_s}$$

where $s = y-px+2x^3$, $t = p-3x^2$.

Another choice of ω and one which leads eventually to the pedal transformation is $\omega = k$, $p = k$, $\theta = p\mu = \int k\, dx = kx = px$. Then

$$x' = \alpha(y - px, p)$$
$$y' = \beta(y - px, p)$$
$$p' = \frac{\beta_p - x\beta_s}{\alpha_p - x\alpha_s},$$

where $s = y-px$.

The pedal transformation, in which

$$x' = -\frac{p(y - px)}{1 + p^2}$$
$$y' = \frac{y - px}{1 + p^2}$$
$$p' = \frac{xp^2 - x - 2py}{yp^2 - y + 2px},$$

is seen to be a special case of the above as far as x' and y' are concerned and the form of p' is readily verified.

Still another choice, and one which leads to the dilation, is

$$\theta = x \pm \frac{ap}{\sqrt{1+p^2}} = k.$$

Then

$$p = \frac{k-x}{\pm\sqrt{a^2-(k-x)^2}} = \omega$$

and

$$\mu = \int \frac{(k-x)dx}{\pm\sqrt{a^2-(k-x)^2}} = \pm\sqrt{a^2-(k-x)^2} = \frac{\pm a\sqrt{1+p^2+p^4}}{1+p^2}$$

if k is replaced by its equivalent $x \pm ap/\sqrt{1+p^2}$. Also

$$x = k \mp \frac{ap}{\sqrt{1+p^2}} = \sigma$$

$$M = -\frac{\theta_x}{\theta_p} = \mp\frac{1}{a}(1+p^2)^{3/2}$$

$$\nu = \mp\int pa(1+p^2)^{-3/2}dp = \frac{\pm a}{\sqrt{1+p^2}}.$$

Then we may write

$$x' = \alpha(y-\mu, \theta)$$
$$y' = \beta(y-\nu, \theta)$$
$$p' = \frac{\beta_p}{\alpha_p},$$

where μ, ν and θ have the values indicated above.

The dilation,

$$x' = x \pm \frac{ap}{\sqrt{1+p^2}}$$
$$y' = y \mp \frac{a}{\sqrt{1+p^2}}$$
$$p' = p,$$

is seen to be a special case of this type as regards x' and y' and the form of p' is readily verified.

Although the subject of contact transformations has an extended litera-

ture, it seems that the problem has not been thoroughly treated from the standpoint taken in this paper. The author is treating the three dimensional case in a similar manner and hopes that still other categories of contact transformations in solved form may be discovered.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

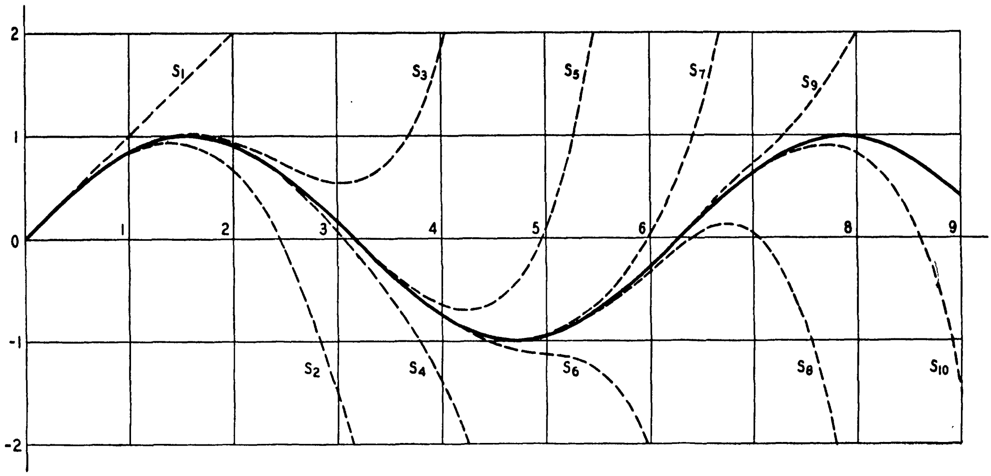
The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

SINE AND COSINE APPROXIMATION CURVES

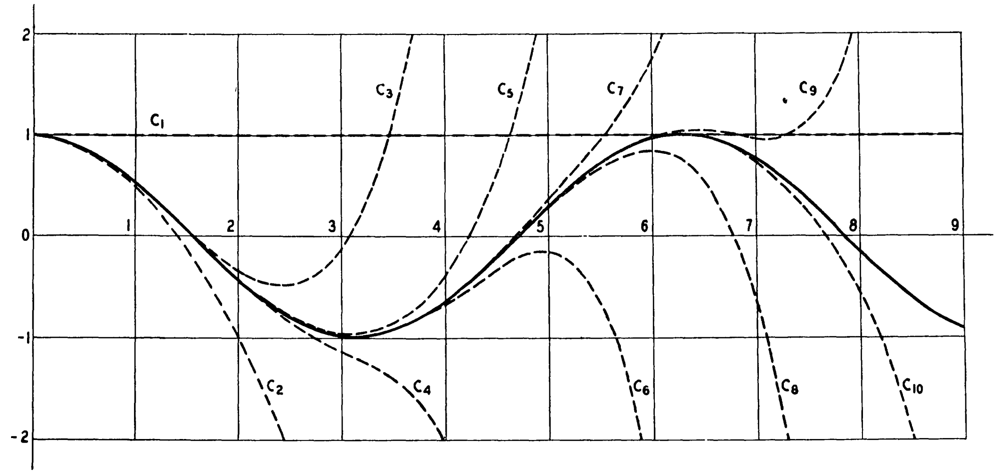
By HELEN M. KAMMERER, Bell Telephone Laboratories, New York

The curves in the accompanying figures show the first ten approximations to $y=\sin x$ and to $y=\cos x$, using the partial sums of the ordinary power series expansions. These interesting curves are not to be found in any textbook with which I am familiar.

The rapidity of approach of the partial sums to the first half period of the sine function is illustrated by the values of the successive approximations at $x=\pi$, the first positive real root of $y=\sin x$. The values $S_6(\pi)$ and $S_8(\pi)$ differ from zero by less than one part in a thousand and one part in a million, respectively. Similarly, at $x=\pi/2$, the cosine approximations $C_4(\pi/2)$ and $C_6(\pi/2)$ differ from zero by less than one part in a thousand and one part in a million, respectively.



$$S_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^{2k-1}}{(2k-1)!}, \quad S_n(x) \to \sin x \text{ as } n \to \infty$$



$$C_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^{2k-2}}{(2k-2)!}, \quad C_n(x) \to \cos x \text{ as } n \to \infty$$

On Hildebrandt's Example of a Function without a Finite Derivative

By E. G. Begle and W. L. Ayres, University of Michigan

T. H. Hildebrandt* has recently given a simple example of a continuous function which has a finite derivative at no point. At the end of this example the Editor inserted a note asking for what values, if any, the function has a definite infinite derivative. We propose to answer this question here in so far as it seems possible to give a solution.

Hildebrandt's function $\phi(x)$ was defined as follows: Let $f_1(x)$ be periodic of period 1 such that†

$$\begin{aligned} f_1(x) &= x \quad &\text{for } 0 \leqq x \leqq \tfrac{1}{2} \\ &= 1 - x \quad &\text{for } \tfrac{1}{2} \leqq x \leqq 1, \end{aligned}$$

and let

$$f_n(x) = f_1(2^{n-1}x)/2^{n-1} \quad \text{and} \quad S_n(x) = \sum_{1}^{n} f_i(x).$$

Then

$$\phi(x) = \lim_{n\to\infty} S_n(x).$$

The function $\phi(x)$ is so defined that for a point $x = k/2^n$ (k an integer),

* This Monthly, vol. 40 (1933), p. 547.

† For convenience in notation we start with $f_1(x)$; that is, our $f_n(x)$ is Hildebrandt's $f_{n-1}(x)$.

$\phi(x)=S_n(x)$, i.e. for this point $f_{n+i}(x)=0$ for every $i>0$. Furthermore, if $k/2^n<x<(2k+1)/2^{n+1}$, then the slope of $S_{n+1}(x)$ is equal to the slope of $S_n(x)$ plus 1, and equal to the slope of $S_n(x)$ minus 1 if $(2k+1)/2^{n+1}<x<(k+1)/2^n$. From this it follows that the right-hand derivative of $\phi(x)$ is $+\infty$ and the left-hand derivative is $-\infty$ at each point of the form $m/2^n$. Then $\phi(x)$ has no derivative at such a point $x=m/2^n$.

Now let x_0 be a point not of the form $m/2^n$. Then for each value of n we have $s/2^n<x_0<(s+1)/2^n$. To investigate $\phi'(x_0)$ we make use of the lemma

$$\text{(A)} \qquad \lim_{h,k\to 0} \frac{\phi(x_0+h)-\phi(x_0-k)}{h+k} = \phi'(x_0) \qquad h,\, k \geqq 0$$

and use the special set of values $x_0+h=(s+1)/2^n$, $x_0-k=s/2^n$. If the general limit (A) exists, then the limit using this special set of values must exist and have the same value. We shall show now that if the limit using this special set of values exists for the Hildebrandt function, then the limit in (A) exists and has the same value. As we remarked above, for these special points $\phi(x)=S_n(x)$ and this ratio in (A) is the slope of $S_n(x)$ between the two points. Then as the slope of $S_{n+1}(x)$ differs from the slope of $S_n(x)$ by ± 1, if the limit exists it can be $+\infty$ or $-\infty$ only. Let us consider the $+\infty$ case only, the other case being entirely similar. We will show that the general limit (A) is $+\infty$ by proving that $\phi'(x_0)=+\infty$. We consider Δx positive, the negative case being entirely similar. We must prove that given any g (no matter how large) there exists a $\delta>0$ such that

$$\frac{\phi(x_0+\Delta x)-\phi(x_0)}{\Delta x} > g$$

whenever $\Delta x<\delta$. Since the limit (A) for the special set of values is $+\infty$ there exists an integer n^* such that for $n>n^*$ the slope of the interval of $S_n(x)$ containing x_0 is greater than $g+10/3$. Let δ be the smaller of $x_0-s/2^{n^*}$ and $(s+1)/2^{n^*}-x_0$, where s is the integer that makes both the differences positive. There exists an integer n such that $r/2^n<x_0<(r+1)/2^n<(x_0+\Delta x)<(r+2)/2^n$ and either x_0 lies in the first half of its interval, or $x_0+\Delta x$ lies in the last half of its interval. Then since $\Delta x<\delta$, we have $n>n^*$. Let $a=r/2^n$, $b=1/2^n$. Then $x_0-a<b$ and $\Delta x>b/2$. As $n>n^*$, the slope of $S_n(x)$ between a and $a+b$ is $>g+10/3$ and between $a+b$ and $a+2b$ is $>g+4/3$, since the difference of the slopes of $S_n(x)$ on adjacent intervals $\leqq 2$. Also $\phi(x)-S_n(x)<2b/3$ for every x. Then $\phi(x_0)<\phi(a)+(g+10/3)(x_0-a)+2b/3$ and $\phi(x_0+\Delta x)>\phi(a)+(g+10/3)b+(g+4/3)(x_0+\Delta x-a-b)$. Then

$$\frac{\phi(x_0+\Delta x)-\phi(x_0)}{\Delta x} > g+4/3-\frac{(2x_0-2a-4b/3)}{\Delta x} > g,$$

since $x_0-a<b$ and $\Delta x>b/2$ makes the last term $<4/3$.

Now let x_0 be expressed in the dyadic system and let I_n and O_n represent the number of 1's and 0's in the first n terms of the expression for x_0, and let D_n

$= O_n - I_n$. Then from the remarks of the second paragraph, we see that the slope of $S_n(x)$ at x_0 is D_n. Now since the general limit (A) is given by the special set of values of h and k, we have

$$\phi'(x_0) = \lim_{n\to\infty} \frac{\phi((s+1)/2^n) - \phi(s/2^n)}{1/2^n} = \lim_{n\to\infty} D_n.$$

Thus if $\lim D_n = +\infty$, then $\phi'(x_0) = +\infty$; if it equals $-\infty$, then $\phi'(x_0) = -\infty$; and if the limit does not exist, then $\phi'(x_0)$ does not exist.

Now as the dyadic representation of each x_0 is defined, we may determine $\lim D_n$. Further, whenever x_0 is a rational number not of the form $k/2^n$, then its dyadic representation is a repeating fraction. In this case it is merely necessary to count the 1's and 0's for one period to compute $\lim D_n$. If in a complete period the number of ones exceeds the number of zeros then $\phi'(x_0) = +\infty$; if the number of zeros exceeds the number of ones, $\phi'(x_0) = -\infty$; and if the two are equal, $\phi'(x_0)$ does not exist.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Elements of Projective Geometry. By E. E. Watson and Margaret M. Watson. New York, D. C. Heath and Company, 1935. viii+236 pages. $2.60.

Teachers of projective geometry will look to this little book, hopeful that it may prove to be the badly needed elementary text in synthetic projective geometry, one of the most attractive courses in the undergraduate curriculum. The book contains numerous well-chosen exercises, and may well serve as a useful adjunct to the course in question; but as a textbook it will prove disappointing.

Since the authors plan to make the book accessible to sophomores, they wisely make no attempt to build on a postulational basis, but lay their foundations frankly on metric geometry and trigonometry. After the usual preliminaries, introducing the concepts of projection and section, duality, and ideal elements at infinity, the theorem of Desargues is made the basis of a development of harmonic relations. Then cross-ratio is defined metrically, and proved projectively invariant. We then have the standard treatment of perspective and projective forms, leading to the projective generation of conics, the theorems of Pascal and Brianchon, poles and polars, foci, and the usual theorems dependent on these topics. Surprisingly, this normal development of projective geometry is interlarded with extensive passages of metric geometry; the two subjects are intermingled, but related to each other hardly at all. For instance, Chapter X

is devoted to the geometry of the triangle; it is excellently done, but has no relevance to any other part of the book. In fact, this chapter contains no single reference to any other chapter, and is itself nowhere else referred to. As an illustration, Desargues's theorem on perspective triangles, which is of course the basis of the projective geometry, is again proved late in Chapter X by means of the theorem of Ceva. One seems to detect here the dual authorship of the book.

One is reluctantly forced to conclude that clarity and accuracy of thought do not characterize the portion of the book dealing with projective geometry. There is much confusion in definitions and in the order of theorems, especially in the earlier and more fundamental parts. We get an idea where we stand as early as page 6, where we read "if x_1 approaches 0 as a limit, $1/x_1$ approaches ∞ as a limit." It is hardly profitable to trace the confusion and repetition in the early chapters; but it may be noted that the proofs of three successive theorems (28, 29, 30), including "Von Staudt's Fundamental Theorem," are fallacious. Unproved statements are repeatedly used in the proofs of essential theorems. In a later chapter the conic is defined as the locus determined by intersection of projective (but not perspective) pencils. Nothing whatever connects these loci with the ellipse and other curves previously known to the student; but presently it is naïvely assumed that they are identical with the latter. One of the joys of projective geometry is that of classifying conics according to their intersections with the line at infinity, and from projective theorems deriving metric properties of ellipse, parabola, and hyperbola. This classification is missing entirely. Presently, however, we find: "in the study of the conic there are certain limiting conditions that are important. Thus . . . . a pair of straight lines is the limiting condition [sic] of the hyperbola when the branches of the hyperbola coincide with the asymptotes." On this basis the authors deduce the theorem of Pappus as a "limiting condition" of that of Pascal, regarding a line pair as a "degenerated" [sic] conic. Comment is superfluous.

The chapter on involutions is characterized by four separate definitions of involution in a line or pencil, very tenuously related to one another, and all more or less subject to criticism. The natural definition, as a projectivity wherein one pair correspond mutually, appears only by implication from one of the theorems.

In using the phrase "infinitely distant points," the authors are only following the tradition of earlier texts. The reviewer here and now proposes that the term be relegated to an infinitely distant limbo. Let us recognize that a point at infinity is an ideal element, with regard to which the concept of distance is entirely without meaning. To speak of the distance from a finite point to such a point as infinite is not only misleading but is logically unsound.

The authors have tried hard to produce a text which shall bring the subject easily within the reach of undergraduates. With this objective, the strictest rigor is not to be desired; but pedagogical expediency should not be an excuse for erratic and illogical development of a subject one of whose chief pleasures is its logical coherence and self-sufficiency.

R. A. JOHNSON

Financial Mathematics. By A. W. Richeson. New York, Prentice-Hall, Inc., 1935. xiv+361 pages. $2.50.

This text is intended for students majoring in business administration and commerce. It is "not to prepare actuaries," as one can readily see; and the reviewer would be better pleased to see included a treatment of amortized and partly amortized mortgages, and serial bonds. But we find only the standard elementary material of such a course, presented with all the formulas required for the topics treated. Many a student is deterred by complicated formulas, and would prefer a direct analysis and a solution by simple and easy logical steps. Thus he is better prepared to treat a strange type of problem. Many practical, but more complicated problems, might have been treated in a way quite within the grasp of students with only the amount of preparation demanded by the author.

Professor Maximilian Philip has shown how this can be done in Part II of his text published, in 1932, by the same company. Someone in reviewing the latter text said of it, "So clear it reads like a novel." The present reviewer wonders why the publishers offer three years later a text with far less claim to originality in material, treatment or procedures presented.

The text has a good number of exercises, many of which have novelty and interest beyond the formal type.

By permission, five of the tables of *Tables of Applied Mathematics* by Professor James W. Glover are included for values of n up to 100, which makes the text more attractive as a reference book for practical work in finance.

C. C. Grove

MATHEMATICS CLUBS

Edited by F. W. Owens and Helen B. Owens, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 463 East Foster Ave., State College, Pa.

HISTORY OF MATHEMATICS CLUBS

Attention of all mathematics clubs is again called to the effort being made to compile a list of all such organizations at present active in the United States and Canada. The dead line for entry in this year's record is June 1, 1936.

In April a letter to Mathematics Clubs was mailed from this office to all organizations listed in our files. If that letter has failed to reach your club please communicate with this department at once. If it has been received and not answered the reply is much needed.

CLUB PAPERS FOR PUBLICATION

In February the suggestion was made that each club submit its best paper for the year 1935–6 for consideration by this department. Such papers will be

(*Greek Chorus enters singing or chanting*)

CHORUS: Oh, Evolu! Oh, Evolu!
He had naught at all to fear from you.
He had no monkeys perching in the family tree.
We see 'tis true.
As great as was his wrong, so great his joy shall be.
His eye is on the future, a challenge doth he see.
To make himself more worthy ever of his ancestry
Is his undying, fervent resolution.

(*Curtain.*)

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

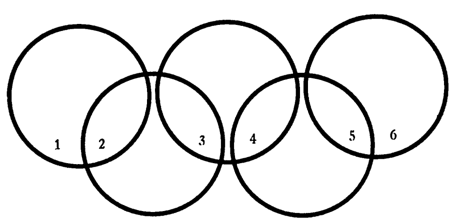
E 211. *Proposed by V. Thébault, Le Mans, France.*

One liter of wine is drawn from a full cask, and replaced by water. Then one liter of the mixture is drawn and replaced by water. This is repeated until thirty-five liters have been drawn off and replaced, when an analysis determines that the cask contains equal parts of water and wine. What is the capacity of the cask?

E 212. *Proposed by C. W. Trigg, Cumnock College, Los Angeles.*

Any point P on the minor circle of an ellipse is connected to the foci of the ellipse. Show that the sum of the areas of the circles constructed on these connectors as diameters, is equal to one-half the area of the major circle of the ellipse.

E 213. *Proposed by Franz Denk, Erlangen, Germany.*



Redistribute the six digits in the same areas in the accompanying diagram

so that after the rearrangement: (a) no two digits now in one circle will be in one circle; (b) no three digits now in two linked circles will be in two linked circles; (c) no four digits now in three linked circles will be in three linked circles; and finally, (d) no five digits now in four linked circles will be in four linked circles.

E 214. *Proposed by D. L. MacKay, Evander Childs H. S., New York City.*

Two non-congruent, similar triangles have two sides of one respectively equal to two sides of the other. Between what limits must the ratio of similitude lie?

E 215. *Proposed by H. T. R. Aude, Colgate College.*

Show that for positive integers written in the scale of five, the sum of any odd number and the subsequent number gives a number whose digit-sum is three, or a number whose digit-sum is a number whose digit-sum is three, or . . . etc.

E 216. *Proposed by J. Rosenbaum, Hartford Federal College.*

Find

$$\lim_{x\to\infty}\left[x\sin\frac{1}{x}+\frac{1}{x}\right]^{x}.$$

E 217. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

After an exact long division had been completed, one particular digit was replaced by the letter c wherever it occurred. Then all the other digits indiscriminately were each replaced by the letter x. Here is the result:

$$\begin{array}{ccccccccccccccc}
x & c & x &) & x & x & x & x & c & x & x & (& x & x & c\ x \\
 & & & & x & x & c & c & & & & & & & \\ \cline{5-8}
 & & & & & x & x & x & c & & & & & & \\
 & & & & & & x & x & c & & & & & & \\ \cline{6-9}
 & & & & & & x & x & x & x & & & & & \\
 & & & & & & x & c & x & x & & & & & \\ \cline{7-10}
 & & & & & & & x & x & x & x & & & & \\
 & & & & & & & x & x & x & x & & & & \\ \cline{8-11}
\end{array}$$

Reconstruct all the original figures and show that the solution is unique.

SOLUTIONS

E 180 [1935, 566]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Find a number written in the scale of seven, with less than forty digits, such that if the digit "3" is moved from its extreme right to its extreme left, the result (still in the scale of seven) is four-fifths of the original number.

Solution by W. E. Buker, Leetsdale High School, Pennsylvania.

All numbers in this solution are written in the scale of seven.

If the original number is x and the changed number is y, then $4x = 5y$. We then construct a multiplication table of products of 4 and 5 in the scale of seven. We then proceed to find the digits of x, $4x$, $5y$ and y, one digit at a time in that order, working from right to left. When we come to a 3 in y, we note whether by stopping there all the conditions of the problem will be satisfied. The smallest solution is $x = 4364604155323020625113$, $4x = 24155323020625113436455 = 5y$, and $y = 3436460415532302062511$.

Since x contains 22 digits, and the next smallest solution would contain 44, which exceeds 40, the solution is unique.

E. P. Starke, in his solution, comments on its similarity to E 63, which was solved by H. T. R. Aude (1934, 268), and refers to the paper of Solomon Guttman in this MONTHLY (1934, pp. 159–166) as a source of further information on such problems.

Also solved by H. T. R. Aude, J. E. Burnam, Mary L. Constable, O. E. Eggert, Walter Penney, C. W. Trigg, Simon Vatriquant, G. W. Wishard and the proposer.

E 181 [1935, 621]. *Proposed by A. Gloden, Luxembourg.*

If the feet of the altitudes of the triangle ABC are the vertices of a second triangle PQR, such that the perimeter of PQR equals the diameter of the circumcircle of ABC, prove that

$$\text{(a)} \qquad \sin A \sin B \sin C = \tfrac{1}{2}$$

and that

$$\text{(b)} \qquad \sin^3 A \cos (B - C) + \sin^3 B \cos (C - A) + \sin^3 C \cos (A - B) = \tfrac{3}{2}.$$

Solution by J. W. Clawson, Ursinus College

It is well known and easily proved that lines joining the feet of the altitudes of a triangle are antiparallel to the corresponding sides. Hence, applying the law of sines to the appropriate triangles, we have $PQ = PC \sin C/\sin B = b \cos C \sin C/\sin B = 2R \sin C \cos C$. Hence the perimeter of PQR is $2R(\sin A \cos A + \sin B \cos B + \sin C \cos C) = 2R$ by hypothesis. Consequently $\sin A \cos A + \sin B \cos B + \sin (A+B) \cos (A+B) = 1$. This easily reduces to $\sin A \sin B \sin C = \frac{1}{2}$, proving (a).

The expression on the left side of (b) is easily reduced, using the fact that $C = \pi - (A+B)$, by means of the addition formulas and of the fact that $\sin^2 x + \cos^2 x = 1$, to the form, $3 \sin A \sin B \sin C$, which by (a) is equal to $3/2$.

Also solved by M. W. Aylor, O. E. Eggert, Hansraj Gupta, Cornelia Strong, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 182 [1935, 621]. *Proposed by D. H. Lehmer, Bethlehem, Pa.*

Show that the infinite product

$$\left(1-\frac{i}{3}\right)^4\left(1-\frac{i}{17}\right)^4\left(1-\frac{i}{99}\right)^4\left(1-\frac{i}{577}\right)^4\cdots$$

in which the successive denominators satisfy the recurrence $D_n=6D_{n-1}-D_{n-2}$, is purely imaginary.

Solution by E. P. Starke, Rutgers University

Define x_n by $x_n=(D_{n+1}-3D_n)/4$. So $x_1=2$, $x_2=12$, etc. Since D_n is easily shown to be of the form $4q+(-1)^n$, x_n is always an integer. Then follow directly $D_n+2x_n=2x_{n+1}-D_{n+1}$ and $D_n=(x_{n+1}-3x_n)/2$. Note that

$$(1)\qquad D_n^2-2x_n^2=1$$

holds true for $n=1$ and for $n=2$. But $D_{n+1}^2-2x_{n+1}^2=(4x_n+3D_n)^2-2(3x_n+2D_n)^2=D_n^2-2x_n^2$. Hence (1) holds for every value of n. (The D_n and x_n form a complete set of integral solutions of (1); cf. solution of problem 3677 [1935, 573].)

The following are easily computed directly:

$$(2)\qquad 2\arctan(1/D_1)=\arctan 2D_1/(D_1^2-1)$$
$$=\arctan(D_1+2x_1-1)/(D_1+2x_1+1),$$

$$(3)\qquad 2\arctan(1/D_n)=\arctan 2D_n/(D_n^2-1)=\arctan(D_n/x_n^2),$$

$$\arctan(D_n+2x_n-1)/(D_n+2x_n+1)$$
$$-\arctan(D_{n-1}+2x_{n-1}-1)/(D_{n-1}+2x_{n-1}+1)$$
$$(4)\qquad =\arctan(D_n+2x_n-1)/(D_n+2x_n+1)$$
$$-\arctan(2x_n-D_n-1)/(2x_n-D_n+1)$$
$$=\arctan 4D_n/(8x_n^2+2-2D_n^2)=\arctan(D_n/x_n^2).$$

Here (2), (3) and (4) constitute a proof by induction that

$$2\sum_{j=1}^{n}\arctan(1/D_j)=\arctan(D_n+2x_n-1)/(D_n+2x_n+1).$$

Then

$$2\sum_{j=1}^{\infty}\arctan(1/D_j)=\lim_{n\to\infty}\arctan(D_n+2x_n-1)/(D_n+2x_n+1)$$
$$=\arctan 1=\pi/4.$$

The amplitude of the given infinite product is $-4\sum_{j=1}^{\infty}\arctan(1/D_j)$ which, according to the above, equals $-\pi/2$. Thus the infinite product is purely imaginary.

The convergence of the infinite product may be shown thus: the modulus of $(1-i/D_n)^4$ is $(1+1/D_n^2)^2$. But $D_n=4x_{n-1}+3D_{n-1}$. So $3D^2_{n-1}<D_n^2$, and therefore $(1+1/D_n^2)^2=1+2/D_n^2+1/D_n^4<1+3/D_n^2<1+1/D_{n-1}^2$. Repetition of this

argument will give $1+1/D_n^2<(1+1/D_1^2)^{2^{1-n}}$. Hence the modulus of the infinite product is less than

$$(1+1/D_1^2)^2(1+1/D_1^2)(1+1/D_1^2)^{1/2}\cdots = (1+1/D_1^2)^{2+1+1/2+\cdots}$$
$$= (1+1/D_1^2)^4 = (10/9)^4.$$

In place of the first part of the above we might as easily have shown that $x_{2n}=2x_nD_n$ and $D_{2n}=D_n^2+2x_n^2$, so that as a consequence $\sum_{j=1}^{2n}\arctan(1/D_j)$ $=\arctan x_n/(x_n+D_n)$. But $x_n/(x_n+D_n)$ is the $2n$th convergent of the infinite continued fraction

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots = \sqrt{2}-1 = \tan(\pi/8).$$

E 183 [1935, 621, *corrected* 1936, 46]. *Proposed by V. Thébault, Le Mans, France.*

In triangle PQR, on PQ lay off $PA=PR$, on PR lay off $PB=PQ$, on QR lay off $QC=QP$, on QP lay off $QD=QR$, on RP lay off $RE=RQ$, and on RQ lay off $RF=RP$. Now if the areas of the triangles PQR, PDE, QFA and RBC, with proper signs, are denoted by k, l, m and n respectively, show that $1/k=1/l$ $+1/m+1/n$.

Solution by M. J. Turner, Ball State Teachers College

Denote the interior angles of triangle PQR by P, Q and R. Likewise, let p, q and r represent the sides opposite the angles in the order named.

Angle $DPE=P$ or its supplement, so

$$2l = PE\cdot PD\sin P = (p-q)(p-r)\sin P.$$

Likewise

$$2m = QA\cdot QF\sin Q = (q-r)(q-p)\sin Q$$
$$2n = RB\cdot RC\sin R = (r-q)(r-p)\sin R.$$

Moreover

$$2k = qr\sin P = pr\sin Q = pq\sin R,$$

so that

$$\sin P = 2k/qr,\quad \sin Q = 2k/pr,\quad\text{and}\quad \sin R = 2k/pq.$$

Hence

$$1/2l+1/2m+1/2n$$
$$= [1/2k]\left[\frac{qr}{(p-q)(p-r)}+\frac{pr}{(q-r)(q-p)}+\frac{pq}{(r-p)(r-q)}\right].$$

Which reduces readily to $1/l+1/m+1/n=1/k$.

Also solved by W. E. Buker, W. B. Clarke, J. W. Clawson, W. R. Hardman, C. A. Murray, E. P. Starke, C. W. Trigg, Simon Vatriquant, G. A. Williams and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTIONS

3783. *Proposed by J. Barinaga, Madrid University.*

Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix} = 1!2!3!\cdots n!\left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n}\right].$$

3784. *Proposed by J. M. Feld, New York City.*

Prove that for any positive integer k the following determinant is zero:

$$\begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(2k+1)!} & \frac{1}{(2k+2)!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(2k)!} & \frac{1}{(2k+1)!} \\ 0 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(2k-1)!} & \frac{1}{(2k)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2!} \end{vmatrix}$$

3785. *Proposed by V. Thébault, Le Mans, France.*

The six spheres of similitude of four spheres with centers at the vertices of a tetrahedron and with radii proportional to given numbers pass through two fixed points on a diameter of the circumsphere of the tetrahedron.

3786. *Proposed by V. Thébault, Le Mans, France.*

With the vertices of a tetrahedron $ABCD$ as centers spheres are described such that the square of the radius of the sphere for a given vertex is equal to half the sum of the squares of the lengths of the edges through that vertex. (a) The sphere (w), orthogonal to the four spheres, has for its center the Monge point Ω of the tetrahedron. (b) Express the radius of the sphere (w) as a function of the radius of the circumsphere (o) and the lengths of the edges. (c) Show that the polar plane of the centroid G of the tetrahedron, with respect to the sphere (w), is the transform of the radical plane of the spheres (o) and (w) in the transformation of similitude $(G, 3)$.

3787. *Proposed by V. Thébault, Le Mans, France.*

In a polyhedron with n faces inscribe another polyhedron having a vertex in each face of the first such that the sum of the squares of the distances between pairs of its vertices, the pairs being taken in all possible manners, is a minimum.

SOLUTIONS

3703 [1934, 522]. *Proposed by V. Thébault, Le Mans, France.*

Prove that the integral part of the fourth root of the product of eight consecutive integers is equal to x^2+7x+6, where x is the smallest of the eight integers. This result may be used to show that the product of eight consecutive integers is never the fourth power of an integer.

Solution by the Mathematics Club of the New Jersey College for Women, New Brunswick, N. J.

The given product may be written

$$\begin{aligned} P &= x(x+7)(x+1)(x+6)(x+2)(x+5)(x+3)(x+4), \\ &= (x^2+7x+6-6)(x^2+7x+6)(x^2+7x+6+4)(x^2+7x+6+6,) \\ &= a^4+4a^3-36a^2-144a = a^4+4a(a+3)(a-12). \end{aligned}$$

where $a=x^2+7x+6$. Hence $a^4<P$, since $a>12$ for $x\geqq 1$. Also

$$(a+1)^4 - P = 42a^2+148a+1>0,$$

and we have finally

$$a^4 < P < (a+1)^4,$$

and the theorem is proved.

Solved also by W. H. Gage, Raymond Garver, Hansraj Gupta, S. B. Littauer, F. L. Manning, R. E. Moritz, A. Ollivier, A. Pelletier, W. T. Short, E. P. Starke, C. W. Trigg, F. Underwood, B. C. Zimmerman, and the proposer.

Editorial Note. Most of the solutions were similar to the above. Manning used twice the process for extracting the square root of a polynomial. Gage set $a=x^2+7x$, then $P=a(a+6)(a+10)(a+12)$. Since the geometric mean of unequal positive quantities is less than the arithmetic mean, he derived the inequality

$$P < \left(\frac{4a+28}{4}\right)^4 = (a+7)^4.$$

One solution noted that the reasoning, similar to that of the solution above, holds if $x \leqq -8$. If $x=-1$ or -6, both P and x^2+7x+6 are zero. For $x=0$, -2, -3, -4, -5, -7 the proposition is not true, since for these values $P=0$ while x^2+7x+6 is not zero. But this does not extend essentially the validity of the theorem; for when x runs through the decreasing negative integral values of $-\infty < x \leqq -8$, P and x^2+7x+6 run through the same corresponding pairs of values as when x runs through the increasing values in $1 \leqq x < \infty$. Starke considered a generalization of the product of n consecutive integers, but encountered a difficulty in setting a lower limit for x except in certain special cases.

3704 [1934, 522]. *Proposed by V. Thébault, Le Mans, France.*

Show that in any tetrahedron the sum of the squares of the distances of the center of the hyperboloid of altitudes (Monge's point) to the vertices is equal to the square of the diameter of the circumscribing sphere.

Solution by R. Goormaghtigh, Bruges, Belgium

It is well known that, when the sum of the squares of the distances of a variable point P to n fixed points in space has a given value, the locus of P is a sphere having its center at the center of gravity of n equal masses placed at the n given points. Hence when the line segment between two points is bisected by the centroid of a tetrahedron, the sums of their distances to the vertices have the same value.

But the center H of the hyperboloid of altitudes is the image of the center of the circumscribing sphere through the centroid; therefore the sum of the squares of the distances of H to the vertices equals four times the square of the radius of the circumscribing sphere.

Solved also by J. E. LaFon, Roy MacKay, and the proposer.

Editorial Note. The solution by the proposer states that the theorem of the problem is a generalization of a similar theorem for an orthocentric tetrahedron by N. A. Court in Mathesis, 1928, p. 342. The solution by LaFon referred to certain theorems on the Monge point in the article by N. A. Court *On the isodynamic points of four spheres*, in this MONTHLY, vol. 39 (1932), p. 196. The final theorem in the solution regarding the Monge point H was proved in the solution by Goormaghtigh of problem 3697 [1936, 191], with a reference to Rouché et de Comberousse, *Traité de Géométrie*, 7th. ed. vol. 2, p. 664. The

elementary theorem in the first part of the above solution is obviously true by the use of vectors, and it may be more familiar in the following wording: The moment of inertia of a system with respect to a point P is equal to the moment of inertia of the same system with respect to the center of gravity G of the system increased by the product of the total mass and the square of the distance between the two points PG^2.

3713 [1934, 634]. *Proposed by R. E. Gaines, University of Richmond.*

Determine the position of a normal chord of a conic which forms a segment of minimum area. Find the area of such a segment of an ellipse.

I. *Solution by E. P. Starke, Rutgers University*

The segment cut from a conic by a normal chord has a minimum area when the chord intersects the axis of the conic at an angle of $\pm\pi/4$.

I. The area of the segment cut from the parabola $y^2=ax$ by the chord PQ is given by $(q-p)^3/6a$, where p and q are the ordinates of P and Q respectively, with $p<q$ (see solution of E128 [1935, 323]).

If PQ is normal to the parabola at Q, we must have $p=-q-a^2/2q$, so that the area of the segment is given by

$$\frac{1}{6a}\left[2q+\frac{a^2}{2q}\right]^3, \qquad q>0,\ p<0.$$

Upon differentiating, we find the condition for minimum area to be $q=a/2$, $p=-3a/2$. The slope of the normal is then -1, and Q is an end of the latus rectum. The other end would result if we set $p>q$. The area of the resulting minimum segment is $4a^2/3$.

II. Let θ_1 and θ_2, $0<(\theta_1-\theta_2)<2\pi$, be the values of the parameter corresponding to two points on the ellipse $x=a\cos\theta$, $y=b\sin\theta$. It is easily shown by the usual methods of the calculus that the area of a segment cut off by the chord joining the two points is given by

$$A = ab[(\theta_1-\theta_2)-\sin(\theta_1-\theta_2)]/2. \tag{1}$$

For this area to be a minimum we must have $1-d\theta_2/d\theta_1=0$. The chord will be normal to the ellipse at the point θ_1 if

$$\frac{a}{b}\tan\theta_1=\frac{b(\sin\theta_2-\sin\theta_1)}{a(\cos\theta_2-\cos\theta_1)}=\frac{-b}{a}\cot\frac{\theta_1+\theta_2}{2} \qquad \text{or} \tag{2}$$

$$\theta_2 = 2\operatorname{arccot}\left[-a^2(\tan\theta_1)/b^2\right]-\theta_1.$$

From the combination of these conditions we have

$$1 = d\theta_2/d\theta_1 = (2a^2b^2\sec^2\theta_1)/(a^4\tan^2\theta_1+b^4)-1, \tag{3}$$

which reduces to $\tan\theta_1=\pm b/a$.

The chord is then a normal of slope ± 1. If we take $0<\theta_1<\pi/2$, (2) becomes $\theta_2=-3\theta_1$, and the area of the segment is

$$(4) \qquad A = ab(4\theta_1 - \sin 4\theta_1)/2 = 2ab\left\{\arctan\frac{b}{a} - \frac{ab(a^2-b^2)}{(a^2+b^2)^2}\right\}.$$

III. Let θ_1 and θ_2, $0<(\theta_1-\theta_2)$, be the values of the parameter corresponding to two points of the same branch of the hyperbola $x=a\cosh\theta$, $y=b\sinh\theta$, $b<a$. The above argument for the ellipse applies here practically unchanged except for the substitution of hyperbolic functions. The computations (numbered to correspond to the steps of II) result as follows.

$$(1) \qquad A = ab[\sinh(\theta_1-\theta_2) - (\theta_1-\theta_2)]/2.$$

$$(2) \qquad -\frac{a}{b}\tanh\theta_1 = \frac{b(\sinh\theta_2 - \sinh\theta_1)}{a(\cosh\theta_2-\cosh\theta_1)} = \frac{b}{a}\,\mathrm{ctnh}\,\frac{\theta_1+\theta_2}{2} \quad \text{or}$$

$$\theta_2 = 2\,\mathrm{arcctnh}\,[-a^2(\tanh\theta_1)/b^2] - \theta_1.$$

$$(3) \qquad 1 = d\theta_2/d\theta_1 = (2a^2b^2\,\mathrm{sech}^2\,\theta_1)/(a^4\tanh^2\theta_1 - b^4) - 1,$$

which reduces to $\tanh\theta_1=\pm b/a$.

The chord is then a normal of slope ± 1.

$$(4) \qquad A = ab(\sinh 4\theta_1 - 4\theta_1)/2 = 2ab\left[\frac{ab(a^2+b^2)}{(a^2-b^2)^2} - \mathrm{arctanh}\,\frac{b}{a}\right].$$

II. *Solution by F. Underwood, University College, Nottingham*

The ellipse with semi-axes a and b is the orthogonal projection of a circle of radius a whose plane passes through the major axis of the ellipse and makes an angle of $\cos^{-1}(b/a)$ with the plane of the ellipse. The normal to the ellipse at $(a\cos\theta, b\sin\theta)$ meets the x-axis at $(ae^2\cos\theta, 0)$, where e is the ellipse eccentricity and the origin of rectangular coordinates is at the center. The chord of the circle which projects into this normal passes through this latter point and $(a\cos\theta, a\sin\theta)$. The length p of the perpendicular from the center to the chord of the circle is

$$(1) \qquad p = \frac{ae^2\sin\theta\cos\theta}{[1-e^2(2-e^2)\cos^2\theta]^{1/2}},$$

and S, the area of the smaller segment of the circle, is

$$(2) \qquad S = a^2\cos^{-1}(p/a) - p\sqrt{a^2-p^2}.$$

Then

$$(3) \qquad \frac{dS}{dp} = -2\sqrt{a^2-p^2}, \qquad \frac{R^3}{ae^2}\frac{dp}{d\theta} = [1-e^2\cos^2\theta][(2-e^2)\cos^2\theta - 1],$$

$$R^2 = 1 - e^2(2-e^2)\cos^2\theta > 0.$$

The minimum for S is when $\cos^2\theta=(2-e^2)^{-1}$, $\sin^2\theta=(1-e^2)/(2-e^2)$. The multiplication of this minimum by b/a then gives the minimum segment of the ellipse

$$ab\{\cos^{-1}[e^2/(2-e^2)]-2e^2\sqrt{1-e^2}/(2-e^2)^2\}.$$

Solved also by the proposer.

Editorial Note. It is easy to see why solution I excludes hyperbolas for which $b\geqq a$. As the point P moves from the vertex V to infinity on the lower part of the right-hand branch, the slope m of the normal increases from zero and approaches a/b as a limit. If the normal for a finite position of P cuts the same branch, we must then have $a/b>m>b/a$, or $a^2>b^2$. Only for such hyperbolas does the problem have a meaning.

In the note to the solution of 3676 [1935, 523] is an elementary proof of the theorem: All segments of the same parabola having the same breadth, or the same altitude, have the same area, and conversely. When the altitude is the chord of the segment, the area of the segment and its altitude increase simultaneously, and this must be the case for any segment. In the notation of solution I the altitude h is given by

$$h=q-p=2q+\frac{a^2}{2q},\tag{1}$$

and the problem reduces to finding the minimum of h. The right side of (1) is the sum of two positive quantities whose geometric mean is a. An elementary theorem in regard to the arithmetic and geometric mean tells us that this sum, or h, is greater than or equal to $2a$; it is equal to $2a$ if and only if $2q=a^2/2q=a$. The minimum altitude is then $h=2a$. In order to find the corresponding minimum area, it suffices to take the segment with chord through (a, a) and $(a, -a)$. The area T of the triangle inscribed in this segment is a^2, and hence the minimum segment area S is given by $S=4a^2/3$, as shown on page 524, loc. cit.

The case for the ellipse may also be treated without the use of calculus, and to show this the method of solution II will be used. A straight line in the plane of the circle with the slope m projects into one with the slope bm/a. Let Q be a point on the circle in its first quadrant; let θ be the inclination to the common axis of the radius $OQ=a$; and let the tangent at Q cut this axis in T. Then QT has the slope $-\cot\theta$. If Q projects into P of the ellipse, then PT is its tangent at P and its slope must be $-b\cot\theta/a$. If the normal at P cuts the axis in N, it must be the projection of QN. Since the slope of PN is $a\tan\theta/b$, the slope of QN is $a^2\tan\theta/b^2$. Let γ be the inclination to the axis of QN so that $\tan\gamma=a^2\tan\theta/b^2$, and let the complete chord of the circle, of which QN is a part, subtend the angle 2ψ at the center. Then $\pi/2-\psi=\gamma-\theta$; and, after taking the tangent of each member of this equality and making a slight reduction with the equation above, we have

$$\frac{a^2-b^2}{a^2}\tan\psi = \tan\theta + \frac{b^2}{a^2}\cot\theta. \tag{2}$$

The smaller segment of the circle cut off by the chord along QN has a minimum area when ψ, or $\tan\psi$, is a minimum. The right member of (2) is the sum of two positive terms whose geometric mean is b/a. Hence the minimum of the left member occurs when and only when the two terms are equal, i.e., when $\tan\theta = b/a$ and then the slope of PN is unity. For the minimum ψ (2) becomes

$$\tan\psi = \frac{2ab}{a^2-b^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta, \qquad \psi = 2\theta. \tag{3}$$

The area of the segment of the circle is then easily computed, and this result multiplied by b/a gives the desired area of the ellipse segment.

The hyperbola is not handled so easily, but there is a complete analogy in the equations. If Q_1 and Q_2 are points on the equilateral hyperbola $x = a\cosh\theta$, $y = a\sinh\theta$, we obtain the area of the sector OQ_1Q_2 by the fiction that the arc Q_1Q_2 is the arc of a circle with the radii $OQ_1 = OQ_2 = a$ subtending the angle $\theta_2 - \theta_1 > 0$ at the center, since this area is actually $a^2(\theta_2-\theta_1)/2$. By the same fiction the area of the triangle OQ_1Q_2 is $\frac{1}{2}a^2\sinh(\theta_2-\theta_1)$, where we replace sin by sinh. Thus if we set $\theta_2-\theta_1 = 2\psi$, the area of the segment is

$$\frac{a^2}{2}[\sinh 2\psi - 2\psi]. \tag{4}$$

It is obvious that the area of the segment increases with ψ, since this is the case when Q_2 and Q_1 are symmetrically situated with respect to the axis, where $\theta_2 = \psi$, $\theta_1 = -\psi$.

If we regard the hyperbola for which $b < a$ as being the orthogonal projection of a rectangular hyperbola with the parameter θ, we begin as before with the slope $\tanh\theta$ of OQ. But here the slope of QT is the positive reciprocal of that of OQ, and then the same reasoning leads to the result that the slope of QN is $-a^2\tanh\theta/b^2$. At this point we are without the aid of the simple properties of the circle for finding the difference 2ψ of the parameters at the ends of the chord along QN; but with the aid of hyperbolic formulas we find that

$$\frac{a^2+b^2}{a^2}\tanh\psi = \tanh\theta + \frac{b^2}{a^2}\coth\theta, \qquad \theta > 0; \tag{5}$$

and the rest of the work is the same as before. If we solve the problem for the ellipse there is really no necessity for an independent solution for the hyperbola. For, suppose that we write out the analytical solution for the ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqq \theta \leqq \pi/2$; and then in this solution we replace θ by $i\theta$ and b by $-ib$. The result is that we shall be considering $x = a\cosh\theta$, $y = b\sinh\theta$, where θ is positive, and x and y are each real and positive. Moreover, there will be no essential change in the argument, our result will be the desired minimum, and the

condition for the minimum will be that the slope of the normal is -1. The minimum area for the hyperbola is then obtained by replacing b by $-ib$ in the expression for the minimum area for the ellipse. It will be observed that this transformation carries (2) into (5).

An interesting method of solution was indicated by the proposer in the following quotation from his letter; " . . . let the normal chord touch the evolute at M. Then, if M is the mid-point of the chord, the area of the segment will be increased if the chord is moved either way. So we get the mid-point of a normal chord at (x', y') (by use of a diameter); find the length of the half chord, and make this equal to the radius of curvature at (x', y'). For any conic the chord makes an angle of 45° with the axes."

We may restate the argument as follows: Let that part of the evolute be taken so that the chord makes a positive acute angle ϕ with the principal axis. For the extremes of the variation of ϕ the area of the segment is a maximum, finite or infinite. Hence there exists an absolute minimum. If the chord in any position touches the evolute at M, the point M divides the chord into two parts of lengths c_1 and c_2. Then, if S is the area of the segment, we have $\Delta S = (c_1^2 - c_2^2)\Delta\theta/2$, to terms of the first order. Hence

$$2\frac{dS}{d\phi} = c_1^2 - c_2^2 .$$

If M is the point of contact for the chord giving the absolute minimum, we must have $c_1 = c_2$, since all conditions of continuity are satisfied. We now show that there exists only one such point M. Let P be the point of the conic at which the chord is a normal, and P' its other end. The circle with PP' as a diameter is not in all cases the osculating circle; it cuts the conic in a third point Q. If a circle and a conic, which is not a circle, intersect in four real points, any pair of opposite chords of intersection together with a principal axis of the conic form an isosceles triangle with its base along the axis. This useful theorem has been used before in this MONTHLY; its analytical proof is quite easy. If then the point M of contact of the normal PP' with the evolute is its mid-point, the circle with the diameter PP' is the osculating circle and PQ must fall along the tangent at P. Hence the tangent and normal at P must form an isosceles triangle with the axis, the slope of the normal must be unity, and that of the tangent must be negative unity. There can be only one such point M for the branch of the evolute considered, since the slope of the normal is increasing. The rest of the solution follows easily.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The Twentieth Summer Meeting of the Mathematical Association of America will be held by invitation at Harvard University on Monday, August 31, in conjunction with the summer meeting and colloquium of the American Mathematical Society which follow during the week. The meeting will be held during the first week of the Tercentenary Celebration of Harvard University. The Association will hold two sessions on Monday for the reading of papers by invited speakers, including Professors G. A. Bliss of the University of Chicago and Edward Kasner of Columbia University. On Monday evening there will be a popular address by Professor G. H. Hardy on the Indian mathematician Ramanujan under the joint auspices of the two organizations.

Among the distinguished scholars whom Harvard University is inviting as lecturers are Professors Rudolf Carnap of the Deutsche Universität, Prague, E. J. Cartan of the University of Paris, L. E. Dickson of the University of Chicago, R. A. Fisher of the University of London, G. H. Hardy of Cambridge University, and Tullio Levi-Civita of the University of Rome. Through the courtesy of Harvard University, their lectures will be delivered before the Society. These lectures are open only to members of the Society and the Association who have registered and obtained badges and to those others especially invited by the Tercentenary Conference and furnished tickets by Harvard University. The Colloquium Lectures, entitled "Topics in general analysis," are to be delivered by Professor E. W. Chittenden of the State University of Iowa. By the invitation of the Committee on Program of the Society, Professor G. C. Evans will give an address entitled "Methods of modern analysis in potential theory." Public lectures by prominent scientists have been arranged by the Tercentenary Committee for the evenings of this week.

Accommodations for a considerable number of the members of the Association and Society and their families and guests will be furnished in the dormitories of Harvard University, the University generously offering free of charge rooms for 300 persons. Selection from those applying will be based on seniority of membership in the Society and the Mathematical Association. Meals will be served at Harvard Union at rates totalling $1.75 per day. About July 1, program and reservation cards will be sent to all members of the Association.

Mr. G. Waldo Dunnington, who is spending the year at the University of Göttingen in order to complete a biography of Gauss, calls attention to the fact that copies of an exquisite reproduction of the Brunswick monument of Gauss are available at the price of 30 marks. The statuettes are 25 inches high and are of plaster of Paris. Orders may be sent directly to Mr. Dunnington, through

whom arrangements for shipping can be made. He can be addressed in care of the Sternwarte.

According to a statement recently issued by Colorado College, this college has become, during recent years, increasingly a center at which visiting research workers have been carrying on their investigations during the summer months. To encourage this movement, the college is prepared to offer during the summer free room accommodations to qualified investigators. Further information may be obtained from the members of the various departments or from the president of the college.

Professor J. W. Alexander, of the Institute for Advanced Study, has been appointed Rouse Ball lecturer in mathematics at the University of Cambridge for the present academic year.

At Harvard University, Professor G. D. Birkhoff has been appointed dean of the Faculty of Arts and Sciences, for three years from September 1, 1936. Professor Birkhoff is serving as acting dean during the current academic year.

The Lamme Medal of the American Institute of Electrical Engineers has been awarded to Professor Vannevar Bush, vice-president of the Massachusetts Institute of Technology and dean of the School of Engineering, for his development of methods and devices for application of mathematical analysis to problems of electrical engineering.

Dr. T. C. Fry, of the Bell Telephone Laboratories of New York City, addressed the University of Cincinnati chapter of Sigma Xi on February 14 on "A Mathematical Theory of Rational Inference."

The following courses in Mathematics are announced for the Summer of 1936.

University of California at Los Angeles. In addition to the usual elementary courses the following advanced courses will be offered: By Professor Daus: Elementary mathematics from an advanced standpoint; By Professor Whyburn: Foundations of geometry.

University of Chicago. First term, June 22 to July 24: Second term, July 27 to August 28. In addition to Integral calculus, Synthetic projective geometry, Solid analytic geometry, Elementary differential equations, and Infinite series and definite integrals, the following courses will be offered: By Professor L. E. Dickson: Analytic theory of numbers. By Professor G. A. Bliss: Theory of functions of real variables. By Professor E. P. Lane: Metric differential geometry; Modern theories of differential geometry. By Professor L. M. Graves: Modern theories of integration. By Professor R. W. Barnard: Algebraic invariants; Linear functional operators. By Professor Walter Bartky: Dynamics; Celestial mechanics. By Dr. Ralph Hull: Elementary theory of numbers.

University of Colorado. In addition to the usual elementary courses the following courses will be offered: By Professor Light: Mathematics of insurance (first term); History of mathematics (second term); Differential equations (both terms); Fundamental differential equations of mathematical physics (both terms). By Professor Hutchinson: Projective geometry (both terms). By Associate Professor Kendall: Theory of equations (second term). By Assistant Professor Hazard: Teachers' course in mathematics (first term).

Columbia University. In addition to the usual elementary courses the following advanced courses will be offered: By Professor Kasner: Survey of mathematics; Applications of calculus to geometry. By Professor J. F. Ritt: Differential equations. By Professor R. G. Archibald: Theory of numbers. By Professor A. C. Berry: Functions of a complex variable.

Cornell University. July 6 to August 16. In addition to the usual elementary work, the following advanced courses will be offered: By Professors Carver and Hulse: Teachers' course. By Professor Hurwitz: Theory of numbers, Advanced calculus. By Assistant Professor Jones: Analytic projective geometry. By Dr. Black: Projective geometry. By Dr. Randolph: Functions of a complex variable. Reading and research will be directed by Professors Carver and Hurwitz; Assistant Professors Jones, Agnew, and Lawrence; Dr. Black and Dr. Randolph.

Duke University. In addition to the usual elementary courses, the following will be offered: By Dr. Dressel: Differential equations. By Professor Elliott: Integral equations; Analytic geometry of space. By Professor Miles: Complex variable; Fourier series. By Professor Rankin: The teaching of mathematics; Modern geometry. By Professor Roberts: Number theory; Foundations of geometry. By Professor Thomas: Galois theory of equations; Thesis seminar.

University of Illinois. June 15 to August 8. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor Carmichael: The division of the circle. By Professor Emch: Geometry; Theory of equations. By Professor Crathorne: Theory of statistics; Calculus of variations. By Assistant Professor Levy: Analytic projective geometry; Analysis. By Assistant Professor Bailey: Teachers' course; Advanced calculus. By Dr. Bourgin: Applications of mathematics. By Dr. Bower: Statistics. By Dr. Miles: Fundamental concepts.

University of Iowa. First term, June 5 to July 16. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Miss Ruth Lane: Content and teaching of mathematics. By Assistant Professor Conkwright: Differential equations; Theory of numbers. By Associate Professor Woods: Modern geometry; Elliptic integrals. By Professor Reilly: Selected topics in algebra; Introduction to partial differential equations. By Professor Chittenden: Advanced calculus; General analysis; Theory of sets of points. By the staff: reading and research.

Second term, July 20 to August 20. By Professor Chittenden: General analysis; Theory of functions. By Associate Professor Ward: Geometrical transformations; Differential equations. By Associate Professor Wylie: Mathematics of finance; History of astronomy; The sidereal universe. By Assistant Professor Craig; Matrices and determinants; Statistics. By the staff: reading and research.

Johns Hopkins University. June 22 to August 1. In addition to courses in college algebra, analytic geometry and elementary calculus, a course in Group theory will be given by Professor F. D. Murnaghan.

University of Kansas. June 10 to August 5. In addition to the usual elementary courses, the following advanced courses are offered: By Professor Jordan: Analytic mechanics. By Professor Wheeler: Modern analytic geometry. By Professor Mitchell: History of mathematics; Teaching of high school mathematics; Theory of numbers. By the staff: Seminar.

University of Kentucky. In addition to the usual elementary courses, the following advanced courses will be offered. First Term. By Dean Boyd: Modern solid geometry. By Professor LeStourgeon: Definite integrals; Elliptic integrals; Fourier's series.

Second term. By Professor Latimer: Theory of numbers. By Professor Downing: Differential equations.

University of Maine. July 6 to August 14. In addition to the usual elementary work, the following advanced courses are offered. By Associate Professor Bryan: Teachers' course; Modern higher algebra; Seminar in secondary school mathematics. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Advanced calculus; Differential equations; Theory of functions of a complex variable.

Massachusetts Institute of Technology. First term, June 16 to July 29. In addition to courses in calculus and differential equations covering the work of the first two years, the following advanced courses are offered: By Dr. Crout: Advanced calculus. By Professor Franklin: Theory of functions of a complex variable. By Professor Struik: Differential geometry; Theory of probability.

Second term, July 29 to September 9. Courses covering the first two years repeated. By Professor Hopf: Advanced calculus. By Professor Struik: Differential geometry; Theory of probability. By Professor Zeldin: Vector analysis.

University of Michigan. June 26 to August 21. In addition to elementary courses in college algebra, trigonometry, plane and solid analytic geometry, calculus, differential equations, statistics, and finance, the following advanced courses will be offered: By Professor Anning: Solid analytic geometry. By Professor Ayres: Advanced calculus. By Professor Bradshaw: Descriptive geometry; Higher geometry. By Professor Carver: Theory of statistics; Finite differences. By Professor Churchill: Fourier series. By Professor Coe: Analytic me-

chanics. By Professor Craig: Theory of probability; Advanced theory of statistics. By Dr. Dushnik: Empirical formulas. By Dr. Elder: Theory of equations and determinants. By Professor Field: Applied mathematics—Engineering problems. By Professor Hildebrandt: Theory of functions of a complex variable; Partial differential equations. By Professor Karpinski: Teachers' seminar in algebra; History of geometry and trigonometry. By Professor Nyswander: Algebraic theory. By Professor Poor: Vector analysis. By Professor Wilder: Introduction to the foundations of mathematics; General spaces. In addition there will be a seminar in pure mathematics conducted by Professors Hildebrandt and Wilder, and one in statistics by Professor Craig.

University of Minnesota. First term, June 15 to July 25. In addition to the usual elementary work, the following advanced courses will be offered. By Professor Elizabeth Carlson and Gladys Gibbens: Reading in senior college mathematics. By Professor Anthony L. Underhill: Differential equations; Advanced calculus. By Professor Dunham Jackson: Advanced algebraic theory; Introduction to the theory of small samples. By Professors Jackson and Underhill: Reading in advanced mathematics.

University of North Carolina. First term, June 11 to July 22. In addition to the usual courses in algebra, trigonometry, analytic geometry and the calculus, the following advanced work will be offered: By Professor Henderson: Differential equations; Solid analytic geometry. By Professor Browne: Modern higher algebra. By Professor Machie: Advanced calculus. By Professor Winsor: College geometry.

Second term, July 23 to August 29. By Professor Lasley: Analytic projectivs geometry; Higher plane curves. By Professor Linker: Differential equatione (continued). By Professor Hoyle: Advanced calculus (continued).

Northwestern University. June 22 to August 14. In addition to the usual elementary courses the following advanced courses will be offered: By Professor Simmons: Solid analytic geometry; By Professor Rutt: Fundamental principles of algebra and geometry; By Professor Curtiss: Ordinary differential equations.

Ohio State University. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor Rasor: Theory of functions of a complex variable; Fourier's series. By Professor Radó: Vector analysis; Theory of fields. By Dr. Wylie: Projective geometry; Differential geometry.

University of Pittsburgh. In addition to the usual courses in elementary subjects, the following more advanced courses will be offered: By Professor Foraker: Modern synthetic geometry; Solid analytic geometry. By Professor Taylor: Advanced calculus; Introduction to the theory of relativity. By Professor Culver: Differential equations.

University of Southern California. First term, June 19 to July 31. The following advanced courses are offered: By Professor Ames: Theory of probability and statistics; Modern higher algebra; History of mathematics. By Associate Professor Steed: Essentials of applied mathematics; Vector analysis; Advanced analytic geometry.

Second term, August 1 to September 4. By Professor Gurney: Differential equations; Seminar (subject to be announced during first term).

Syracuse University. In addition to the regular courses in algebra, trigonometry, analytic geometry, and the calculus, the following advanced courses will be offered: By Professor Decker: Modern higher algebra or Fundamental concepts of mathematics. By Professor Campbell: College plane geometry or Elementary vector analysis. By Professor Carroll: Methods in the teaching of mathematics; Field work in mathematics. By Professor Harwood: Advanced analytic geometry or Algebraic analysis.

Teachers College, Columbia University. July 7 to August 14. In addition to the major course on the teaching and supervision of mathematics and one on the teaching of algebra given by Professor Reeve, the following courses will be offered. By Miss Sutherland: Teaching arithmetic in primary grades; Teaching arithmetic in intermediate grades; Professionalized subject matter in junior high school mathematics. By Professor Schuster: Modern business arithmetic; Field work in mathematics. By Dr. Sanford: Teaching algebra in junior high schools; The history of mathematics. By Mr. Smith: Teaching geometry in secondary schools; A demonstration class in plane geometry. By Dr. Swenson: Professionalized subject matter in senior high school mathematics; A demonstration class in eleventh year mathematics. By Dr. Wolff: Teaching mathematics in the secondary schools of Germany; The correlation of secondary school mathematics with science and art.

The University of Vermont. In addition to the usual elementary subjects, the following more advanced courses will be offered: By Professor Bullard: Differential equations. By Professor Butterfield: History of mathematics. By Professor Nicholson: Teaching of algebra and plane geometry in secondary schools.

University of Wisconsin. Six weeks session, June 29 to August 7. In addition to the usual elementary subjects, the following advanced courses will be offered: By Professor Bennett: College geometry. By Professor Hartung: Content of secondary mathematics; The teaching of mathematics. By Professor MacDuffee: Algebraic invariants; Theory of numbers. By Professor Sokolnikoff: Advanced calculus; Differential equations; Vector analysis.

Special nine weeks session for graduates, June 29 to August 28. These courses may be taken for six weeks. By Professor March: Theory of analytic functions; Fourier series.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N. Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Summer Meeting, Harvard University, Aug. 31, 1936.

Twenty-first Annual Meeting, Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, March.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S. C., April 17–18.
SOUTHERN CALIFORNIA, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, May 9.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 6, JUNE-JULY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The sixteenth regular meeting of the Southern California Section of the Mathematical Association of America was held at Fullerton Junior College, Fullerton, California, Saturday, March 7, 1936. Professor C. G. Jaeger presided.

The attendance was seventy-four, including the following thirty-six members of the Association: L. J. Adams, O. W. Albert, E. E. Allen, L. D. Ames, Harry Bateman, Mable L. Beckwith, Clifford Bell, E. T. Bell, L. T. Black, Myrtie Collier, P. H. Daus, D. C. Duncan, Iva B. Ernsberger, Harriet E. Glazier, E. R. Hedrick, C. G. Jaeger, Glenn James, G. R. Kaelin, G. R. Livingston, W. E. Mason, G. F. McEwen, A. D. Michal, William Orange, Lena E. Reynolds, J. M. Robb, G. T. Sellew, G. E. F. Sherwood, Marcus Skarstedt, D. V. Steed, F. C. Touton, C. W. Trigg, S. E. Urner, L. E. Wear, Mabel G. Whiting, W. M. Whyburn, Euphenia R. Worthington.

The following officers were elected for the next year: Chairman, F. C. Touton, University of Southern California; Vice-Chairman, S. E. Urner, Los Angeles Junior College; Program Committee, C. F. Jaeger, Pomona College and W. M. Whyburn, University of California at Los Angeles, Chairman. The next meeting was tentatively scheduled for March 6, 1937, at Los Angeles Junior College.

The following seven papers were read:

1. "Characteristics of mathematicians" by Professor E. T. Bell, California Institute of Technology.

2. "Elements of non-analytic functions" by Professor E. R. Hedrick, University of California at Los Angeles.

3. "The role of objective in teaching junior college mathematics" by Professor F. C. Touton, University of Southern California.

4. "The solution of linear equations" by I. E. Highberg, California Institute of Technology, introduced by Professor Michal.

5. "Elementary congruences" by Professor C. G. Jaeger, Pomona College.

6. "Elements of a general theory of analytic functions" by A. E. Taylor, California Institute of Technology, introduced by Professor Michal.

7. "The mathematics of the torsion balance" by Professor S. E. Urner, Los Angeles Junior College.

Abstracts of the papers follow, the numbers corresponding to those in the list of titles.

1. As the result of an intensive study of the lives of thirty-five mathematicians (many more were included but rejected from the final exposition), Dr. Bell was led to certain conclusions at variance with the popular conception of mathematicians in general and of *great* mathematicians in particular. A selection from the mass of accumulated data will appear in book form, probably

in 1936, as a series of biographies, from which the non-technical reader will be able to draw his own conclusions (although to economize space much evidence has been suppressed). A few of the popular myths exploded by a historico-critical examination of the facts are these: *all* great mathematicians were lovers of music—the majority were not; a rigid adherence to mathematical truth makes a man both intellectually honest and truthful—in a significant percentage of cases the pursuit of mathematics has not in any way improved the pursuer's moral or intellectual integrity; the inexorable justice and impersonality of mathematics induces the like in its devotees—it does not, and good mathematicians, as a class, probably surpass artists and actors in hyper-neurotic touchiness and self-conceit. Battles over priority in mathematics probably surpass in ferocity any similar brawls in the arts. It is sheer rubbish to claim that a course in mathematics is of any benefit whatever in what is called "character building." Such is the verdict of scientific history.

2. If in the complex plane, $w=f(z)$, and if the derivative is defined in the usual manner, imposing the conditions that the given function be analytic places *strong* restrictions on the function f, which for many theorems are not necessary. Professor Hedrick outlined a program of investigation in the theory of non-analytic functions in order to determine what part of the theory of analytic functions of a complex variable is still applicable, and in general, to determine the minimum hypotheses which are necessary for the important theorems of analysis.

3. In Professor Touton's paper consideration was given to the place of purpose and method in achieving the goals of a course. These purposes and methods, it was maintained, should be clear in the thinking of the instructor and consciously approached by the learner. Statements of the purposes were proposed for junior college mathematics curriculum, and of a semester course in junior college algebra. A list of six groups of users of mathematics was given with a suggestion as to varied types of content suited to these groups. Consideration was given to the equipment of the teacher essential to the efficient direction of the learning process in mathematics. Finally a statement was given in some detail showing the procedures of the learner in achieving his maximum mastery of the content presented in the mathematics curriculum as one of his several approaches to his attainment of the goals of junior college education.

4. In this paper Mr. Highberg discussed the conditions under which the linear equation $y=F(x)$ is soluble with a continuous inverse. A linear function is defined to be one which is additive and continuous. He considered linear functions of the real and complex numbers and also the linear vector function as examples of the finite dimensional cases. The solution of the Fredholm integral equation as an example of a linear equation in an infinitely dimensional space led to the discussion of linear equations in complete normed vector spaces. He gave the theorem of Schauder on the solution of a linear function defined on one complete vector space to another, and he also discussed briefly the recent work of Hausdorff on this subject.

5. In his paper Professor Jaeger considered some of the well-known but more or less neglected phases of congruences, with undergraduate courses in Number Theory especially in mind. In particular, the relation between groups and the Euler ϕ function was given in an elementary manner.

6. It was the aim of Mr. Taylor's talk to sketch the nature of the problems which present themselves when one attempts to generalize the Cauchy-Weierstrass theory of analytic functions by substituting for the complex number field a complete, complex vector space (in the sense of Banach).

In its broad essentials the theory carries over the group of theorems beginning with Cauchy's integral theorem, and ending with the Taylor's series. A discussion of singularities reveals, however, that Picard's theorem is not true in general. The nature of both poles and essential singularities is more complex. There is a generalization of Mittag-Leffer's theorem.

7. The torsion balance was developed by Baron Eötvös at the University of Budapest, during the years 1890–1919, as a purely scientific instrument for precise measurement of the variations of gravity. It has recently come into commercial use as an aid to the geologist in mapping underground structures, in connection with the search for oil and other mineral deposits. Essentially, the instrument consists of a beam carrying two weights at different levels, suspended at its center by a fine wire. When the beam comes into equilibrium, the moment of the gravitational field is balanced by the restoring moment of the wire. Knowing the coefficient of torsion of the wire, one calculates the torque, which is expressible analytically, in terms of the neutral position of the beam, and of the four Eötvös quantities U_{xz}, U_{yz}, $2U_{xy}$, $U_{yy}-U_{xx}$, U being the potential function of the field. Since these four quantities would be zero if the earth were a homogeneous sphere, one separates out the portions of these quantities which are due to the ellipsoidal figure of the earth, to the rotation effect, and to the irregularities of the nearby ground-surface. The remaining distortions are due to underground irregularities. Geological interpretation is difficult, but in general depends upon comparison with the Eötvös quantities for assumed irregularities of underground density. In Dr. Urner's paper, an effort was made to indicate the types of mathematics used in setting up the equation of equilibrium, and in the more difficult task of making the corrections described above.

P. H. DAUS, *Secretary*

THE THIRTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at State Teachers College and Mississippi Woman's College, Hattiesburg, Mississippi, March 13–14, 1936. The chairman, Professor H. L. Smith of Louisiana State University, presided and introduced the guest speaker, Professor J. O. Hassler, University of Okla-

homa. Professor Hassler's subject at the banquet was "The spirit of discovery—a motivating power in mathematics and science."

The attendance was approximately sixty-five including the following twenty-six members of the Association: G. A. Baker, T. A. Bickerstaff, H. E. Buchanan, Mrs. A. P. Daspit, D. S. Dearman, Virginia I. Felder, Elizabeth Freas, Margaret E. Harris, J. O. Hassler, D. May Hickey, J. R. Hitt, Dorothy McCoy, Janet McDonald, Elsie McFarland, B. E. Mitchell, I. C. Nichols, Arthur Ollivier, S. T. Sanders, H. L. Quarles, Alta H. Samuels, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker.

The following officers were elected for the year 1936–37: Chairman, D. S. Dearman, State Teachers College, Hattiesburg; Vice-Chairman for Louisiana, B. A. Tucker, Southeastern Louisiana College; Vice-Chairman for Mississippi, J. R. Hitt, Mississippi College; Secretary, Dorothy McCoy, Belhaven College.

The following six papers were read:

1. "True probable errors" by Professor G. A. Baker, Mississippi Woman's College.

2. "A discussion of teaching problems, with special reference to the superior student" by Professor R. L. O'Quinn, Louisiana State University.

3. "A geometric definition of the torsion of a skew curve" by Professor H. L. Smith, Louisiana State University.

4. "Real and complex roots of an algebraic equation by the method of Graeffe" by Professor J. F. Thomson, Tulane University.

5. "A topic on cyclic groups of general linear fractional transformations" by Professor T. A. Bickerstaff, University of Mississippi.

6. "An evaluation and comparison of objective and subjective tests in mathematics" by Professor J. O. Hassler, University of Oklahoma.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In finding the distributions of estimates of statistical constants in random samples from hypothetical populations it is usual to assume that the deviations are measured from the constants of the assumed population. Professor Baker assumes that the deviations are measured from the statistical constants of a first sample and measured in terms of the constants of a first sample. Under these conditions the distributions of the means and standard deviations of samples of n drawn from a normal population are given and made the basis for "true" probable errors. It is found that the "true" probable errors are considerably in excess of conventional probable errors for small samples.

2. In this paper Professor O'Quinn presented in detail a number of problems in teaching arising in our great modern colleges and universities, with suggested solutions. He gave data to prove the case of those who favor segregation into special sections of the more gifted students, and made suggestions as to methods that may prove beneficial in the teaching of the more gifted students, in order that they may build up their ability to achieve desired results. He discussed the question of conscious efforts along the line of improving the effectiveness of our

thinking, as well as how we may direct our students into modes of thought that shall prove of permanent benefit.

3. In this paper Professor Smith defined the torsion of a skew curve at a point P_0 of it as the limit, as three points P_1, P_2, P_3 simultaneously approach P_0 along the curve, of a certain function $f(P_0, P_1, P_2, P_3)$. He showed that when the derivatives at P_0 of the coordinates exist up to and including those of the third order, then the above definition yields the usual formula for the torsion at P_0 provided the denominator in the formula does not vanish.

4. Professor Thomson showed how Graeffe's method consists in forming a new equation whose roots are some high power of the roots of the original equation. The roots are then widely separated and the equation can be solved at once numerically. By this method all roots, both real and complex, are determined at once, in one sequence of operations, without a preliminary determination of the approximate position of the roots and without carrying a large number of significant figures. He discussed the method and then applied it to solve two equations, one with real roots, the other with real and complex roots.

5. Professor Bickerstaff considered the powers of a fractional linear transformation. With $T(z)=(az+b)/(z+d)$ as a generating transformation,

$$T^{\alpha}(z) = \frac{a_{\alpha}z + b}{z + a_{\alpha} + d - a},$$

where $a_{\alpha}=(r_2K^{\alpha}-r_1)/(K^{\alpha}-1)$, r_1 and r_2 being the unequal fixed points of the transformation T, and K being $(a-r_1)/(a-r_2)$. Now $S(z)=(lz+m)/(z+n)$ belongs to the cyclic group T^{α} if

$$n - l = d - a, \qquad l = a_{\alpha}, \qquad m = b;$$

and $S\equiv T^{\alpha}$ if $K^{\alpha}=(l-r_1)/(l-r_2)$ and hence

$$\alpha = \frac{\log (l - r_1) - \log (l - r_2)}{\log (a - r_1) - \log (a - r_2)}.$$

6. Professor Hassler reported on two experiments with objective and subjective tests, and discussed the claims of those proposing objective tests. He showed a change in their trend of thinking as far as mathematics is concerned. He reached the following conclusions:

a. The errors in the students' grades due to the effect of a teacher's judgment in grading the traditional problem examination in mathematics make it imperative that some improvement be made in methods of testing.

b. Objective testing does not completely eliminate the variability of a pupil's grades. It helps, but the grade is a function of other variables besides the teacher's judgment and the student's knowledge.

c. A single examination is not a safe criterion for determining what a student knows.

d. Objective tests are changing their nature so as to include longer and more

complicated problems, each containing many elements, with recent attempts being made to break each problem up into one-step units.

e. Objective tests tend to raise the grades of poor students more than the good students.

f. Objective tests are best to use in the early stages of a new subject for diagnostic purposes or to smooth the path for the beginner.

g. Objective tests do not test for some of the principal values in mathematics. Some proponents of objective testing admit that subjective tests must still be used to supplement the objective tests.

h. Objective tests are not needed as badly in mathematics as in many other subjects where one's opinion plays a more prominent role, if we will break up long problems into separate elements as we grade.

i. We should use both types of tests and introduce the principles of objectivity into our grading. We should never forsake developing and testing for *the ability to solve a problem* rather than to fill in a blank or choose the correct word out of a submitted list.

DOROTHY McCOY, *Secretary*

THE THIRTEENTH ANNUAL MEETING OF THE MICHIGAN SECTION

The thirteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan on Saturday, March 21, 1936, in association with the Michigan Academy of Sciences, Arts and Letters. There were sessions for the reading of papers both morning and afternoon with the chairman of the Section, Professor J. B. Brandeberry, presiding at both times.

The total attendance was about one hundred, including the following forty members of the Association: W. L. Ayres, W. D. Baten, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, R. V. Churchill, C. C. Craig, S. E. Crowe, Wayne Dancer, W. W. Denton, P. S. Dwyer, J. P. Everett, Peter Field, C. H. Fischer, K. W. Folley, W. B. Ford, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, H. S. Kaltenborn, Theodore Lindquist, C. E. Love, Roy MacKay, E. D. McCarthy, A. L. Nelson, J. A. Nyswander, H. L. Olson, L. C. Plant, G. Y. Rainich, Gladdis E. Richards, C. C. Richtmeyer, L. J. Rouse, C. J. Shires, E. R. Sleight, A. G. Swanson, E. T. Welmers, R. L. Wilder, J. B. Winslow.

The annual luncheon and business meeting was held during the noon recess at the Michigan Union with fifty-eight present. The following officers were elected for the year 1936–37: Chairman, C. C. Richtmeyer, State Teachers College, Mt. Pleasant; Secretary-Treasurer, C. C. Craig, University of Michigan; third member of Executive Committee, J. B. Brandeberry, University of Toledo.

The Committee on Interesting Undergraduate Students in Mathematics reported favoring the publishing of bound mimeographed copies of outstanding essays and other pieces of work by undergraduate students, each institution paying for the expense for the papers of its own students. The papers will be bound in one cover and distributed to interested students and others. Also it is proposed to hold meetings at which these undergraduates may present their pieces of work as papers. After considerable discussion it was voted that the Executive Committee should proceed with the organization of this project as outlined by the Committee.

The principal paper of the meeting was a talk by Professor S. A. Goudsmit of the Department of Physics of the University of Michigan on the development of modern theoretical physics.

The complete program follows:

1. "The use and construction of factor stencils" by Dr. J. D. Elder, University of Michigan, introduced by the Secretary.

2. "On a certain problem concerning geodesics" by J. W. Zimmer, Michigan State College, introduced by Professor Grove.

3. "On the number of ways to triangulate a convex polygon" by Donat Kazarinoff, University of Michigan, introduced by the Secretary.

4. "A proof of the fundamental theorem of integral calculus by use of Taylor's expansion" by Richard Fowler, Albion College, introduced by Professor Sleight.

5. "On the definition of determinants" by Professor L. J. Rouse, University of Michigan.

6. "Teaching moment of inertia to engineering students" by Professor H. E. Mayrose, Department of Engineering Mechanics, University of Detroit, introduced by Professor L. S. Johnston.

7. "Geometric proofs of multiple-angle theorems" by Professor Wayne Dancer, University of Toledo.

8. "The new entrance requirements and their effect on mathematics in high school" by Professor E. R. Sleight, Albion College.

9. "Some new ideas in physics" by Professor S. A. Goudsmit, Department of Physics, University of Michigan, by invitation of the Program Committee.

10. "An application of the Point Calculus of Grassman" by Professor J. W. Bradshaw, University of Michigan.

Abstracts of some of the papers follow:

2. Mr. Zimmer's problem was to find among the admissible arcs, $x=x(s)$, $y=y(s)$, $z=z(s)$ $(s_1 \leqq s \leqq s_2)$ satisfying the equations

$$\phi(x, y, z) = 0, \qquad x'^2 + y'^2 + z'^2 - 1 = 0,$$

and having end points satisfying the equations

$$s_1 = x_1 - \alpha_1 = y_1 - \beta_1 = z_1 - \gamma_1 = x_2 - \alpha_2 = y_2 - \beta_2 = z_2 - \gamma_2 = 0,$$

one which minimizes the arc-length integral. Every extremal arc for this prob-

lem is normal. The minimizing arc is free of corners and if an arc is to be a geodesic it need only satisfy the Euler Lagrange multiplier rule and have no point conjugate to 1 between 1 and 2 or at 2.

4. Mr. Fowler gave a proof of the fundamental theorem of integral calculus by use of Taylor's expansion and direct summation.

5. Since the definition of determinants ordinarily given in textbooks on college algebra is unsatisfactory when dealing with determinants of the fourth and higher orders, and the idea of expansion by minors must be introduced, Professor Rouse suggested that it would be well to define a determinant of order n initially as a symbol denoting the algebraic sum of n products formed by multiplying each element of the first row by its minor, the element being taken with its sign unchanged or changed according as the number of the column in which it lies is odd or even. With this definition it was pointed out that the elementary properties of determinants may easily be derived.

6. Professor Mayrose stated some of the difficulties which must be overcome in a satisfactory explanation of moment of inertia. First: There is no agreement in the definition of the term; some use weight, and some use mass in their definitions and arrive at the same results by modification of the formulas in which moment of inertia appears. Some declare that it is only a mathematical form, which will be found in numerous formulas in science. Second: There is no adequate name for the complicated unit in which we must express the result of our calculation for the isolated quantity, moment of inertia. If one expresses the quantity as an abstract number, giving no unit at all, there is no clue to the units used in calculation, and this is very important if the number is to be substituted in other formulas. The student has great difficulty in mastering this idea of complicated units. It seems, therefore, that the most satisfactory time to teach the subject will be that time when it can be put to work; when the student can use it for a time without knowing what he is using, and then learn that there is a certain economy in isolating the expression which we know as moment of inertia from his formulas, and in setting up tables of values for the more common geometrical forms. If he knows how to use the moment of inertia of bodies, or the moment of inertia of areas, the student will find little difficulty in understanding their respective meanings.

7. Utilizing a classical construction for trisecting an angle with the use of a marked straight-edge, Professor Dancer demonstrated a novel development of the formulas for $\sin 3x$ and $\cos 3x$. The same figure illustrated the double angle formulas, and other trigonometric identities.

8. Mathematics in the past has held a very prominent place in the curricula of most high schools, but for some time the importance of this subject in the preparatory schools has been questioned. Two events of recent occurrence have tended to decrease its importance, legislative enactment and changing entrance requirements. Professor Sleight's paper attempts to show the effect of both of these upon the amount of mathematics which we may expect to find in the high school of the future.

9. Professor Goudsmit discussed the historical development of quantum mechanics. He pointed out that its progress is primarily guided by and based on the results of physical experiments. He sketched briefly matrix mechanics and wave mechanics and showed their relationship. He mentioned some of the most recent discoveries about the nucleus of the atom. These discoveries, it is believed, will not affect the more fundamental results and ideas introduced by the present quantum theory, such as the uncertainty principle of Heisenberg.

10. The relation between the two foci of the conics tangent to three lines is a quadratic transformation. If one focus traces a straight line, the one-parameter family of conics thus obtained involves in general the cube of the parameter. The focal curve of a range of conics is a circular cubic. Professor Bradshaw exhibited a projective generalization of these well known facts obtained by the methods of the point calculus.

W. L. AYRES, *Secretary*

HEAVISIDE'S OPERATIONAL CALCULUS—ITS APPLICATIONS AND FOUNDATIONS*

By HILLEL PORITSKY, General Electric Company, Schenectady, N. Y.

When Hill was bold enough to use infinite determinants in researches in the lunar theory, his work met with ridicule, until Poincaré proved the convergence of such a procedure and thus started formally the theory of an infinite number of linear equations in an infinite number of unknowns. A somewhat similar history followed the publication by Heaviside of certain methods of solving linear differential equations known nowadays as Heaviside's operational calculus, except that in this case, a much longer period—almost a generation—elapsed between the appearance of Heaviside's work and the completion and justification of his methods by Bromwich, Carson, Jeffreys, Wiener, March, and others. During such a long period of time what one might call two opposing schools of thought have grown up. On the one hand, those impressed by the convenience of Heaviside's methods for solving certain problems have followed in his footsteps, but have neglected to strengthen the foundations, even when at times shaky. On the other hand, the pure mathematicians, horrified by the lack of rigor and almost empirical nature of the methods, have either denounced them or passed over them in haughty silence. Now that the healing of the breach has started, the views of Heaviside as the scientific martyr and saint or as the mathematical villain may properly be relegated to history. With the smoke of battle cleared, it behooves us to utilize the practical advantages of the operational methods but without exempting them from critical mathematical reasoning.

The Heaviside operational calculus is concerned with linear functional operators. These operators operate on a function of a variable x, say $h(x)$, leading to another function of x, $g(x)$:

* Address delivered by invitation before the Association at Ann Arbor, September 10, 1935.

$$O[h(x)] = g(x). \tag{1}$$

The operators O with which the calculus is concerned are expressed in terms of the differential operator

$$p = \frac{d}{dx}, \tag{2}$$

thus

$$O = f(p). \tag{3}$$

We shall now define the operators O for certain simple functions f on the right of (3) and discuss some of their properties and their applications.

If O is an operator, we denote by O^n, $n=1, 2, 3, \cdots$, its n-th repetition, by $1/O$ or O^{-1} its inverse, by $O^{1/2}$ an operator whose repetition leads to O. Similar interpretations can be given to O^n for rational $n \neq 0$. The existence of O^n for integer n is tied up with the requirement that the result of the operation O belong to the same class of objects on which O operates. The existence of a unique inverse and its iterations depend upon the above requirement as well as the condition that the transformation effected by O be one to one. The "identical" operation which transforms the objects into themselves is denoted by O^0 or 1.

Applying the above to the operator p, we interpret p^0 or 1 as the identity; p^n for positive integer n as the operation of n successive differentiations:

$$p^n = \left(\frac{d}{dx}\right)^n = \frac{d^n}{dx^n}. \tag{4}$$

The inverse operation p^{-1} or $1/p$ we interpret as the definite integral with zero as the lower limit:

$$p^{-1}h(x) = \int_0^x h(s)ds, \tag{5}$$

while p^{-n} for positive integer n is the n-th iterated integral. Thus,

$$p^{-2}h(x) = \int_0^x \int_0^s h(s_1)ds_1 ds. \tag{6}$$

The region of integration in (6) is shown in Fig. 1; the vertical lines with arrows indicating the first integration. If the s integration be carried out first, its limits will be s_1 to x (see dotted line in Fig. 1); thus (6) becomes

$$\begin{aligned} p^{-2}h(x) &\int_0^x \int_{s_1}^x h(s_1)ds ds_1 \\ &= \int_0^x h(s_1)(x - s_1)ds_1 = \int_0^x (x - s)h(s)ds, \end{aligned} \tag{7}$$

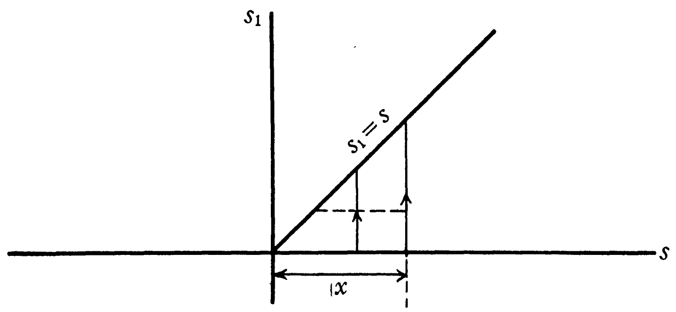


FIG. 1

since a variable of integration in a definite integral can be interchanged at will. Starting with (7), one proves similarly $p^{-3}h(x) = \int_0^x (x-s)^2 h(s)ds/2$ and, by induction, for any positive integer n,

$$p^{-n}h(x) = \int_0^x (x - s)^{n-1}h(s)ds/(n - 1)!. \tag{8}$$

The operator p^{-1}, it will be noted, is not an inverse in the sense that both

$$pp^{-1} = p^0 = 1 \tag{9}$$

and

$$p^{-1}p = p^0 = 1 \tag{10}$$

hold. The relation (9) always holds, but as will be noted from

$$p^{-1}ph(x) = \int_0^x \frac{d}{ds} h(s)ds = h(x) - h(0),$$

(10) holds only when applied to functions h that vanish at $x=0$. To avoid the necessity of right-sided and left-sided reciprocals, one may consider the whole interval $-\infty < x < \infty$ replacing (5) by

$$p^{-1}h(x) = \int_{-\infty}^x h(s)ds \tag{5'}$$

but confining oneself to functions that vanish identically for negative x:

$$h(x) = 0 \quad \text{for} \quad x < 0. \tag{11}$$

The relation (10) is now seen to hold for all $h(x)$ to which p may be applied in the extended interval. From (9), (10) follows

$$p^m p^n = p^{m+n} \tag{12}$$

for arbitrary integers m, n of opposite sign. (For m, n of similar sign (12) obviously holds.)

The relation (8) suggests the following definition for p^{-n} for arbitrary positive n:

$$p^{-n}h(x) = \int_0^x (x - s)^{n-1}h(s)ds/\Gamma(n), \quad n > 0. \tag{13}$$

Thus, in particular,

$$p^{-1/2}h(x) = \int_0^x (x - s)^{-1/2}h(s)ds/\sqrt{\pi}. \tag{14}$$

The integral (14) is improper but convergent (for integrable h); likewise, for (13) for $0<n<1$. One now proves that

$$p^{-1/2}p^{-1/2} = p^{-1}$$

and, more generally, that (12) holds for arbitrary negative m, n; this is done by interchanging the order of the two integrations as in connection with (6) and Fig. 1, and utilizing the Eulerian integral of the first kind.

For negative n, (13) does not apply since the integral diverges. The definition of p^n for non-integer positive n is carried out by following (13) by p, p^2, $\cdots$ (that is, by using (12) for $m=1, 2, \cdots ; n<0$).

The relation (12) may now be proved for any real m, n for functions satisfying (11) and for which the operators apply for all x. From (11) follows

$$p^m p^n = p^n p^m.$$

This constitutes a special instance of the general relation

$$f_1(p)f_2(p) = f_2(p)f_1(p) \tag{15}$$

presently to be noted.

We now define addition of operators and multiplication of an operator by a constant c as follows from the linearity property:

$$[cf(p)]\cdot h(x) = c[f(p)\cdot h(x)], \tag{16}$$

$$[f_1(p) + f_2(p)]\cdot h(x) = f_1(p)\cdot h(x) + f_2(p)\cdot h(x). \tag{17}$$

Using (16) and (17), it is easy to define polynomials in p. For two such polynomials $f_1(p)$, $f_2(p)$ the relation (15) is readily proved, each sequence of operations being equivalent to the operations represented by the product of the two polynomials. Similarly, definition for polynomials in p^{-1} are readily established.

We consider next operators arising from linear differential equations with constant coefficients. As the simplest example consider the differential equation

$$\frac{dg(x)}{dx} + ag(x) = h(x), \tag{18}$$

where a is a constant. Writing it in the form

$$(p + a)g = h, \tag{18'}$$

we obtain upon dividing non-ceremoniously by $(p+a)$

$$g = \frac{1}{p+a} h. \tag{19}$$

Thus introduced, the operator $1/(p+a)$ merely denotes the reciprocal of $(p+a)$ and (19) is but the equivalent of (18). To obtain a more explicit meaning for this operator, consider its expansions in powers of either p or p^{-1}:

$$\frac{1}{p+a} = \frac{1}{a} - \frac{p}{a^2} + \frac{p^2}{a^3} - \cdots, \tag{20}$$

$$\frac{1}{p+a} = \frac{1}{p} - \frac{a}{p^2} + \frac{a^2}{p^3} - \cdots. \tag{21}$$

When (19) is interpreted by means of (20) or (21) a formal series solution of (18) is obtained. However, while (20) can only be applied to functions h that possess all derivatives, and even for analytic h the resulting series will often not converge, (21) is readily shown to lead to a convergent series and to a true solution if h is integrable. We adopt (21) as the interpretation of (19). Thus defined, it yields the particular solution of (18) that vanishes at $x=0$. This is entirely in agreement with the restriction (11).

As an example, if

$$h(x) = 1, \tag{22}$$

we obtain

$$g(x) = \frac{1}{p+a} 1 = \left(\frac{1}{p} - \frac{a}{p^2} + \cdots\right)1, \tag{23}$$

and since

$$p^{-n}1 = x^n/n! \tag{24}$$

(23) becomes

$$g(x) = x - \frac{ax^2}{2!} + \frac{a^2x^3}{3!} - \cdots$$

which will be recognized as

$$(1 - e^{-ax})/a. \tag{25}$$

We have thus arrived at the result

$$\frac{1}{p+a} 1 = \frac{1 - e^{-ax}}{a}. \tag{26}$$

While (22), (26) could be considered for any real x, it is customary in operational calculus to impose (11) on the various functions under consideration (it

is this restriction that serves to pick a unique solution of (18)). Thus (22), (26) will be considered to apply for $x>0$ only, while both $h(x)$, $g(x)$ vanish for $x<0$. The resulting $h(x)$ is then known as the "Heaviside function" or the "unit step function" and denoted by 1 or $H(x)$:

$$H(x) = 1 = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases} \tag{27}$$

the relations (24), (26) being written as

$$p^{-n}1 = \frac{x^n}{n!}, \tag{24'}$$

$$\frac{1}{p+a}\,1 = \frac{1-e^{-ax}}{a}, \tag{26'}$$

while the value 0 is understood on the right hand side for negative x. Heaviside himself frequently omitted the symbol for $H(x)$ altogether.

In a similar way any linear differential equation with constant coefficients, such as

$$(p^n + a_1p^{n-1} + \cdots + a_n)g = D(p)g = h, \tag{28}$$

or system of simultaneous differential equations with constant coefficients

$$\sum_j a_{ij}g_j = h_i; \qquad i, j = 1, \cdots, n, \tag{29}$$

where a_{ij} are linear or of higher order in p can be treated by solving for g or g_i algebraically, expanding the resulting rational fractions in p

$$N(p)/D(p) \tag{30}$$

in powers of p^{-1}, interpreting each term as a multiple integral, and summing. In case of (28), the result is the solution vanishing with its first $(n-1)$ derivatives at $x=0$.

To express the result in a more convenient form the fraction (30) is broken up into *partial fractions*. The most convenient way of carrying this out is not the familiar algebraic form, but the form

$$\frac{N(p)}{D(p)} = \frac{pN(p)}{pD(p)} = \frac{N(0)}{D(0)} + \sum_i \frac{N(p_i)}{p_iD'(p_i)}\,\frac{p}{p-p_i} \tag{31}$$

obtained by resolving $N(p)/p\,D(p)$ into partial fractions and multiplying the result by p; here p_i are the roots of the denominator $D(p)$, assumed simple. N is supposed to be of lower degree than D.

To illustrate the above, consider (28) with h given by (27). The solution under consideration, using (31), becomes

$$g(x) = \frac{1}{D(p)}\,1 = \left(\frac{1}{D(0)} + \sum \frac{1}{p_iD'(p_i)}\,\frac{p}{p-p_i}\right)1. \tag{32}$$

This is now interpreted by means of

$$\frac{p}{p+a}\,1 = e^{-ax}, \tag{33}$$

a formula that follows from (26′) by differentiation. In the more general case (29) with h_i equal to either 0 or 1 the solution is similarly obtained as $[N(p)/D(p)]1$ and using (31), (33), interpreted as

$$\frac{N(p)}{D(p)} = \frac{N(0)}{D(0)} + \sum_i \frac{N(p_i)}{p_i D'(p_i)} e^{p_i x}. \tag{34}$$

Repeated roots can be similarly handled. This explicit representation in exponentials is known as the "Heaviside expansion." Its use can be extended to proper entire functions $D(p)$.

The frequent choice of $h(x)$ as the function 1 is entirely intentional. The case of arbitrary $h(x)$ can be reduced to it by utilizing the *translational* or *shifting property* of the various operators thus far considered, expressed as follows: if

$$f(p)h(x) = g(x) \tag{35}$$

then, for any real constant c,

$$f(p)h(x+c) = g(x+c). \tag{36}$$

This is verified for the various operators considered thus far if (5′), (11) be adopted.*

If now an arbitrary $h(x)$ be approximated by means of a step function and the area under it be cut up into horizontal strips as shown in Fig. 2, the approximation to $h(x)$ is seen to be given by

$$h(x) = H(x)h(0) + \sum_i H(x - x_i)(\Delta h)_i, \tag{37}$$

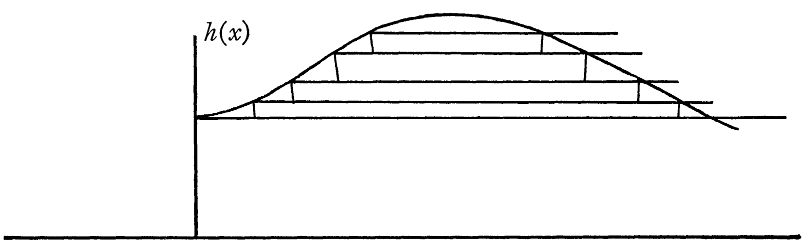


FIG. 2

where $(\Delta h)_i$ is the magnitude of the i-th step. Application of the linearity property of the operators $f(p)$ and the shifting property (35)–(36) reduces the case

* To avoid conflict between the translation in (36) and (11), c may be restricted to positive values.

of arbitrary $h(x)$, or rather its step-wise approximation, to the case of Heaviside unit functions. Upon passing to the limit by decreasing the size of the steps, (37) yields

$$h(x) = h(0)H(x) + \int_0^x H(x-s)\frac{dh}{ds}\,ds \tag{38}$$

and this suggests the formula

$$f(p)h(x) = f(p)H(x)h(0) + \int_0^x [f(p)H(x-s)]\frac{dh}{ds}\,ds, \tag{39}$$

which constitutes the reduction of the case of an arbitrary function $h(x)$ to that of the function 1.* This can be verified directly for the various operators thus far considered.

For the above reasons operational calculus treatments confine themselves to tabulations of the right hand members $g(x)$ of

$$f(p)1 = f(p)H(x) = g(x) \tag{40}$$

for various $f(p)$, thus deriving $g(x)$ by a proper operation $f(p)$ applied to the function $H(x)$. A more convenient treatment of arbitrary $h(x)$ that avoids the integrations (39), is often resorted to; it consists in using the operational tables backwards, finding, if possible, what operation $\phi(p)$ acting on 1 yields $h(x)$. If this be found, then one puts

$$f(p)h(x) = f(p)\phi(p)1 \tag{41}$$

and the table is now used directly for the result of the operation $f(p)\phi(p)$ acting on 1. Applied to (28) or (29) for the case where the right hand member is a linear combination of polynomials, exponentials, and trigonometric functions (for $x>0$), this method yields the solution in question as readily as for the case $h(x)1$.

A system of the form (29) arises in connection with electric circuits where g_i are the currents flowing in various branches or meshes, h_i the applied voltages, and a_{ij} operators of the form

$$a_{ij} = L_{ij}p + R_{ij} + 1/pC_{ij} \tag{42}$$

the L's, R's, C's being constant. The integro-differential system resulting from (29) is readily handled by the above methods, the solution obtained representing the currents that ensue upon the introduction in an initially quiescent state of the voltages h_i.

Without stopping over proofs of convergence, legitimacy of use of partial

* In terms of Stieltjes integrals, (38) and (39) take on the forms

$$h(x) = \int_{-\infty}^x d[h(s)H(x-s)], \qquad f(p)h(x) = \int_{-\infty}^x f(p)d[h(x)H(x-s)]$$

respectively.

fractions, etc., we shall point out instances of very similar procedures that are more familiar to present day mathematicians. Consider the integral equation of the second kind:

$$(1 + \lambda K)u = f, \tag{43}$$

where Ku stands for either of the operations

$$\int_a^x K(x, s)u(s)ds, \tag{44}$$

$$\int_a^b K(x, s)u(s)ds, \tag{45}$$

according as we have a Volterra or Fredholm integral equation. The familiar solution by means of an infinite series of iterated integrals can be obtained by dividing (43) by $1+\lambda K$, obtaining

$$u = \frac{1}{1 + \lambda K} f \tag{46}$$

and expanding $1/(1+\lambda K)$ in positive powers of K. For the Volterra case the solution is valid everywhere; for the Fredholm case, in general, only for small enough $|\lambda|$. Thus the expansion similar to those of (20, (21) is useful in obtaining the classical solution of (43). Similar infinite series solutions are obtained for arbitrary simultaneous linear differential equations by converting these into integral equations and utilizing matrix notation. The successive terms are now obtained by repeating the operation consisting of matrix multiplication and integration. For the case (29) (and zero initial values) this series solution is identical with the expansions in powers of p^{-1} considered above.

Another instance where the above notions are found illuminating is in solving Abel's integral equation. Writing it in the form

$$\int_0^x (x - s)^{n-1} h(s)ds/\Gamma(n) = g(x), \qquad 0 < n < 1, \tag{47}$$

we recognize it as

$$p^{-n}h(x) = g(x). \tag{47'}$$

To solve, operate on both sides with p^{n-1}, then with p, obtaining

$$h(x) = pp^{n-1}g(x) \tag{48}$$

or

$$h(x) = \frac{d}{dx}\int_0^x (x - s)^{-n} g(s)ds/\Gamma(1 - n).* \tag{48'}$$

* See Whittaker and Watson, *Modern Analysis*, 3rd ed., p. 229.

More spectacular but harder to justify by synthetic methods are the applications of operational calculus to partial differential equations such as the "heat conduction equation"

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \tag{49}$$

and the "wave equation"

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}. \tag{50}$$

These equations also occur in the propagation of signals in a non-inductive cable and on a transmission line (with x representing the time, y the distance). Writing (49) in operational form so far as $\partial/\partial x$ is concerned:

$$pu = \partial^2 u/\partial y^2, \tag{49'}$$

and solving as if p were constant, one obtains

$$u = Ae^{-p^{1/2}y} + Be^{p^{1/2}y}. \tag{51}$$

When considering the region $y>0$ with boundary conditions applied at $y=0$, it is customary to put $B=0$ obtaining

$$u = Ae^{-p^{1/2}y}. \tag{52}$$

Putting $y=0$ there results,

$$A = u\big|_{y=0}, \tag{53}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = -Ap^{1/2}. \tag{54}$$

Writing A last in (54), substituting from (53), one obtains

$$u\Big|_{y=0} = -p^{-1/2}\left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{54'}$$

This relation, interpreted as a fractional derivative, may be proved to be entirely correct. It describes the relation between the temperature at the end of a rod and the heat inflow, or, again, between the potential at the end of a non-inductive cable and the current entering it. Similarly (53) is given the form

$$u = e^{-p^{1/2}y}(u\big|_{y=0}). \tag{53'}$$

A correct interpretation of the operator in (53′) is obtained by expanding in powers of $p^{1/2}$ and *neglecting the integer powers*. The procedure, discovered by Heaviside, was never justified by him.

At this stage the need for a thorough recasting of the foundations of the operational methods becomes evident. A universal definition of the operators

$f(p)$ is needed. Such a definition was given by J. R. Carson. Carson interprets (40) by means of the integral relation

$$\frac{f(p)}{p} = \int_0^\infty e^{-xp} g(x) dx. \tag{55}$$

That is, the statement that the function $g(x)$ is obtained as a result of the operation $f(p)$ acting on the unit function $H(x)$, is *defined* as meaning that (55) is satisfied by $g(x)$ and $f(p)$ for every (real positive) p.

A more direct interpretation for operational formulas was obtained by Bromwich. He interprets (40) as follows:

$$g(x) = \frac{1}{2\pi i}\int_L \frac{f(p)}{p} e^{px} dp = f(p)H(x), \tag{56}$$

where the integration is carried out in the complex-plane along a path L parallel to the axis of imaginaries, and to the right of $p=0$ and of the singularities of $f(p)$.

To show that the definitions (55), (56) agree with some of the previous definitions, consider the case of the identical operation:

$$f(p) = 1 \tag{57}$$

in which case (40) yields, of course,

$$g(x) = H(x). \tag{58}$$

The Carson and Bromwich integrals reduce to

$$\frac{1}{p} = \int_0^\infty e^{-xp} dx, \tag{59}$$

$$\frac{1}{2\pi i}\int_L \frac{1}{p} e^{px} dp = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases} \tag{60}$$

respectively. The former is immediately verified, the latter is readily proved by moving the path of integration in the direction of real positive p for $x<0$, but in the opposite direction for $x>0$. The integral approaches zero due to the real part of the exponential approaching $-\infty$, but when $p=0$ is crossed (for $x>0$) the integral changes by the residue of the integrand.

The simple case of the identical operation just disposed of can now be used to prove the general mutual consistency of (55) and (56). Replacing x by $(x-s)$ in (60) yields the Bromwich integral for $H(x-s)$:

$$H(x-s) = \frac{1}{2\pi i}\int_L \frac{1}{p} e^{p(x-s)} dp = \frac{1}{2\pi i}\int_L \frac{e^{px}}{p} e^{-ps} dp. \tag{61}$$

If now an arbitrary $g(x)$ be approximated by means of (37) one obtains from (61)

$$g(x) = \frac{1}{2\pi i}\int_L \frac{e^{px}}{p}\left[g(0) + \sum_i (\Delta g)_i e^{-px_i}\right]dp. \tag{62}$$

As the steps are made smaller the bracket in (62) approaches

$$g(0) + \int_0^\infty e^{-ps}\frac{dg}{ds}\,ds$$

and upon integrating by parts and assuming

$$\lim_{x\to\infty} e^{-px}g(x) = 0, \tag{63}$$

it becomes

$$\left[p\int_0^\infty e^{-sp}g(s)ds\right], \tag{64}$$

so that the following identity results for an arbitrary $g(x)$:

$$g(x) = \frac{1}{2\pi i}\int_L \frac{e^{px}}{p}\left[p\int_0^\infty e^{-sp}g(s)ds\right]dp. \tag{65}$$

Upon defining (64) as $f(p)$, (65) is broken up into (55) and (56).

When applied to differential equations such as (19) with $h(x) = H(x)$:

$$(p + a)g(x) = H(x), \tag{19'}$$

the Bromwich integral robs the operational solution of a great deal of its mystery. Suppose first that the right hand side of (19′) is e^{px}, where p is a constant. A solution of (19′) is then

$$g(x) = \frac{e^{px}}{p + a}. \tag{66}$$

Now (60) represents $H(x)$ as a sum, or rather as a limit of a sum, of exponentials. Building a similar (limit of) sum of the corresponding solutions yields

$$g(x) = \frac{1}{2\pi i}\int_L \frac{e^{px}}{p(p + a)}\,dp. \tag{67}$$

In (60) the path of integration L could be anywhere to the right of the origin; in (67) it is taken to the right of $p = -a$ as well. When this is done $g(x)$ as given by (67) vanishes for $x<0$ as may be seen by moving the path of integration L to the right. For $x>0$, L is moved to the left and poles are encountered at $p=0$, $p = -a$. The residues at these poles furnish the constant and the exponential components of (26′) respectively.

Thus the Bromwich integral interprets the operational solution of (19) in the light of *superposition of exponentials*; both the right hand member of (19) and the function sought are expressed as definite integrals whose integrand is

an exponential (in x). The appearance of the exponential in any comprehensive treatment of operational calculus is not surprising. The consistent attempt to confuse the operation of differentiation with that of multiplication by p for which all kinds of excuses had to be offered is now justified once for all due to the validity for a constant p of the relation

$$\frac{d}{dx} e^{px} = pe^{px}.$$

The Carson integral in (55) is also known as the "Laplace integral," and both it and the Bromwich integral are, in fact, related to the Fourier integral. Indeed, if $f(p)/p$ is such that the path of integration in (56) can be made to coincide with the pure imaginary axis, then (55), (56) become completely equivalent to the ordinary Fourier integral for a function $g(x)$ which vanishes for negative x. Actually, for the important function $H(x)$ such a reduction to the Fourier integral is not possible, and $H(x)$, not being integrable in the interval $-\infty, +\infty$ cannot be expanded into a Fourier integral. Thus the Bromwich integral must be considered as a *slight generalization* of the Fourier integral obtained at the price of introducing complex values but capable of handling more general functions. Thus viewed, the operational calculus becomes but an outgrowth of harmonic analysis.

The notion of analytic functions $f(p)$ is foreign to the Fourier integral

$$\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}dx$$

which in general converges only for real α; its appearance in the present case is due to the vanishing of $g(x)$ for negative x. This renders $f(p)$, as given by (55), analytic in a half plane $R(p) > c$, where c is a constant.

If the Carson integral alone were used, a restriction to real p is still possible, but the advantages of the Bromwich integral in replacing $f(p)H(x)$ by means of an *explicit* operation, in yielding the Heaviside expansions in a natural manner, in clarifying the operational treatment of differential equations (both ordinary and partial), and in many other ways, render it invaluable. Without it and the use of analytic function theory, the non-existence of Heaviside expansions for operators like $p^{-1/2}$ or $e^{-p^{1/2}y}$, finds no simple explanation. On the other hand, the appearance of the *branch point* in the above analytic functions of p shows why it is hopeless to try to move L off to the left: the branch point serves as an impregnable barrier to such a motion; the path of integration can be wound around it, but cannot enclose it and be moved past it. However, by collapsing the path of integration onto the real negative axis one can obtain an explanation for the omission of the integral powers of p in the expansion of $e^{-p^{1/2}y}$. On the other hand, the Carson integral yields the explicit solution of the inverse problem of the operational calculus of finding the operation $f(p)$ which (applied to $H(x)$) yields a given function $g(x)$; it is also best suited for establishing the com-

mutativity of any two operations $f_1(p)$, $f_2(p)$ and for verifying the equivalence of the above p^{-n} with the new definitions for arbitrary $n>0$.

Operational calculus can be adapted to solutions of differential equations such as (29) with non-vanishing initial values, but it still prefers to join them on to functions vanishing for $x<0$. In this connection the following degenerate functions enter:

$$pH(x),\quad p^2H(x),\ \cdots\ ; \tag{68}$$

these are described as having a hump of area 1 at $x=0$, two humps of opposite sign there, etc.; or as the "unit impulse" at $x=0$, a "doublet" at $x=0$, $\cdots$. Such functions are generally outlawed by mathematicians who would consider $dH(x)/dx$ as zero for $x\neq 0$ and as nonexistent for $x=0$. However, a logical treatment in terms of *limits* of differentiable function that approximate to $H(x)$, can (and still remains to) be given.

Operational calculus effects a connection between linear functional transformations enjoying the "shifting property" (36) and analytic functions, by expressing such operations as analytic functions of p; it is thus related to the modern developments of transformations in Hilbert space. It further effects a correspondence between real functions $g(x)$, vanishing for $x<0$, and functions $f(p)$ analytic in a half plane.

The above treatment has been suggestive rather than exhaustive, many proofs have been sketched but lightly, and many interesting topics have been omitted. It is hoped that it will serve to popularize the subject and that it will induce mathematicians (particularly of this country) to include it more often in their research activities and curricula. It is felt that the Heaviside notation will persist side by side with the modern interpretations by virtue of its simplicity and charm.

THE DIAGRAM METHOD FOR DETERMINANT EXPANSIONS

By D. G. BOURGIN, University of Illinois

The literature is rich in special methods for deriving particular determinant expansions. Perhaps the most effective formal developments are associated with the tensor density* $\epsilon^{ij\cdots kl}$. However, these various procedures obscure to some extent the exact nature of the developments obtained; and, except in the case just mentioned, seem rather of the nature of verifications than motivated general methods. This note presents a simple scheme based on Taylor's theorem and indeterminate multipliers. The principles involved are, of course, well known in analysis, but the power of their combination in determinant theory seems to have been overlooked.† Incidentally, there is presented the notion

* Veblen: *Invariants of Quadratic Differential Forms.*

† Cf., for instance, Muir: *Theory of Determinants*; vols. I–IV; and Muir-Metzler: *Theory of Determinants.*

of the "diagram" of a determinant expansion. The diagram provides a classification and suggests a direction for further systematic investigation of determinants and their expansions.

The basic idea is, merely, that if *arbitrary* terms are given λ multipliers, the determinant may be expanded in the form

$$\Delta = \Delta(\lambda)\Big|_{\lambda=1} = \Delta(0) + \cdots \frac{1}{n!}\frac{\partial^n \Delta(0)}{\partial\lambda^n}, \tag{1}$$

$(1/i!)(\partial^i\Delta(0)/\partial\lambda^i)$ represents the sum of all the different determinants arising from the differentiation of i distinct rows (columns). The choice of rows (columns) as the exclusive groups differentiated for $(1/i!)(\partial^i\Delta(0)/\partial\lambda^i)$ of course leaves entirely arbitrary the choice made for any other term $(1/i!)(\partial^i\Delta(0)/\partial\lambda^i)$ in the expansion. The point of the method is the disposition of these multipliers. The "diagram" characterizing a particular type of expansion is defined as being the λ distribution.

The Cayley* expansion about elements of the main diagonal is immediate. Its diagram, illustrated sufficiently for the fourth order determinant, is

$$\begin{vmatrix} \lambda & \cdot & \cdot & \cdot \\ \cdot & \lambda & \cdot & \cdot \\ \cdot & \cdot & \lambda & \cdot \\ \cdot & \cdot & \cdot & \lambda \end{vmatrix}.$$

Hence

$$\Delta = \Delta(0) + \frac{\partial}{\partial\lambda}\Delta(0) + \cdots \frac{1}{4!}\frac{\partial^4\Delta(0)}{\partial\lambda^4}$$

$$= \begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \begin{vmatrix} \cdot & 0 & 0 & 0 \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ 0 & 0 & 0 & \cdot \end{vmatrix}$$

$$+ \begin{vmatrix} \cdot & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \begin{vmatrix} \cdot & 0 & 0 & 0 \\ \cdot & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \begin{vmatrix} \cdot & 0 & 0 & 0 \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ 0 & 0 & 0 & \cdot \end{vmatrix} + \begin{vmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{vmatrix} + \cdots$$

where the dots indicate unchanged elements.

The Laplace† development has as its associated diagram r rows (columns) multiplied by λ. The differentiations are carried out according to columns (rows).

---

* Muir-Metzler: p. 107; ibid

† Muir-Metzler: p. 93; ibid.

Evidently developments in terms of any arbitrary grouping of elements are immediate by this method. Most of the customary determinant expansions have a homogeneity character, namely: in the resulting sum each term consists of a product of r elements multiplied by a product of $n-r$ elements. Perhaps the method is seen to best advantage in the example below where the homogeneity condition is waived and the diagram is decidedly unsymmetrical:

$$\Delta(\lambda)=\begin{vmatrix} \lambda & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \lambda & \lambda & \cdot & \cdot \\ \cdot & \lambda & \lambda & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \rightarrow \begin{vmatrix} 0 & \cdot & & & \cdot \\ \cdot & & & & \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & & & \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 \\ \cdot & 0 & 0 & & \\ \cdot & \cdot & \cdot & & \end{vmatrix}$$

$$+\begin{vmatrix} 0 & & & & \\ \cdot & & & & \\ \cdot & 0 & 0 & & \\ 0 & & & 0 & 0 \\ \cdot & & & & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & 0 & 0 & 0 & 0 \\ \cdot & & & & \\ 0 & \cdot & \cdot & 0 & 0 \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & 0 & 0 & 0 & 0 \\ \cdot & & & & \\ \cdot & 0 & 0 & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \\ 0 & \cdot & \cdot & 0 & 0 \\ 0 & \cdot & \cdot & 0 & 0 \\ \cdot & & & & \cdot \end{vmatrix} +,$$

when the rows are differentiated. If the columns are differentiated, one gets

$$\begin{vmatrix} 0 & & & & \\ \cdot & & & & \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & & & & \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & & & \\ 0 & \cdot & & & \\ 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot \\ 0 & \cdot & & & \end{vmatrix} + \begin{vmatrix} 0 & 0 & \cdot & & \\ \cdot & 0 & & & \\ & \cdot & 0 & \cdot & \cdot \\ & \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & & & \end{vmatrix} + \begin{vmatrix} 0 & \cdot & 0 & & \\ & \cdot & 0 & & \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ & \cdot & 0 & & \end{vmatrix}$$

$$+\begin{vmatrix} \cdot & 0 & & & \\ 0 & 0 & & & \\ 0 & \cdot & 0 & \cdot & \cdot \\ 0 & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & & & \end{vmatrix} + \begin{vmatrix} \cdot & & 0 & & \\ 0 & & 0 & & \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & & 0 & & \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \end{vmatrix} +.$$

In accordance with an earlier remark, we may differentiate rows (columns) for $\partial\Delta/\partial\lambda$ and columns (rows) for $\partial^2\Delta/\partial\lambda^2$ above. In general, if q of the terms $\partial\Delta(0)/\partial\lambda\cdots\partial^n\Delta(0)/\partial\lambda^n$ differ from 0, there are 2^q equivalent expansions of the given type (i.e. same diagram).

For each λ diagram there is suggested in a natural way a complementary diagram in which the λ factors are now adjoined to those terms previously free of such multipliers and vice versa. If we divide each row of $\Delta(\lambda)$ by λ and replace $1/\lambda$ by $\bar{\lambda}$ there results

$$\Delta(\lambda) \equiv \lambda^n \Delta'(\bar{\lambda}) \tag{2}$$

where Δ' indicates that the complementary diagram for the λ is involved. From this identity follows at once

$$\frac{1}{i!} \frac{\partial^i \Delta(0)}{\partial \lambda^i} \equiv \frac{1}{n-i!} \frac{\partial^{n-i} \Delta'(0)}{\partial \lambda^{n-i}}. \tag{3}$$

Hence the expansions (cf. Equation 1) associated with two complementary diagrams are identical in detail except for inversion of term order.

Incidental to the paper's purpose it is of interest to put the question as to the number of expansions of different types possible for the general nth order determinant. There is, of course, ambiguity regarding the criterion for distinction of types, but on the basis of the simple considerations of this note, any natural rendering would proceed from the notion of the diagram. A broad definition would, in fact, identify type as diagram and accordingly, bearing in mind the essential equivalence of two complementary diagrams, the enumeration gives

$$\tfrac{1}{2}\sum_1^{n^2-1} {}_{n^2}C_i \quad \text{or} \quad 2^{n^2-1} - 1 \tag{4}$$

It would perhaps be preferable to count only those diagrams whose symmetry properties are distinct according to an arbitrary criterion. For instance, one may plausibly refuse to differentiate among diagrams consisting of a single λ column. Furthermore it would be natural to invoke the evident duality between row and column—such considerations would lead us afield. The estimate given in Equation 4 is evidently an upper limit.

It is expected that the adaptations to the three and n way determinants and permanents would evidence the utility of the "diagram" method even more strikingly.

ON THE CONVERSE OF FERMAT'S THEOREM

By D. H. LEHMER, Lehigh University

In many research problems in the theory of numbers it is desirable to have a rapid, universal and positive method for the identification of prime numbers. Thus far the most successful tests for the primality of numbers of no special form depend on the converse of the following theorem of Fermat:

THEOREM 1. *If a is any integer, and p a prime, then $a^p - a$ is divisible by p.*

Nearly 25 centuries ago the Chinese,* who discovered this theorem in the case $a=2$, stated also the strict converse,

THEOREM 2. *If n divides $2^n - 2$, then n is a prime.*

An example disproving this theorem was not discovered until 1819, when

* Messenger of Math., vol. 27 (1897–8), p. 174.

Sarrus* noted that $2^{341}-2$ is divisible by $341=11\cdot 31$. Since then, many writers† have given further exceptions to Theorem 2. In fact infinitely many such exceptions exist.‡ The first valid converse of Fermat's theorem was given by Lucas§ in 1876 who later‖ modified it to the following

THEOREM 3. *If n divides a^x-1 for $x=n-1$, but not for x a proper divisor of $n-1$, then n is a prime.*

Further modifications of this theorem as regards both hypothesis and conclusion are necessary before practical tests¶ for primality emerge. However in all cases it is necessary to know some factor (the larger the better) of $n-1$ and this requirement may be rather serious in many cases.

Some years ago it occurred to the writer that it might be feasible to compile a list of all important exceptions to Theorem 2, which lie below some high limit. Armed with such a list the arithmetician would be able to use Theorem 2 as a test for primality. An unimportant exception would be one whose composite character is easily recognized. Such a number would either lie below 10^7, the limit of published factor tables and list of primes,** or else would have a small factor. A rather arbitrary, but fairly reasonable definition of the word "small" is $\leqq 313$. With this definition, existing congruence tables, on which the search for exceptions to Theorem 2 is based, set the high limit mentioned above at 100 million.

More than 90% of the 90 million numbers between 10^7 and 10^8 are multiples of small primes $\leqq 313$. A given 8-digit number can be completely tested for such small factors in about 7 minutes, if use is made of a calculating machine. Hence the numbers below 100 million which offer any real difficulty comprise a set S of about 8,700,000 numbers whose least factor exceeds 313. Incidentally S contains precisely 5,096,885 primes.

In 1926 P. Poulet†† completed a table of all exceptions to Theorem 2 less than 50 million. An examination of the table showed that only 310 of its 1511 entries belong to S and their frequency is relatively low near the end of the table. Encouraged by these facts the present writer made independantly the list presented herewith of the only members of S for which Theorem 2 is false. There are 526 entries in all. With each entry is given its smallest divisor. This table has been compared with that of Poulet.

---

* Annales de Math., vol. 9 (1818–9), p. 320.

† See Dickson, *History of the Theory of Numbers*, vol. 1, pp. 92–95.

‡ For example, let $n=pq$ where p is a primitive factor of 2^k-1 and q is a primitive factor of 2^k+1, k being any odd integer>1. Then 2^n-2 is divisible by n.

§ Assoc. Franç. Avanc. Sc., vol. 5 (1876), p. 6. Amer. Jour. Math., vol. 1 (1878), p. 302.

‖ *Théorie des Nombres*, Paris (1891), p. 441.

¶ Bull. Amer. Math. Soc., vol. 33 (1927), pp. 327–340; vol. 34 (1928), pp. 54–56.

** Carnegie Institution of Washington, Publications 105 and 165.

†† Sphinx-Œdipe, vol. 23 (1928).

TABLE OF COMPOSITE SOLUTIONS n OF FERMAT'S CONGRUENCE $2^n \equiv 2 \pmod{n}$ AND THEIR SMALLEST PRIME FACTOR p

n	p	n	p	n	p	n	p	n	p
10031653	2833	14796289	457	19020191	2671	25909453	3943	32676481	2161
10084177	2593	14899751	3343	19328653	3109	26377921	1549	32701297	3617
10266001	401	15101893	829	19404139	2011	26470501	4201	33146717	1487
10323769	349	15139199	2383	19471033	883	26821601	521	33298337	1217
10331141	691	15188557	1949	19607561	587	26877421	541	33600533	3347
10386241	1861	15268501	2089	19734157	937	26886817	2593	33627301	3217
10402237	1613	15479777	587	19985269	3061	26977001	4241	33704101	2053
10403641	2281	15525241	1609	20081953	3169	27108397	4657	33840397	2909
10425511	2441	15583153	3121	20117467	2243	27219697	2609	34003061	2381
10505701	1621	15621409	2113	20234341	3181	27271151	503	34100821	2833
10610063	2351	15757741	1621	20261251	2251	27279409	1249	34581457	1009
10712857	2143	15802681	1103	20417311	367	27331921	3697	34856167	2287
10763653	409	15976747	3739	20647621	1429	27380831	1511	34890481	4177
10974881	1913	15978007	1999	20770621	1861	27392041	2341	35498467	3331
11115037	971	16070429	1637	21303343	1999	27409541	3023	35576599	1609
11335501	1321	16132321	613	21306157	3769	27491237	2203	35820937	2677
11367137	2753	16324001	2333	21355951	2311	27492581	2141	35851037	4889
11541307	1699	16349477	1699	21359521	1009	27509653	3709	36255451	3011
11585293	2153	16360381	541	21397381	2671	27664033	3037	36307981	4261
12032021	2003	16435747	907	21400481	1889	27798461	3229	36448387	3019
12263131	811	16705021	1669	21623659	1163	27808463	4567	36721021	2857
12273769	1873	16717061	3541	21654533	2687	28011001	4001	36724591	2143
12322133	2027	16773121	433	21907009	3457	28325881	3301	36919681	2297
12327121	3511	16818877	1609	22087477	709	28406953	3769	36974341	3511
12376813	389	16822081	397	22137809	3137	28527049	2389	36981601	3041
12498061	1021	16853077	2053	22215961	521	28572961	1483	37109467	2833
12659989	1453	16879501	1453	22351249	577	29143633	2017	37376509	1579
12711007	1783	16973393	3761	22369621	2731	29214541	541	37439201	3533
12854437	823	17116837	2069	22591301	3881	29581501	4441	37469701	3061
13057787	467	17134043	1097	22669501	2381	30058381	3877	37962541	4357
13073941	2557	17208601	1151	22953673	2143	30185569	1657	38010307	3083
13295281	1489	17327773	2633	23247901	3251	30219757	2749	38118763	5347
13338371	3163	17375249	3229	23261713	4177	30295141	1741	38210323	2473
13446253	509	17405537	1753	23315977	2857	30388753	1103	38404501	601
13448593	1297	17590957	1033	23464033	2797	30529693	4177	38439523	6043
13635289	739	17759681	1217	23577497	1619	30662497	1013	38903287	3119
13694761	2617	17777191	1553	23734901	2297	30740417	827	38971661	2549
13747361	2141	17870561	2441	23822329	1993	30881551	4201	39016741	1289
13773061	2143	18067501	1607	23828017	2143	30894307	971	39052333	2857
13856417	2689	18073817	3881	23872213	1093	31040833	3217	39117439	1399
13991647	2143	18366937	1429	23963869	2617	31150351	2791	39465091	1571
13996951	1871	18443701	3037	25080101	953	31166803	1777	39512773	1063
14026897	1873	18454921	373	25150501	1151	31436123	1619	39655153	1747
14179537	2917	18487267	571	25276421	2903	31735621	1699	39684157	1627
14282143	2857	18535177	2153	25326001	2251	31759121	2657	40325041	3889
14324473	1693	18653353	2857	25457833	4513	31766983	2857	40361197	6043
14589901	1801	18736381	3061	25629913	1013	32091781	3271	40629601	2851
14709241	631	18740971	1531	25696133	2927	32095057	2833	40778989	733
14794081	2221	18985627	2179	25768261	1531	32168117	2383	40782589	1597

n	p	n	p	n	p	n	p	n	p
40827473	953	48064021	4003	59953741	2927	69128641	2017	80142761	7753
40987201	1601	48191653	4909	59955331	1291	69176647	4159	80146909	919
41121433	1283	48269761	617	59999011	4297	69228967	4297	80375707	4483
41262073	457	48316969	3109	60352921	3313	69231061	1861	80556337	3061
41568101	1051	48448661	5581	60547831	1471	69485281	4813	80687881	937
41604109	1613	48462301	1861	60566431	1231	69612061	1499	80891009	353
41642681	761	48506581	1033	60581401	601	69678307	1867	80918281	6361
41662297	2887	48551161	1933	60696661	3181	69705529	5113	81433591	1303
41840809	3061	48563089	3361	60738257	6037	69885649	3739	81480601	1321
41866001	881	48656869	2393	60998653	397	69917371	5531	81954133	5227
41987111	1871	49075417	6553	61201009	3499	70149631	1423	82139471	1511
42009217	641	49303801	2341	61219789	1429	70461331	2099	82279741	461
42344609	2657	49411801	1553	61754941	5557	70593931	2971	82452061	6421
42485119	6361	49664207	1439	61794709	2089	70626301	6353	82870517	7433
42623017	3541	50155733	1657	61832377	3517	70695769	3433	82882753	5441
42697873	2017	50376019	3257	62289541	5581	70728121	811	83204801	3041
42702661	4621	50523661	2687	62633371	1979	72498253	2693	84164033	5297
42709591	2311	51129781	2341	63001801	6481	72543547	4259	84421081	1511
42984589	2677	51283501	3581	63065281	2203	72595951	6529	84487457	479
43224397	887	51302353	1987	63167743	4409	72680431	2791	84998503	1279
43363601	5101	51500521	1741	63318169	3559	72805981	1613	85207669	5653
43661257	2203	51627817	3593	63328469	467	72884701	6037	85328717	727
43798457	2417	52072021	1021	63346999	1327	73562833	457	85400701	4621
44314129	1489	52119289	3229	63388033	4597	73645001	2861	85519337	3083
44465221	2011	52181407	5711	63781901	6521	74217487	1759	85759147	6353
44482901	3851	52204237	3613	64009261	421	74411131	6521	86027329	2797
44671001	2729	53154337	5953	64148717	1283	74927161	6121	86114117	1013
44912701	421	53283169	6529	64605041	6961	75140137	3877	86438857	4649
45100177	2239	53542147	3659	64735897	6361	75143251	6553	86484269	3797
45175201	3361	53656021	2377	65144501	3391	75187669	2857	86999837	4817
45219329	5209	53675623	6553	65254393	3613	75565873	5861	87211573	853
45414433	379	54029741	1733	65301013	1429	76725091	1171	87499651	2339
45563027	563	54449431	4519	65350801	2161	76745101	1201	87558127	4679
45819541	2141	55109401	1061	65359477	2557	76839733	1019	87694261	1997
45879941	3911	55176097	929	65565457	4049	76923461	3581	88099339	2347
46055851	1283	55318957	6073	66096253	5749	77295961	6217	88256449	2833
46094401	4801	55610837	6089	66296401	1201	77533123	499	88368853	3433
46256489	2777	55729957	3733	66790057	6673	77594653	6229	88661861	1319
46325029	5381	56420033	4337	66849091	331	77648941	3331	88735921	2081
46386589	1759	56479897	2377	66932851	4091	77812153	2447	88930463	8167
46469809	3049	56687821	853	67194401	4733	77817979	3529	89244901	6529
46517857	5569	56810137	3769	67642513	3109	78206917	2797	89308771	5113
46657181	2789	58003213	2879	67928221	487	78671651	1811	89784581	5471
46679761	2161	58422409	3331	68075177	457	78795181	6277	90014653	6709
47220367	1087	58449847	3823	68102641	3691	78939089	3391	90270613	919
47253781	4861	58509977	1933	68165761	3121	79398901	6301	90665789	2459
47356171	1987	58679941	4423	68462551	5851	79417801	2971	90803429	6353
47744209	593	58755877	1759	68512867	4139	79464533	5147	91433281	5521
47903701	3461	59631211	1931	68621701	4783	79539197	3257	91587289	2393
47918581	5233	59840537	2579	68839597	6427	79786523	2579	92139433	1753
47930023	1999	59913157	457	69030901	2221	79854409	1753	92645437	4813

n	p	n	p	n	p	n	p	n	p
93431521	5581	94502701	4861	96904081	6961	97676723	1427	98735209	4057
93541537	397	95451361	5641	96925921	2731	97796953	4423	99115297	3319
93571633	1009	95452781	3989	97255801	1801	97863529	1979	99486889	643
93591721	6841	96618397	6217	97496449	3221	97924217	3299	99789673	571
93926197	8191	96888641	8681	97655933	5413	98523877	5437	99898801	8161
								99945007	4999

The application of this table is illustrated by the example*: Let $n = 38417699$. In the first place we note that this number is not in the table. Next we test n for small factors. For factors >13 it is best to proceed as follows. The product† of all primes p for which $13 < p \leqq 101$ is

$$61259163450953351685331342103097 1751.$$

The remainder on division of this number by $n = 38417699$ is 12213651 which is prime‡ to n. Hence n has no factor $\leqq 101$. Proceeding in the same way with

$$\prod_{103 \leqq p \leqq 199} (p) = 33495846626702153279106918423727531393773883$$

and

$$\prod_{211 \leqq p \leqq 313} (p) = 7830453835247682568266635066915629203174553741$$

we find in less than 8 minutes that n has no small factor. We finally determine the remainder on division of $2^n - 2$ by n. It is best to carry out this calculation as follows

k	$2^k \pmod n$	k	$2^k \pmod n$
38417699	2	18758	19481823
19208849	38417698	9397	58510488
9604424	15330547	4689	39810844
4802212	2114190	2344	298964
2401106	34240341	1172	12770894
1200553	41191813	586	6505293
600276	14017841	293	70634070
300138	1135428	146	16552223
150069	16487276	73	64619142
75034	17260691	36	28630924
37517	64435532	18	262144

Each entry in the column k is the greatest integer in half the preceding entry,

* This number is a divisor of the 83rd term of the sequence 720, 1698, 1710, $\cdots$, in which each term is the sum of the aliquot parts of its predecessor.

† This large number and the two following should be separated for convenience into periods of 8, 9, or 10 figures according to the keyboard capacity of the calculating machine.

‡ The application of the Euclid's algorithm can be made rapidly without putting pen to paper by always retaining the two latest remainders in the product register.

the first entry being n. The other column is now constructed from the bottom entry $262144 = 2^{18}$ by successive squaring. Each square is divided by n and the remainder, or its double in case the corresponding entry k is odd, is the next entry. In this way the above table can be completed in 15 minutes. Since on arriving at the top, where $k = n$, we obtain the entry 2, we see that $2^n - 2$ is divisible by n. But n belongs to S and is not in the table. Hence n is a prime.

If $2^n - 2$ were not divisible by n, then by Theorem 1, n would be composite and some other method* would have to be applied if the factors of n were needed

Method of Constructing the Table.† The composite members of S may be separated into 3 classes.

Class I. Numbers having square factors >1.

Class II. Products of two distinct primes.

Class III. Products of three‡ distinct primes.

The following theorem disposes of Class I.

THEOREM 4. *If n is a member of Class I for which $2^n - 2$ is divisible by n, then* $n = 12327121$.

Proof. Let $n = p^2 m$ where p is an odd prime. Let ϵ be the exponent to which 2 belongs modulo p. Then since n divides $2^n - 2$ we have

$$2^{n-1} = 2^{p^2 m - 1} \equiv 1 \pmod{p^2}. \tag{1}$$

Hence we may write

$$p^2 m - 1 = h\epsilon \tag{2}$$

where h is an integer.

Now let $2^\epsilon = 1 + kp^\alpha$, where k is prime to p. If $\alpha = 1$, we have by (2)

$$2^{n-1} = 2^{p^2 m - 1} = (2^\epsilon)^h = (1 + kp)^h \equiv 1 + hkp \pmod{p^2}.$$

Hence by (1) we conclude that p divides h. But this contradicts (2). Therefore $\alpha \geqq 2$. Writing $p - 1 = \epsilon\delta$ we have

$$2^{p-1} = (2^\epsilon)^\delta \equiv 1 \pmod{p^\alpha}. \tag{3}$$

But the only solutions§ $p < 14000$ of (3) even when $\alpha = 2$ are $p = 1093$ and 3511. Now since $n = p^2 m$ is a member of S we have either $m = 1$ or $m > 313$. The latter alternative violates the inequality $n < 10^8$, while the former violates $n > 10^7$, except when $p = 3511$. In this case $n = p^2 = 12327121$ and $2^n - 2$ is divisible by n since

$$2^{n-1} = 2^{p^2 - 1} = (2^{p-1})^{p+1} \equiv 1 \pmod{p^2}.$$

* The best method to use for numbers of this size is Legendre's quadratic residue method as simplified by D. N. Lehmer's factor stencils. For other methods see Dickson, loc. cit., ch. 14.

† For the properties of the binomial congruence used in what follows the reader may consult Mathews Theory of Numbers, ch. 1.

‡ The product of 4 or more primes each greater than 313 would exceed 10^8.

§ Proved by Beeger; Messenger of Math., vol. 55, (1925), pp. 17–26.

Hence $n=12327121$ is the only member of Class I which belongs to our table.

The numbers of Class II which belong in the table were found by applying the following theorem.

THEOREM 5. *If p and q are odd distinct primes, then $2^{pq}-2$ is divisible by pq if and only if $p-1$ is divisible by the exponent to which 2 belongs modulo q and $q-1$ is divisible by the exponent to which 2 belongs modulo p.*

Proof of Necessity. Let $2^{pq}-2$ be divisible by pq, then by Fermat's theorem

$$0 \equiv 2^{pq} - 2 \equiv 2^q - 2 \pmod{p}.$$

Hence $q-1$ is divisible by the exponent to which 2 belongs modulo p. Interchanging p and q in the argument we complete the proof of necessity.

Proof of Sufficiency. Let the exponents to which 2 belongs modulo p and modulo q divide $q-1$ and $p-1$ respectively. Then

$$2^{p-1}\cdot 2^{q-1} \equiv 1 \pmod{pq}. \tag{4}$$

Then by Euler's generalization of Fermat's theorem and by (4) we may write

$$0 \equiv 2^{\phi(pq)+1} - 2 = 2^{pq}\cdot 2^{p-1}\cdot 2^{q-1} - 2 \equiv 2^{pq} - 2 \pmod{pq}.$$

Hence the condition is sufficient.

In applying this theorm extensive use was made of the tables* of Cunningham and Woodall and Kraitchik giving the exponents to which 2 belongs modulo p for $p<300000$, and also the tables† of the factors of 2^k-1. Altogether there are 525 members n of Class II for which 2^n-2 is divisible by n.

Class III was disposed of with the help of

THEOREM 6. *If p, q, and r are distinct odd primes and if $2^{pqr}-2$ is divisible by pqr, then $r(p+q-1)-1$ is divisible by the least common multiple of the exponents to which 2 belongs modulo p and modulo q.*

Proof. Since $2^p\equiv 2 \pmod{p}$ we have

$$2^{pqr} \equiv 2^{qr} \equiv 2 \pmod{p}.$$

Hence if ϵ is the exponent to which 2 belongs modulo p we have

$$p - 1 \equiv 0 \pmod{\epsilon} \tag{5}$$

$$qr - 1 \equiv 0 \pmod{\epsilon}. \tag{6}$$

---

* Cunningham and Woodall, Messenger of Math., vols. 37, 42, 44, and 45. These tables are for $p<100000$.

Kraitchik, *Recherches sur la Théorie des Nombres*, Paris (1924), Table 1, pp. 131–191. This table is for $p<300000$. Errata in this table for $p<100000$ have been tabulated by Cunningham and Woodall Messenger of Math., vol. 54, p. 70.

† Cunningham and Woodall. Factorization of $y^n\pm 1$, London (1925), pp. 1–9, as augmented by numerous contributions from various writers since 1925.

Multiplying (5) by r and adding (6) we see that ϵ divides $r(p+q-1)-1$. But this is symmetric in p and q, hence the theorem.

There are 431 pairs of primes (p, q) for which a prime r can be found such that $313<p<q<r$ and $10^7<pqr<10^8$. Theorem 6 was used in each of these cases to restrict the form of r. In only a few cases was an r found to satisfy this theorem. In these cases p and r were interchanged and the theorem applied again. As a result* no member n of the Class III divides 2^n-2. This means that all the entries in the table are products of two primes. Therefore the set S does not contain one of those remarkable composite numbers n, like 561, for which a^n-a is divisible by n for every a. This follows from a known theorem† to the effect that such a number must be the product of at least three distinct primes.

ZEROS OF THE HERMITIAN POLYNOMIALS

By E. R. SMITH, Iowa State College

The Hermitian polynomials of this paper are defined by the relation

$$(1) \qquad H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2})$$

and are of the form

$$H_n(x) = x^n - \frac{n(n-1)}{2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} x^{n-4} - \cdots .$$

They satisfy the relations

$$(2) \qquad H_n(x) - xH_{n-1}(x) + (n-1)H_{n-2}(x) = 0,$$

$$(3) \qquad H_n'(x) = nH_{n-1}(x),$$

$$(4) \qquad H_n''(x) - xH_n'(x) + nH_n(x) = 0.$$

These polynomials are the Hermitian polynomials as given by Jahnke and Emde,‡ and differ slightly from those given in Hermite's original paper.§ He defined

$$U_n(x) = e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

The relation between $H_n(x)$ as defined above and Hermite's $U_n(x)$ is

$$U_n(x) \equiv (-\sqrt{2})^n H_n(\sqrt{2}x),$$

---

* This result has been confirmed by Poulet in a recent letter.

† See for instance Carmichael: this MONTHLY, vol. 19, (1912), pp. 22–27. More than 200 of these numbers are known. All but a very few of them were found by Poulet.

‡ *Funktionentafeln mit Formeln und Kurven*, 2nd edition, 1933, p. 105.

§ Hermite, *Sur un nouveau développement en séries des fonctions*, Comptes Rendus, 58 (1864), pp. 93–100; pp. 266–273.

and hence the zeros of the original $U_n(x)$ would be $1/\sqrt{2}$ times the zeros of the $H_n(x)$ of this paper.

The "functions of the parabolic cylinder" are closely connected with the Hermitian polynomials* by the equation

$$u_n(x) = e^{-x^2/4}H_n(x) \tag{5}$$

and they satisfy the differential equation

$$u_n''(x) + \left(n + \frac{1}{2} - \frac{x^2}{4}\right)u_n(x) = 0. \tag{6}$$

It is obvious that $H_n(x)$ and $u_n(x)$ vanish for the same values of the variable.

The reader is referred to a paper by W. Hahn† for a summary of the work which has been done on the location of the roots of the Hermitian polynomials. A method of computation, different from the one used here, has been suggested by Zernike‡ and he derived a formula for obtaining an approximation for the largest root, which is very good for large values of n. The author has not been able to find any list of numerical values of the roots of the Hermitian polynomials and apparently none have been computed except for special cases.

By use of the methods due to Sturm it will be shown that if $r_{n,i}$ and $r_{n,i+1}$ are successive zeros of (5) then

$$r_{n,i+1} - r_{n,i} = \frac{2\pi}{\sqrt{4n+2-\mu^2}},$$

where μ is an appropriately chosen value between $r_{n,i}$ and $r_{n,i+1}$.

Consider the equation

$$v_n''(x) + k^2 v_n(x) = 0, \tag{7}$$

where k is a constant whose value will be determined by later considerations. From (6) and (7) it follows that

$$\frac{d}{dx}(v_n u_n' - v_n' u_n) + \left(n + \frac{1}{2} - k^2 - \frac{x^2}{4}\right)v_n u_n = 0. \tag{8}$$

For a particular solution of (7) take

$$v_n(x) = \sin k(x - r),$$

where r and k are chosen so that $r + \pi/k$ are successive roots of $u_n(x) = 0$. On

---

* Whittaker, *On the functions associated with the parabolic cylinder in harmonic analysis*, Proceedings of the London Mathematical Society, 35 (1903), pp. 417–427.

† W. Hahn, *Bericht über die Nullstellen der Laguerreschen Polynome*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 44 (1934), pp. 215–236.

‡ Zernike, *Eine asymptotische Entwickelung für die grösste Nullstelle der Hermiteschen Polynome*, Proc. Amsterdam, vol. 34 (1931), p. 673.

integrating (8) between the limits r and $r+\pi/k$ we have

$$\int_r^{r+\pi/k}\left(n+\frac{1}{2}-k^2-\frac{x^2}{4}\right)u_n(x)\sin k(x-r)dx = 0.$$

Since $u_n(x)$ and $\sin k(x-r)$ do not change signs in the range of integration it follows from the mean value theorem of integral calculus that

$$\left(n+\frac{1}{2}-k^2-\frac{\mu^2}{4}\right)\int_r^{r+\pi/k}u_n(x)\sin k(x-r)dx = 0,$$

where μ is a constant such that $r<\mu<r+\pi/k$. Consequently

$$k = \tfrac{1}{2}\sqrt{4n+2-\mu^2}$$

and

$$\frac{\pi}{k} = \frac{2\pi}{\sqrt{4n+2-\mu^2}} \tag{9}$$

is the interval between the successive zeros of (5) and also the successive zeros of (1).

In order to obtain numerical values for the zeros of $H_n(x)$ a first approximation for small values of n may be obtained by means of (9), using empirical values of μ. The required values of μ do not differ greatly from the zeros of $H_{n-1}(x)$ which lie between the successive zeros of $H_n(x)$. While this procedure is entirely general it is easier for relatively large values of n to obtain the approximations by extrapolation from the zeros of the polynomials corresponding to $n-1$, $n-2$, $\cdots$, by well known difference methods.

Better values may be obtained from these first estimates by Newton's method of successive approximations. However, direct application of the method is not feasible except for comparatively small values of n. An adaptation of the method more suitable for machine calculation may be obtained by means of relations (2) and (3).

If an approximation of a zero of the polynomial is given by α then the correction obtained by Newton's method is

$$h = -\frac{H_n(\alpha)}{H_n'(\alpha)}.$$

Using (3) this may be reduced to

$$h = -\frac{H_n(\alpha)}{nH_{n-1}(\alpha)}.$$

By means of (2) we obtain

$$\frac{H_n(\alpha)}{H_{n-1}(\alpha)} = \alpha - \frac{(n-1)H_{n-2}(\alpha)}{H_{n-1}(\alpha)} = \alpha - \frac{H_{n-1}'(\alpha)}{H_{n-1}(\alpha)}.$$

Consequently

$$h = -\frac{\alpha}{n} + \frac{H'_{n-1}(\alpha)}{nH_{n-1}(\alpha)}. \tag{10}$$

Let $\pm r_{n-1,1}$, $\pm r_{n-1,2}$, $\cdots$ be the roots of $H_{n-1}(x)=0$. Then

$$H_{n-1}(\alpha) = \alpha \prod_{i=1}^{(n-2)/2} (\alpha^2 - r_{n-1,i}^2), \quad \text{for } n \text{ even},$$

$$= \prod_{i=1}^{(n-1)/2} (\alpha^2 - r_{n-1,i}^2), \quad \text{for } n \text{ odd}.$$

Therefore

$$h = \frac{2\alpha}{n}\left(-\frac{1}{2} + \frac{1}{2\alpha^2} + \sum_{i=1}^{(n-2)/2} \frac{1}{\alpha^2 - r_{n-1,i}^2}\right), \quad \text{for } n \text{ even},$$

$$= \frac{2\alpha}{n}\left(-\frac{1}{2} + \sum_{i=1}^{(n-1)/2} \frac{1}{\alpha^2 - r_{n-1,i}^2}\right), \quad \text{for } n \text{ odd}.$$

This formula is well adapted for obtaining the correction by means of mechanical calculators.

In order to check the accuracy of the approximations the relation

$$\sum r_{n,i}^2 = \frac{n(n-1)}{2}$$

may be used.

The approximate values of the zeros correct to six decimal places up to $n=27$ are here given.

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
0.000000	1.000000	0.000000	.741964	0.000000	.616707	0.000000
		1.732051	2.334414	1.355626	1.889176	1.154405
				2.856970	3.324257	2.366759
						3.750440

$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$
.539080	.000000	.484936	.000000	.444403	.000000	.412590
1.636519	1.023256	1.465989	.928869	1.340375	.856680	1.242689
2.802486	2.076848	2.484326	1.876035	2.259464	1.725418	2.088345
4.144547	3.205429	3.581823	2.865123	3.223710	2.620690	2.963037
	4.512746	4.859463	3.936166	4.271826	3.563444	3.886925
			5.188001	5.500902	4.591398	4.896936
					5.800167	6.087409

$n=15$	$n=16$	$n=17$	$n=18$	$n=19$	$n=20$	$n=21$
.000000	.386761	.000000	.365246	.000000	.346964	.000000
.799129	1.163829	.751843	1.098395	.712085	1.042945	.678046
1.606710	1.951980	1.509883	1.839780	1.428877	1.745246	1.359765
2.432437	2.760245	2.281020	2.595834	2.155503	2.458664	2.049102
3.289082	3.600874	3.073797	3.374737	2.898051	3.189015	2.750593
4.196208	4.492955	3.900066	4.188020	3.664417	3.943967	3.469847
5.190094	5.472226	4.778532	5.054073	4.465873	4.734581	4.214344
6.363948	6.630878	5.744460	6.007746	5.320536	5.578739	4.994964
		6.889122	7.139465	6.262891	6.510590	5.829382
				7.382579	7.619049	6.751445
						7.849383

$n=22$	$n=23$	$n=24$	$n=25$	$n=26$	$n=27$
.331179	.000000	.317370	.000000	.305156	.000000
.995162	.648471	.953422	.622462	.916545	.599354
1.664125	1.299876	1.593480	1.247312	1.531216	1.200683
2.341760	1.957327	2.240468	1.877058	2.151530	1.806045
3.032404	2.624323	2.897729	2.514473	2.780138	2.417684
3.741496	3.305040	3.569307	3.162776	3.420156	3.038150
4.476362	4.004775	4.260384	3.825901	4.075427	3.670473
5.247725	4.730724	4.978041	4.508930	4.750948	4.318418
6.073075	5.493474	5.732747	5.218848	5.453615	4.986906
6.985981	6.310350	6.541675	5.966015	6.193693	5.682761
8.074030	7.214660	7.437891	6.767465	6.988089	6.416155
	8.293386	8.507804	7.656038	7.869427	7.203877
			8.717598	8.923052	8.078350
					9.124421

A NEW METHOD FOR THE EVALUATION OF $\iint_A f(x, y)dydx$

By W. L. MORRIS, Pittsburgh, Pa.

A problem frequently met in engineering practice is the evaluation of

$$\iint_A f(x, y)dydx.$$

If the integrand $f(x, y)$ is one of a somewhat restricted group of functions and if the limits of the area A are simple, evaluation by direct integration is usually possible. In many cases, however, the limits are not simple and the integrand itself may be of such a nature that a method of evaluation of this kind is not possible.

Since the evaluation of such integrals is so frequently necessary, several methods have been devised for their approximate evaluation when the exact evaluation is not feasible. Some of those in most frequent use are: the Prismoidal

Rule; the Trapezoidal Rule; Simpson's Rule; Durand's Rule; the use of special planimeters capable of measuring areal moment and moment of inertia directly.

Another method used is that of transformation of coordinates in an effort to simplify the limits of integration. The application of a transformation of this kind, generally results in a much more complicated integrand. Furthermore, it is only for relatively simple areas that such transformations may be found.

The purpose of this paper is to develop a practical method of evaluating such integrals by means of transformations of the type mentioned, but a method in which the transformations are used to effect an entirely different kind of simplification. Instead of finding transformations by means of which the area A is mapped with simpler limits in the uv plane (which usually results in a more highly complicated integrand), the transformations sought will be the ones which will simplify the integrand itself.

Consider the transformation

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \end{aligned} \tag{1}$$

subject to the condition

$$J\left(\frac{x, y}{u, v}\right) \neq 0.$$

The inverse transformation will be of the form

$$\begin{aligned} u &= u(x, y) \\ v &= v(x, y). \end{aligned} \tag{2}$$

Applying this transformation to $\iint_A f(x, y)dydx$, we obtain

$$\iint_{A'} f[x(u, v), y(u, v)]J\left(\frac{x, y}{u, v}\right)dudv,$$

where A' is the area in the uv plane obtained by mapping A of the xy plane thereon by means of (2).

Now our purpose in effecting this transformation is to simplify the integrand. The simplest integrand possible is, of course, the one which is unity throughout A'.

If we place

$$f[x(u, v), y(u, v)]J\left(\frac{x, y}{u, v}\right) = 1$$

and recall that

$$\frac{1}{J\left(\frac{x, y}{u, v}\right)} = J\left(\frac{u, v}{x, y}\right)$$

we obtain

$$J\left(\frac{u, v}{x, y}\right) = f(x, y). \tag{3}$$

Consequently, if we can find transformations such that (3) is satisfied, our purpose is accomplished. It remains to be shown that it is possible to find such a transformation.

Writing (3) in an expanded form

$$f(x, y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \tag{4}$$

it becomes evident that we may choose $u(x, y)$ almost at will and find a $v(x, y)$. As a matter of fact since (4) becomes a simple partial differential equation after we choose the $u(x, y)$, it is possible to find an infinite number of $v(x, y)$ functions for almost any $u(x, y)$. Thus for any given integrand there is a doubly infinite number of transformations which will accomplish the purpose in question.

It is desirable to use this freedom in a manner which will lead to as simple a transformation as possible. By a method of trial and error fairly simple transformations have been obtained for those types of integrands occurring most frequently in engineering practice. These are as follows:

Integrands	Transformations
$f(x, y) = x^2 + y^2$	(a) $u = \dfrac{x^2 - y^2}{2}, \quad v = xy$
	(b) $u = \dfrac{x^3}{3} + xy^2, \quad v = y$
$f(x, y) = x^2$	(a) $u = \dfrac{x^2}{2}, \quad v = xy$
	(b) $u = \dfrac{x^3}{3}, \quad v = y$
$f(x, y) = x$	$u = \dfrac{x^2}{2}, \quad v = y$
$f(x, y) = xy$	(a) $u = \dfrac{x^2}{2}, \quad v = \dfrac{y^2}{2}$
	(b) $u = \dfrac{x^2y}{2}, \quad v = y$

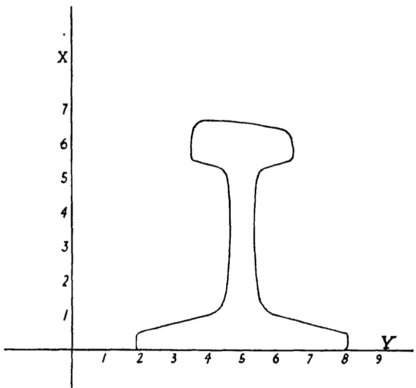


FIG. 1

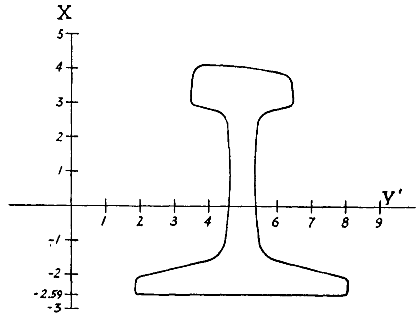


FIG. 3

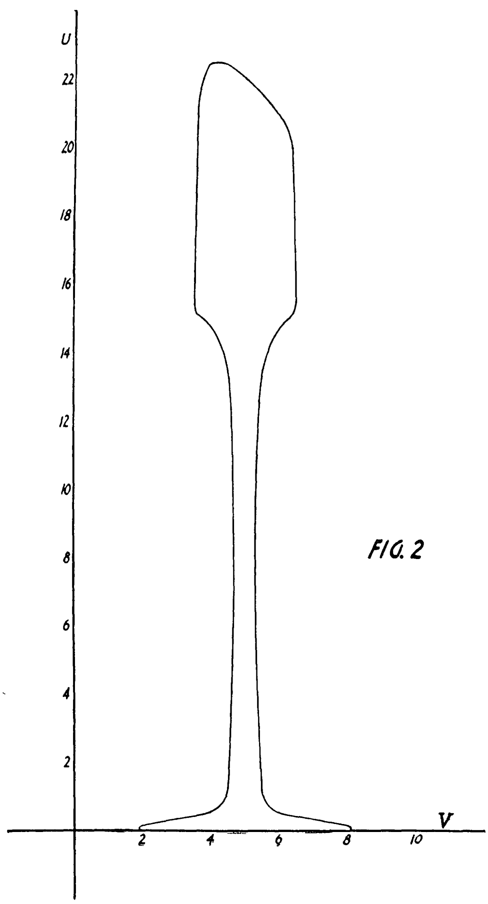


FIG. 2

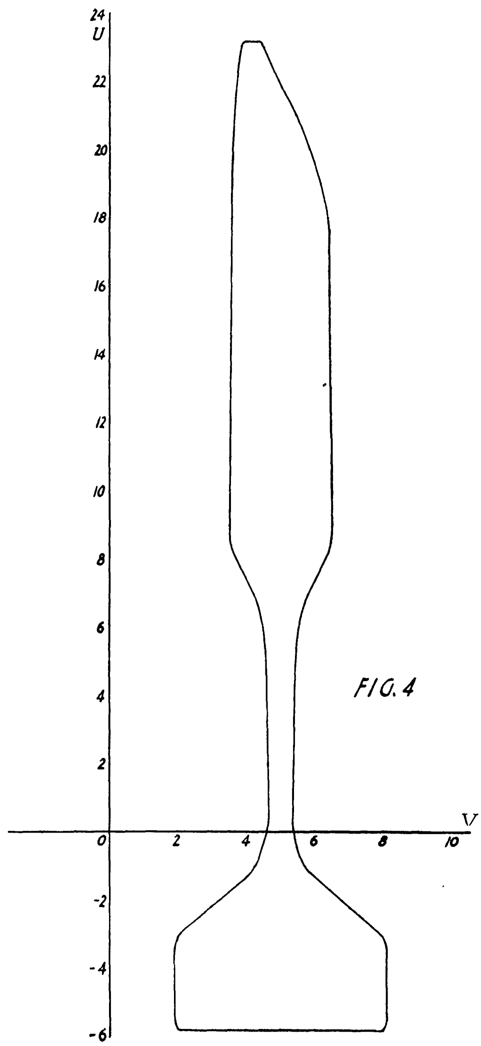


FIG. 4

These transformations are not one-to-one and it is consequently necessary to devise suitable mapping rules. In the cases listed here, such rules are simple and are readily obtained.

It may be noted that, for all except one integrand, there are two transformations listed. Each transformation has its own relative merits and for special application, one may be preferred to another. For example, for $f(x, y)=x^2+y^2$, transformation (b) preserves quadrants and hence maps all of the xy plane on a single coordinate system in the uv plane. For some purposes, this characteristic might cause it to be preferred to transformation (a). The same or similar considerations apply to other transformations.

As an example of the application of the method to an actual engineering problem, consider the cross section of the worn rail (Fig. 1). In order to calculate its bearing capacity, it is necessary to know, among other things, the distance from the base to the center of gravity, or the location of the neutral axis, and the moment of inertia about this axis.

We determine the neutral axis by applying the transformation for $f(x, y)=x$ and obtain Fig. 2 whose area is 34.20 units. The cross sectional area of the rail is 13.20 square inches so the neutral axis is 2.59 inches above the base.

Applying transformation (b) for $f(x, y)=x^2$ to the rail plotted so that its neutral axis coincides with $y=0$ (Fig. 3), we obtain Fig. 4 the area of which is 71.88 units so that we have $\bar{x}=2.59$ inches, and

$$I_{NA} = 71.88 \text{ inches}.$$

The application of the other transformations is similar and this method is believed to be applicable to problems of greater complexity.

Further work along these lines is being undertaken.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

SIMULTANEOUS QUADRATICS SOLVABLE IN QUADRATIC IRRATIONALITIES

By L. C. KARPINSKI, University of Michigan

Most of our college textbooks for freshmen give work on simultaneous quadratic equations in two variables and often also in the texts used in a second course in so-called college algebra this topic is again discussed. However, the fundamental question of the types of simultaneous quadratics which are solvable in terms of quadratic irrationalities is, in general, quite inadequately discussed. While the topic may be treated in connection with the study of Galois resolvents yet the conditions under which the solution of a pair of simultaneous

quadratic equations will reduce to the solution of quadratics are sufficiently simple so that the comprehensive discussions of the conditions can be obtained by elementary considerations.

Given

$$\begin{cases} a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0 \\ a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0 \end{cases} \tag{1}$$

the condition that the quadratic expression,

$$a_1x^2 + 2h_1xy + \cdots + k(a_2x^2 + 2h_2xy + \cdots)$$

should factor into linear factors, reduces to the cubic in k,

$$\begin{vmatrix} a_1 + ka_2, & h_1 + kh_2, & g_1 + kg_2 \\ h_1 + kh_2, & b_1 + kb_2, & f_1 + kf_2 \\ g_1 + kg_2, & f_1 + kf_2, & c_1 + kc_2 \end{vmatrix} = 0. \tag{2}$$

This is, of course, the well known condition that the conic through the intersection of two conics should reduce to pairs of lines through the four points of intersection.

The condition that this cubic should have a root rational in terms of the coefficients of the given equations is the condition that the pair of simultaneous quadratic equations should be solvable in terms of quadratic irrationalities in these coefficients. Thus if one equation is linear either $k=0$ is a root or the cubic reduces to a quadratic; if the equations are of the type

$$\begin{aligned} a_1x^2 + h_1xy + b_1y^2 + k_1(2g_1x + 2f_1y + c_1) &= 0 \\ a_2x^2 + h_2xy + b_2y^2 + k_2(2g_1x + 2f_1y + c_1) &= 0, \end{aligned} \tag{3}$$

then a linear factor in k is obtained. These include practically all the types commonly discussed in American elementary textbooks for freshmen. In any numerical illustration the question of the solution of the simultaneous quadratics reduces immediately to that of the rational root of a cubic.

Incidentally the given expression gives an elementary proof of the fact that the biquadratic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + \alpha_4 = 0$$

reduces to a cubic since the solutions may be obtained as the solutions of the simultaneous quadratics:

$$\begin{aligned} a_0y^2 + 2a_1xy + a_2x^2 + a_3x + a_4 &= 0, \\ y - x^2 &= 0. \end{aligned}$$

These results are sufficiently simple and obvious that it would appear quite certain that they have been given somewhere in the literature of elementary

algebra; I have not found the reference. However, our American elementary textbooks leave entirely erroneous impressions in the minds of pupils about the general conditions under which simultaneous quadratics have an elementary solution. It is desirable, of course, to connect such a discussion as that above with the corresponding topic of geometry concerning problems solvable with ruler and compasses wherein also this reduction to solutions involving only quadratic irrationalities of given constants is involved.

In the appendix (p. 551) fifth edition of Chrystal's *Algebra*, vol. I. (London, 1910), to which my attention was kindly directed by R. E. Gilman of Brown University, one finds in the discussion of Lagrange's solution of the biquadratic, employing the so-called Lagrange Resolvent, the following statement:

"The above method for solving a biquadratic is merely an analytical translation of the geometrically obvious fact that the four points of intersection of two conic sections can be determined by finding the intersections with either of the conics of any one of the three pairs of straight lines which contain all the four points."

Chrystal continues,

"If the cubic resolvent is irreducible, k will involve an incommensurable cube root, and the roots of the biquadratic will not be expressible by means of square roots alone. Hence, in order that the biquadratic may be expressible by means of square roots alone, it is necessary that the cubic resolvent should have a commensurable root, which can always be found as explained" $\cdots$ (elsewhere in Chrystal).

"This condition is obviously sufficient; for, if we use the (rational) value of k, then [the four roots] will all be expressible by means of a single square root $\cdots$; and each of the two quadratics will be soluble by means of an additional square root."

Of course, if the biquadratic with rational coefficients has one or more linear factors, these may be found by a limited number of trials and equally if the cubic has a rational root this may be found by trial.

But the interesting fact is not noted in Chrystal that precisely similar conditions obtain for the condition that a pair of simultaneous quadratic equations in two variables should be solvable in terms of quadratic irrationalities. This same cubic which when solved gives the solution of the fourth degree equation gives also the conditions under which simultaneous quadratic equations in two variables are reducible.

Commonly in our textbooks we find under simultaneous quadratics, the three types:

1. One linear and one quadratic which is solvable immediately by substitution.
2. Two quadratics, each without linear terms; commonly solved by eliminating the constant, although many texts give the substitution of vx for y, and reduce to a quadratic in v by division.
3. Equations one of which is a quadratic (and thus factorable) in some linear function of the two variables involved.

The accidental character of these types is apparent. The treatment leaves unanswered the question as to the conditions that must obtain in order that there should be a solution; in fact most of our college textbooks and high school textbooks do not indicate that, in general, the solution of simultaneous quadratics in two variables (each one having quadratic terms) reduces to a fourth degree equation in one unknown.

The simultaneous quadratics of type 3 above include as a special case with $g_1=g_2=f_1=f_2=0$, the type commonly given in the textbooks as "homogeneous of the second degree." Substituting these values of the coefficients in the cubic for k it appears that the third column (or the third row, but not both) of the determinant will have $k_1+k\cdot k_2$ as a common factor. This gives to k the rational value $k=-k_1/k_2$ and for that value the equations can be solved in terms of quadratic irrationalities.

As an illustration let us take the fourth degree equation which gives four vertices of a regular pentagon; this is the biquadratic factor of x^5-1, set equal to zero. To plot the roots of $x^5-1=0$ you are led to the fourth degree equation, $x^4+x^3+x^2+x+1=0$. The complex roots are given by this equation; when the points are plotted on the complex number diagram, these points with the point (1, 0), corresponding to the real root, give a regular pentagon.

The roots of the fourth degree equations are given by the roots of the system,

$$x^2 + xy + y^2 + x + 1 = 0$$

$$y - x^2 = 0.$$

The condition that

$$x^2 + xy + y^2 + x + 1 + k(y - x^2) = 0$$

should factor into two linear factors is given by the equation,

$$\begin{vmatrix} 1-k & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & k/2 \\ \frac{1}{2} & k/2 & 1 \end{vmatrix} = 0.$$

Expanding this determinant and clearing of fractions we obtain the cubic equation, $k^3-k^2-3k+2=0$, of which the roots are 2 and $(-1\pm\sqrt{5})/2$. Taking the value $k=2$, the polynomial

$$x^2 + xy + y^2 + x + 1 + 2(y - x^2) = 0,$$

factors into the two linear factors,

$$x - \frac{(y+1)(\sqrt{5}+1)}{2} \quad \text{and} \quad x + \frac{(y+1)(\sqrt{5}-1)}{2}.$$

Solving the two systems of equations

$$x - \frac{(y+1)(\sqrt{5}+1)}{2} = 0 \qquad x + \frac{(y+1)(\sqrt{5}-1)}{2}$$

$$y = x^2 \qquad \text{and} \quad y = x^2$$

gives the four complex roots of $x^5-1=0$. It is an instructive exercise to see how this algebraic solution corresponds in many ways to the geometric solutions given by Euclidean geometry.

The cubic in k, numbered (2) above, gives then the answer not simply to the question when the biquadratic is solvable by quadratics, but also to the question which arises even more frequently as to determining whether a given pair of simultaneous quadratics involving only rational coefficients is solvable in terms of quadratic irrationalities, or, in other words, by what our American schools commonly term "high school algebra." The answer is that when the resulting cubic equation has a rational root the simultaneous quadratics are solvable by the elementary methods and the solution by using the above determinant indicates the method of attack.

Since this discussion involves only the expansion of a third order determinant, the formula can be given even to high school students and certainly to freshmen taking the first college course, in algebra. Obviously, too, the use of the formula involves the fact that in an equation given by a polynomial in x, with integral coefficients, equated to zero, the possible rational roots are numbers which in fractional form have the numerator a factor of the constant term and the denominator a factor of the coefficient of the highest power of x. This theorem is commonly included in elementary algebra so that the whole proof would appear to be within the grasp of first year college students and of superior high school students. The use of determinants is only a convenience and not at all a necessity in the discussion.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Elementary College Algebra. By H. W. Kuhn and J. H. Weaver. New York, The Macmillan Company, 1935. xviii+360 pages. $2.00.

The material covered in this book is the same as is usual in the standard texts. A notable innovation is the introduction of the notion of the derivative in the chapter on Integral Rational Functions, and its later use in approximating irrational roots of equations. Horner's method of course is also offered.

It is noted that the authors have given a very complete set of problems illustrating the theory, especially in the first and more elementary part of the book.

Undoubtedly this section of the work compares with other texts more favorably than that part which constitutes properly a course in freshman college algebra.

No particular attempt has been made to improve the presentation of such topics as Mathematical Induction, Complex Numbers, Permutations and Combinations, and the Binomial Theorem. It would seem that these topics have been developed in rather too brief a manner. The chapters devoted to the solution of equations and to determinants include all the standard theorems and operations, and impress the reviewer favorably. As a presentation of a standard subject along mainly conventional lines, the book as a whole seems acceptable.

H. O. BANKS

History of Hindu Mathematics—A Source Book. Part I. *Numeral Notation and Arithmetic.* By Bibhutibhusan Datta and Avadhesh Narayan Singh. Lahore, India, Motilal Banarsi Das, 1935. xx+261 pages. 10s 6d.

Readers of the MONTHLY have become familiar with the name of one of the authors of this work since articles by B. Datta have been appearing over a period of ten years. Through these articles, he has won a place as a reliable research worker in the field of Hindu mathematics. It is gratifying that a work on the history of Hindu mathematics has now come from the hands of these two Hindu scholars; moreover that a complete history to appear in compact form is here begun with the promise of volumes to come. The work under consideration is the first part and deals with the history of the numeral notation and of arithmetic. The second part, we are told, is devoted to algebra and the third part contains the history of geometry, trigonometry, calculus and various other topics such as magic squares, theory of series and permutations and combinations.

Part I is separated into two chapters. In Chapter I is found an account of the various devices employed by the Hindus for denoting numbers. The gradual evolution of the decimal place-value system of notation has been traced and all evidence relating to its use is brought together for the first time. Arithmetic in general is the subject of Chapter II. The evolution of our methods of performing the fundamental operations, which derive from India, is presented. Common rules, in fact, practically the whole of elementary arithmetic finds a place here, since all of it is due to the Hindus.

This work is unusually successful in combining the two features of a history and a source book. While it may be making an invidious distinction to separate these two, it is well known that a distinction exists. Yet the authors succeed in molding the two phases into a whole without offering sharp lines of cleavage. On the one hand the history of the symbols, of word numerals, of alphabetic notations, of the zero symbol is exhaustively presented. On the other, there are outlines of theories on the origin of numerals, Hindu numerals in Arabia, in Europe, miscellaneous references to numerals. In addition, the work contains excellent summaries, such as (p. 86) "Date of invention of the place-value nota-

tion." Chapter I closes with Tables I–XV of numerals, the last one being "Numeral Forms in Modern Hindu Scripts" (16 scripts), in column arrangement for the sake of comparison, a fine feature.

The actual invention of the place-value system is assigned to the beginning of the Christian era, very probably the 1st century B.C. The inability of the writer of this review to pass judgment on dates is complete, but the conservatism of the authors inspires confidence in this respect even though the tendency of Hindu scholars to claim antiquity in their civilization is generally acknowledged.

The foregoing remarks relate to Chapter I but many of them apply to Chapter II, whose subject matter, however, is not generally controversial. The latter half of the work is given over to detailed accounts, with illustrations and explanations taken from the several Hindu authors in which they occur, of the following: addition, subtraction, multiplication, division, square, cube, square-root, cube-root, checks on operations (a complete set of rules for checking by nines is first found in India), fractions, the rule of three, commercial problems (most of the commercial problems are essentially of an algebraic character, but they are included in arithmetic), rule of false position, mixture, and the mathematics of zero. In this chapter, one is startled to learn that the works of Bhâskara II (1150) have held sway in India for eight centuries (as standard textbooks) and that methods which have been taught practically without change for the last 1500 years or more are still employed in the old fashion on a *pâtî* ("board").

The book closes with a "Bibliography of Sanskrit Mathematical Works," among them a number of manuscripts. And thus comes to an end a fine contribution to the history of science and to the history of mathematics, a unique one to the latter. This first volume augurs well for the two which are to follow. The appetite for them is whetted by such glimpses as this: "It will be shown in the section on Calculus, that Bhâskara II has actually used quantities which ultimately tend to zero, and has successfully evaluated the differential coefficients of certain functions."

Datta and Singh's *History of Hindu Mathematics* should be in every library which reaches standards covered by the word "approved." It should be owned by individuals who have any interest whatever in the history of the progress of science. From the standpoint of authoritative subject matter and from that of book-making, it is a notable history.

LAO GENEVRA SIMONS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN E. OWENS, State College, Pa.

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College Pennsylvania.

CLUB TOPICS

The bibliography of the Duplication Problem as discussed in this MONTHLY, vol. 23, p. 106, furnishes a good start for the preparation of a club program.

A competitive examination held recently by the Pi Mu Epsilon of Pennsylvania State College included the following questions:

Prove that the angles A, B, C of any plane triangle satisfy the inequality,

$$\sin A/2 \sin B/2 \sin C/2 < 1/4.$$

Show that the equations of the tangents of the circle $x^2+y^2=r^2$, through the external point (a, b) are $(ar \pm bt)x+(br \mp at)y=r(a^2+b^2)$ where t is the length of the tangents from the point (a, b) to the circle.

Simplify the product

$$(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)(2^{2^3}+1)\cdots(2^{2^n}+1).$$

A machine gun with muzzle velocity V ft./sec. is on an armored train proceeding at velocity v ft./sec. along a straight horizontal track. Find the maximum range, (1) in the direction of motion of the train, (2) opposite to the direction of motion of the train.

In a letter reminiscent of amusing stunts at otherwise serious club meetings are suggestions for a "mock Ph.D. examination" and the labelling of new members. "On the back of each new member," says our correspondent, "is pinned a tag with a mathematical term. By asking questions which can be answered by yes, or by no, he must identify the name on his back. Only when he has correctly guessed it may he wear his label pinned in front."

From Connecticut College Dr. Julia W. Bower sends the following: Before the erection of Eckhart Hall at the University of Chicago, much of the social life of the mathematics faculty and students centered in the "Slaught parties." Professor Slaught opened his home to the group which filled a gay evening with games and mathematical frivolity. Under pressure of hurried preparation for one such party our host drafted a clever mathematical play called *Evolution of Numbers*. The actors provided their own costumes, throwing themselves eagerly into the spirit of the production. Later the play was published in the MONTHLY, vol. 35, p. 146.

This spring at Connecticut College freshmen in mathematics had the pleasure of presenting this play. Like the original actors they had great fun working it up and concocting costumes. A few changes were made: the Continuum was transformed from "a sturdy fellow" into an angel of peace; and the Hoi Polloi was expanded into two characters who spoke their part in unison and separated appropriately for the Dedekind cut. The girls were so enthusiastic about the play that many memorized their parts before the first rehearsal and one drew a crayon portrait of "Ahmes, our Founder" to hang on the wall. News of their activity spread over the campus and aroused such general interest that it was de-

cided to present the play at an open meeting. The reception of the play was as enthusiastic as its preparation. Success was beyond expectation, and many in the audience learned a valuable lesson in the development of our number system.

To other mathematics clubs we heartily recommend trying out this worthy little drama of Professor Slaught's.

Club Reports

1934–35

The Shuttleworth Mathematical Society of the University of Saskatchewan continues its active career. It is now twenty years old.

A very young member among the mathematics clubs is the Kappa Mu Epsilon of the University of New Mexico. While only a year old they report a growing interest and will send programs hereafter.

At the University of Toledo was recently installed a chapter of Pi Mu Epsilon. Professor J. H. Weaver of the Ohio State University was the installing officer.

The Alpha chapter of Pi Mu Epsilon of Colorado was recently installed at the University of Colorado by Professor A. J. Kempner.

Unless otherwise stated, all officers mentioned are for the year 1934–35 *and all programs were given during that year.*

The Mathematics Club of The George Washington University

Besides a banquet, a Christmas party, business meetings and a beach outing, this active club held eleven meetings with the following subjects and speakers:

"Distribution of prime numbers" by T. Berry; "Transcendental numbers" by N. Goldman; "Mathematical logic" by W. Hoppmann; "An elementary problem in geometry" by R. Been; "Statistical problem involving the Lexis theory of dispersion" by W. Hendricks; "N-dimensional geometry" by W. Hoppmann; "Curve tracing" by W. H. Gammon; "Mathematical phases of astronomy" by F. P. Scott; "Waring's problem" by Dr. F. E. Johnston; "Classification of cubic curves" by V. Schult; "Perfect numbers" by C. Lennahan.

The Mathematics Club of Brown University

Six evening meetings were held with the following programs:

"Sections of a cone" by Jean G. Asbury; "Projective intersections of surfaces" by Professor R. C. Archibald; "The number π" by W. V. Loebenstein; "The Naperian base, e" by Margaret E. Mason; "Transcendental numbers" by W. W. Moss, Jr.; "Why is physics mathematical?" by Professor R. B. Lindsay of the department of Theoretical Physics; "The Euler ϕ-function" by T. Tannenwald, Jr.; "Number theory in antiquity" by Natalie Smith; "Euler" by Martha M. Wicks; "The bridges of Koenigsberg" by Julia G. Watson; "Multiply-connected surfaces" by Florence Weber; "The real projective plane" by Van Z. Williams; "The Mayan calendar" by Professor M. H. Stone of Harvard University.

Mathematics Club of Kansas State College

Vada Crawford, President; H. Wierenga, Vice President; Irma Stanbery, Secretary Treasurer; Marjorie Lomas, Chairman of Program Committee.

Professor Emma Hyde presented the "Plans for the year" and the following is a list of subjects and speakers:

"New proofs of the Pythagorean theorem" by J. Barhydt; "History of trigonometry" by Mary Niestadt; "Recreations" by Betsy Sesler; "Trisection of an angle" by T. C. Wherry: "Where we got our digits" by Wilma Byers; "Fallacies" by D. Peterson; "Constellations" by Dean R. W. Babcock; "Telescope construction" by Professor W. T. Stratton; "Our number system" by Marjorie Lomas; "Recreational problems" by June Fearing; "Slide rule" by J. York; "Curve tracing devices" by Professor C. E. Pearce of the Engineering School; "The calendar" by Vada Crawford; "The trignograph" by H. Mariner; "Measurement of distances" by Alma Furman; "Weights and measures" by Aletha Irvine; "An engineering problem" by J. F. Ransom.

An evening social meeting and a play, "A paradox dinner party" completed the year's work.

Mathematics Club of Mount Mary College, Milwaukee

The club is a member of the Milwaukee Intercollegiate Mathematics Association and entertained one of its regular meetings. A picnic at Beringer Farm closed the year. The programs of the monthly meetings and the speakers were as follow:

"Magic squares" by Eleanor Herbst; "Mathematics in chemistry" by Florence Winking; "Pendulums and clocks" by Muriel Nevins; "Primitive mathematics" by Margaret Kreiziger; "Egyptian mathematics" by Mary Catherine Neugent; "Ionian and Pythagorean schools of mathematics" by Adele Sidek; "First Alexandrian school" by Margaret Kreiziger; "Second Alexandrian school" by Muriel Nevins; "The oriental school" by Doris Beringer; "Algebra through the ages" by Mary Neugent; "Mathematics from 1000 to 1300" by Florence Winking; "Mathematics in the 12th and 13th centuries" by Eleanor Herbst.

Mathematics Club of Northwestern University

This lively club held evening meetings, combining a social hour in which students got acquainted and discussed many questions, serious and otherwise, with a carefully prepared talk on a mathematical or astronomical subject. These talks were frequently given by members of the faculty though students gave part of them. Such papers were always followed by discussions of the subjects presented.

The club visited the Planetarium in Chicago and ended the year with a picnic.

Mathematics Club of Eastern Illinois State Teachers College

Harold Diel, President; Wilma Nuttall, Vice President; Avis Pruekt, Secretary Treasurer; Professors E. H. Taylor and H. F. Heller, Sponsors.

This active club of thirty-eight members held fourteen meetings, including a fall wiener roast, a spring picnic, a ciphering contest and a program presented by the mathematics students of the Charleston high school. This last was a complement to the meeting at which the high school students were guests and the program especially prepared for their benefit.

Other subjects and speakers were as follows: "Mathematics in the modern world" by Professor E. H. Taylor; "Construction of our number system" by Professor H. F. Heller; "Trisection of an angle" by S. Funkhouser; "History and development of trigonometry" by Josephine Novotny; "Infinity" by L. Dulgar; "Harmonic analysis" by D. A. Rothschild; "Mathematics in the changing world" by Helen Barr; "Revolving numbers" by B. Osborn; "Value of analytics and calculus in the secondary school" by Anna Balmer; "Development of arithmetic from colonial days to about 1921" by H. L. Metter; "Number scales" by F. Allen.

The program for the high school students included brief talks as follows: "Number series" by Professor E. H. Taylor; "History and development of π" by Elizabeth Jones; "A mathematical proof" by Mary R. Bear; "An algebraic fallacy" by Esther M. Shubert.

Mathematics Club of New York State College for Teachers

Loraine Loder, President; Mildred Facer, Vice President; D. Rogers, Secretary; G. Ungerer, Treasurer.

The club maintained a regular Puzzle Box, presented a comedy skit, Point College, for the newly admitted members, held a picnic and introduced a new idea into programs. A debate was

held on the question, "Resolved that the study of mathematics should have practical applications as its sole aim." The negative side won. Other talks and speakers were as follows:

"Our mathematics club" by G. Ungerer; "Binomial theorem" by D. Rogers; "Map coloring" by Jeanne LaRoque; "Excursions in number" by Laura Hendricks; "Fermat's last theorem" by Professor R. Beaver; "Symedians of a triangle" by W. Fowler; "The structure of the universe" by Dr. R. G. Wilson of the Dudley Observatory; "Calculation of log tables" by Rosa Peters; "Mathematics in aviation" by G. Ungerer.

Pi Mu Epilson of The University of Illinois

R. J. Hickey, Director; J. R. Schaeffer, Secretary; Edith Lytle, Corresponding Secretary; J. Glasser, Treasurer; Virginia Modesitt, Chairman Executive Committee.

The chapter held several social meetings woven round the choice and pledging of new members. At the initiation banquet, Professor A. R. Crathorne acted as toastmaster and Dean M. T. McClure, of the College of Liberal Arts and Sciences delivered an address on "The philosophy of education."

At the regular meetings the speakers and subjects were as follows: "A problem in the fourth dimension" by D. M. Brown; "Finite geometry" by Dr. Beulah Armstrong; "A problem in the algebra of Mendel's law" by R. E. Watson; "Mathematics in the history of music" by Gertrude Hendrix; "The principle of duality" by Shirley Stumbaugh; "Some elementary properties of Galois fields" by R. Thrall; "Authors in analysis" by G. L. Waters; "Derivation of the equation of a caustic curve of the parabola."

Mathematics Club of Haverford College

W. N. Huff, President; J. Haywood, Secretary.

Speakers and subjects were as follows:

"Nature and scope of mathematics" by Professor A. H. Wilson; "Kepler's laws" by W. N. Huff; "Algebraic charts" by A. B. Boggs; "Caustics" by J. Haywood; "The representation of numbers of squares" by Professor J. A. Shohat of the University of Pennsylvania; "Rational triangles and quadrilaterals" by C. B. Watkins; "Pascal's theorem" by W. B. Morgan, Jr.; "Bezout's theorem" by W. L. Azpell; "Semi-linear equations" by Professor C. O. Oakley; "Double star orbits" by P. M. Whitman; "Solution of prize problem" by P. H. Miller.

Officers for 1935–36 are W. B. Morgan, Jr., President and P. M. Whitman, Secretary.

The Mathematics Club of Brown University

This club is the only one as yet reported which issues a printed program in the autumn for all meetings of the current year. The management of the club is in the hands of two committees, Program, Professor A. A. Bennett, Chairman; and Arrangements, Dr. B. C. Getchell, Chairman. Twice each year a social hour follows the address and each member is allowed to invite one guest. The average attendance of undergraduates is fifty. A presiding officer is designated for each evening, either a faculty member or a senior.

For the current year 1935–6 the subjects and speakers follow: "Regular hypersolids" by Ruth A. Oldfield; "The four-color problem" by H. Moses; "Some concepts in group theory" by Grace K. Anderson; "Crystallographic forms" by Lillian M. McCabe; "To solve the quintic equation" by H. Rodin; "Computation by series" by Dr. J. S. Frame; "Circles related to the triangle" by Marie P. Galligan; "Linkages" by G. Smithson; "Continued fractions" by Leonora Grozen; "Magic squares" by I. J. Gruntfest; "Newton's interpolation formula" by R. B. Grannis; "The historical and mathematical development of Cartography" by Dr. B. H. Brown.

Mathematics Discussion Group of Columbia University

From this group which meets semi-monthly comes this topic from one of their meetings and the references used in its study: "Mapping or graphical representation of $f(z)$, exhibiting processes of wide importance in the mathematical and physical sciences." References: H. B. Fine, *Calculus*; W. F. Osgood, *Advanced Calculus;* J. Pierpont, *Functions of a Complex Variable;* A. Knopp, ***Funktionenthorie.***

wafer-thin so that it does not interfere with the paths of any of the particles it has thrown off; or if (b) the wheel does halt the progress of any particle which impinges on it.

SOLUTIONS

E 184 [1935, 622]. *Proposed by M. O. Reade, Brooklyn, New York.*

The cube root of an eight-digit number was extracted in the usual manner, and then each digit was replaced by a code letter, with the following result. Solve the code.

$$\begin{array}{cccccccc|l}
W & P, & W & D & B, & M & W & C & A\;K\;C \\
B & P & & & & & & & \\
\hline
A & S & W & D & B & & & & \\
B & P & K & P & B & & & & A\;M\;K\;M \\
\hline
 & B & C & M & S & M & W & C & \\
 & B & C & M & S & M & W & C & M\;M\;S\quad S\;P\;C \\
\hline
\end{array}$$

Solution by B. C. Zimmerman, Corozal, B. H.

From the first two columns it appears that $S=0$, $A=B+1$, $W=A+B$, and $B+P=9$ or 10. Now since $A^3=BP<100$, $A=3$. [For if $A=4$, $BP=64$ and $A=P$.] So $B=2$, $P=7$ and $W=5$.

From the third column, $K+C=W+9=14$, and from the last multiplication we see that $C=6$ and $K=8$. Therefore from the second multiplication, $M=4$. Finally, from the fourth column, $M+P=7+4=11$, so that $D=1$. Consequently, the original problem was to extract the cube root of 57,512,456, which is 386.

Editorial Note. In his solution, C. W. Trigg points out that the given data were largely superfluous, and that the solution could have been obtained from

$$\begin{array}{cccccccc|l}
x & x & x & x & B & x & x & x & A\,K\,C \\
B & P & & & & & & & \\
\hline
x & x & x & x & B & & & & \\
x & x & x & x & B & & & & x\;x\;x\;M \\
\hline
x & x & x & x & x & x & x & x & \\
x & x & x & x & x & x & x & C & x\;x\;x\;x\;x\;C \\
\hline
\end{array}$$

Similarly, E. P. Starke points out in his solution that the problem would have had a unique solution if it had merely been stated in the form, "Find a perfect, eight-place cube, whose first, third and seventh digits are alike, and the others all different."

Also solved by E. F. Allen, David Blackwell, W. E. Buker, J. E. Burnam, Madge Childre, Mary L. Constable, Fred Discepoli, Wm. Douglas, J. S. Gold, H. E. H. Greenleaf, M. C. Hartley, W. R. Holloway, C. A. Murray, C. F. Pinska, O. M. Rogers, A. W. Smith, Cornelia Strong, Simon Vatriquant, and the proposer.

E 185 [1935, 622]. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A thin, straight wire is marked off into m equal lengths by $m-1$ points. It is then bent at a right angle at each of one or more of these points, making each segment parallel to one of two rectangular axes. The bent wire may be self-intersecting, but not self-coincident over a finite length. How many different shapes may it have? Extend the problem to three dimensions by permitting the segments of the wire to be parallel to any of three rectangular axes.

No solution has been received for this problem. The editors would be glad to learn if the readers consider it too difficult for the Elementary Problem section.

E 186 [1935, 622]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In a geometric progression whose terms are all positive integers, the first five terms contain nine digits each, the next five terms contain ten digits each, the next four terms contain eleven digits each, and the remaining two terms contain twelve digits each. Reconstruct the progression and show that it is unique.

Solution by C. L. Weaver, Boston, Mass.

If we designate the nth term of this progression by $T_n = ab^{16-n}c^{n-1}$, where a, b and c are positive integers, the given information is equivalent to

$$\begin{aligned} T_1 &= ab^{15} \geqq 10^8, & T_5 &= ab^{11}c^4 < 10^9,\\ T_6 &= ab^{10}c^5 \geqq 10^9, & T_{10} &= ab^6c^9 < 10^{10},\\ T_{11} &= ab^5c^{10} \geqq 10^{10}, & T_{14} &= ab^2c^{13} < 10^{11},\\ T_{15} &= abc^{14} \geqq 10^{11}, & T_{16} &= ac^{15} < 10^{12}. \end{aligned}$$

Dividing T_{10} by T_1 tells us that $c/b < 10^{2/9}$, and dividing T_{15} by T_{10} gives $10^{2/10} < c/b$, so that $1.584 < c/b < 1.669$. The smallest possible values for b and c to satisfy this restriction are $b=3$ and $c=5$. Therefore a must be $> 10^8 3^{-15}$, or > 6.96, and since a is an integer, it must be at least 7. Then the largest possible $c < (10^{12}/7)^{1/15}$, or 5.54. Therefore $c=5$ and $b=3$ are the only possibilities. Now $a < 10^{10}3^{-6}5^{-9}$, or 7.02, so a can only be 7. Thus the only possible solution comes from $T_n = 7 \cdot 3^{16-n}5^{n-1}$, and substitution shows that this progression satisfies all the given conditions.

Also solved by David Blackwell, J. E. Burnam, J. W. Clawson, Mary L. Constable, H. E. H. Greenleaf, C. F. Pinska, J. T. Rule, E. P. Starke, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 187. *Proposed by A. Gloden, Luxembourg.*

Find the smallest square whose last six digits follow the pattern *aabaab*.

Solution by E. P. Starke, Rutgers University.

Evidently $(1002)^2=1004004$ is the first square of more than six digits satisfying the requirements, with $a\neq b$. No number with six digits or less can satisfy because *aabaab* is divisible by $1001=7\cdot 11\cdot 13$, and hence is not a square unless $a=b=0$. So 1002^2 is the solution.

The problem can also be solved for particular values of a and b, provided (a, b) is one of the following pairs: (0, 0), (0, 1), (0, 4), (0, 9), (2, 5), (3, 6), (4, 1), (4, 9), (8, 1), (8, 9) or (9, 6). The method of solution is sufficiently illustrated by the following examples.

Show that there exists no square whose last six digits are 776776. Note that $76^2=5776$. If N^2 and 76^2 end in the same three digits, N^2-76^2 is divisible by 1000; hence $N=500c\pm 76$, where c is an integer. Since $N^2=250000c^2\pm 76000c+5776$ should end with the digits 776776, we would have, after subtraction and division by one factor 1000, $250c^2\pm 76c-771$, an odd number still divisible by 1000. But this is impossible.

Find the smallest square ending in 889889. Note that $83^2=6889$. Take, as above, $N=250c\pm 83$. Then $N^2=62500c^2\pm 41500c+6889$ ends with 889889; hence $125c^2\pm 83c-1766$ is divisible by 2000. By use of congruences or otherwise, it is not difficult to determine values of c which satisfy this condition. The smallest is $c=298$, whence $N=74417$ and $N^2=5537889889$.

Also solved by W. E. Buker, J. E. Burnam, Mary L. Constable, Bernard Greenspan, Joseph Milkman, Walter Penney, W. R. Talbot and C. W. Trigg.

E 188. *Proposed by N. A. Court, University of Oklahoma.*

A variable sphere (P) passes through a fixed point and has its center on a fixed sphere (A). Find the envelope of the radical plane of the two spheres (P) and (A).

Note. The analogous problem relative to two circles was solved analytically in the *Nouvelles Annales de Mathématiques*, ser. 2, vol. 15 (1876), p. 284, Q. 1198.

Solution by V. Thébault, Le Mans, France.

Let us first consider two coplanar circles (A) and (B), with centers at A and B, and radii a and b respectively. Assume a third, variable circle (P), whose center P moves on circle (A), and which is forced to cut circle (B) orthogonally. Call the intersections of (P) with (A), C and D, and the intersections of (P) with (B), E and F. Designate by d the distance AB, and by K and Q the respective projections of B on AP and CD. Let M be the projection of A on CD. We then have, both in magnitude and sign:

$$PB^2 = AB^2 + AP^2 - 2\cdot AP\cdot AK, \tag{1}$$

$$PE^2 = PC^2 = AC^2 + AP^2 - 2\cdot AP\cdot AM. \tag{2}$$

Consequently, $BE^2=PB^2-PE^2=AB^2-AC^2-2\cdot AP(AK-AM)=AB^2-AC^2-2\cdot AP\cdot MK$; whence we finally deduce that

$$BQ = MK = (d^2 - a^2 - b^2)/2a. \tag{3}$$

This means that the radical axis CD of the circles (A) and (P) envelopes a circle whose radius is $(d^2-a^2-b^2)/2a$, which is concentric with circle (B). Conversely, if the radical axis CD of two circles (A) and (P) envelopes a circle centered at B, whose radius is given by (3), while the center of (P) moves around (A), then circle (P) is orthogonal to some circle centered at B. If we use the same notation as before, we still have, by (1) and (2), that

$$PB^2 - PE^2 = d^2 - a^2 - 2\cdot AP\cdot MK = a \text{ constant}.$$

If we now rotate our figure about the line AB, the circles (A) and (B) generate the spheres $[A]$ and $[B]$. At each point P on the surface of $[A]$ we may center a sphere $[P]$ orthogonal to sphere $[B]$. Then the radical plane (Q) of the spheres $[A]$ and $[P]$ must envelope a sphere centered at B and with radius $(d^2-a^2-b^2)/2a$. We have thus proved the theorem:

If a sphere $[P]$ so varies that its center P remains on the surface of a fixed sphere $[A]$ while the surface of $[P]$ meets orthogonally the surface of a second fixed sphere $[B]$, then the radical plane of spheres $[A]$ and $[P]$ so moves as to envelope another sphere concentric with $[B]$ and with radius $(d^2-a^2-b^2)/2a$, where a and b are the respective radii of spheres $[A]$ and $[B]$, and d is their central distance.*

The proposed problem is obviously a special case of this theorem, in which the radius b of sphere $[B]$ has shrunk to zero, thus compelling the sphere $[P]$ to pass always through the point B.

Also solved by C. E. Springer, E. P. Starke, Simon Vatriquant, G. A. Williams and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3788. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

There are infinitely many pairs of straight lines, each marking on the sides of a given quadrilateral four segments having their mid-points collinear. Show

---

* This theorem, due to V. Thébault, appeared in the Annales de la Société Scientifique de Bruxelles, 1922, p. 103.

that the locus of the intersections of the straight lines of each pair is a straight line.

3789. *Proposed by J. M. Feld, New York City.*

Show that the equation of the ellipse can be written in the form

$$[\bar{a}(z-c)-b(\bar{z}-\bar{c})][\bar{b}(z-c)-a(\bar{z}-\bar{c})]+(a\bar{a}-b\bar{b})^2=0,$$

where a, b, c are fixed complex numbers, z is a variable complex number, and $\bar{z}$ is the conjugate of z. Find the center, the lengths of the semi-axes, and the position of the major axis.

3790. *Proposed by V. W. Adkisson, University of Arkansas.*

If $F(a, b, c, x)$ represents the hypergeometric series, show that

$$F(s,-p,p+s,1)=\frac{(s+p-1)(s+p-2)\cdots(p+1)}{2(s+2p-1)(s+2p-2)\cdots(2p+1)},$$

where p is in general complex, $R(p)>0$, and s is a positive integer. In particular show that

$$F(1,-p,p+1,1)=1/2, \qquad R(p)>0.$$

3791. *Proposed by I. M. Sheffer, The Pennsylvania State College.*

Let P_1, P_2, P_3 be three points on a circle C, so situated that P_2 and P_3 are on opposite sides of the diameter through P_1, and chords P_1P_2 and P_1P_3 make angles less than 45° with this diameter. Form a lattice-work in the plane with P_1P_2 and P_1P_3 as adjacent sides of a lattice parallelogram. Then of all the lattice points in the plane, only P_1, P_2, P_3 are in or on C.

Corrected statement of 3768 [1936, 50] *Proposed by N. A. Court, University of Oklahoma.*

Construct a tetrahedron, given in position the mid-points of four of the edges. (Correction of an error in editing).

SOLUTIONS

3700 [1934, 521]. *Proposed by J. H. M. Wedderburn, Princeton University.*

Find a basis h_{pq} for matrices of order n such that each element of the basis is idempotent and

$$h_{pq}h_{rs}=k_{pqrs}h_{ps}$$

where k_{pqrs} is a rational number.

Solution by the Proposer.

The conditions are fulfilled by setting

$$h_{pq} = \sum_{i=1}^{p} \sum_{j=1}^{q} e_{ij}/m_{pq}$$

where e_{ij} are the ordinary matric units and m_{pq} is the smaller of p and q. The value of k_{pqrs} is then $m_{qr}m_{ps}/m_{pq}m_{rs}$.

3702 [1934, 521]. *Proposed by W. P. Udinski, University of Texas.*

In the domain of real numbers, let $f(x)$ be a function such that $f(\alpha+mh)$ (h is a constant) is finite and of the same sign or zero for every $m=0, 1, 2, \cdots, n-1$. Further, let $P_s(x)$ be a polynomial of degree s such that

$$\sum_{m=0}^{n-1} f(\alpha + mh)P_s(\alpha + mh)(\alpha + mh)^k = 0 \tag{1}$$

$$\text{for } k = 0, 1, 2, \cdots, s - 1.$$

Finally, we assume, that at some point of the set involved in the summation in (1) neither $f(x)$ nor $P_s(x)$ is zero.

Under the above hypothesis, prove that the roots of $P_s(x)$ are all real and distinct and all lie on the interval $<\alpha, \alpha+(n-1)h>$.

The above lemma is applicable to each member of the family of orthogonal (with respect to a finite summation) polynomials introduced by Tschebyscheff. See: C. Jordan, *Lond. Math. Soc. Proc.* series 2, vol. 20, page 297.

Solution by the Proposer.

The following proof is an adaptation of one given by Stieltjes for a lemma of a similar sort concerning integration. See *Oeuvres* Tome I, page 380. Let

$$P_s(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_s).$$

From the relation (1) of the problem for which $k=0$, we see that $P_s(x)$ must have a root within the given interval. Let $\alpha_1, \alpha_2, \cdots, \alpha_p$; $p \leqq s$, be those roots of $P_s(x)$ which lie in the designated interval, and further define

$$Q_p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_p).$$

Then from (1) we have, if $p<s$,

$$\sum_{m=0}^{n-1} f(\alpha + mh)P_s(\alpha + mh)Q_p(\alpha + mh) = 0;$$

but this is impossible. Hence $p=s$, that is all the roots lie in the given interval. Suppose now that $\alpha_1=\alpha_2$. Then again from (1) we would have

$$\sum_{m=0}^{n-1} f(\alpha + mh)\left[\frac{P_s(\alpha + mh)}{\alpha - \alpha_1 + mh}\right]^2 = 0$$

which also is impossible. Hence all the roots are distinct.

3706 [1934, 581]. *Proposed by G. R. Livingston, State Teachers College, San Diego.*

Using compasses only, find the points which divide a given circle into five equal arcs.

Solution by L. S. Shively, Ball State Teachers College, Muncie, Ind.

Let O be the center and a be the radius of the given circle. With a as radius, lay off in succession on the circle, the equal arcs AB, BC, CD, and DE. With AC as radius and with A and D as centers, draw arcs intersecting in F. With OF as radius and A as center draw an arc intersecting the given circle in G. With the same radius OF, and C and E as centers, draw arcs intersecting in H. Then GH is the radius with which, if arcs be laid off successively from any initial point on the given circle, these arcs will each be one fifth of the circumference.

For the proof, draw the lines OA, OD, OF, AC, AF, AG, CH, and CE. Let K be the point of intersection of OD and CE. The truth of each of the following statements is obvious: A, O, D and H are collinear; $AC=a\sqrt{3}$; $OF=AG=a\sqrt{2}$; G lies on OF and OG is perpendicular to OA; $HK^2=CH^2-CK^2=2a^2-3a^2/4=5a^2/4$; and $HK=a\sqrt{5}/2$; $HO=(\sqrt{5}-1)a/2$. Hence HO is the side of a regular inscribed decagon from which it follows that GH is the side of a regular inscribed pentagon.

Solved also by A. D. Bradley, Abe Gelbart, R. Goormaghtigh, J. E. La Fon, R. K. Morley, Leon Recht, J. Rosenbaum, H. F. Schroeder, E. P. Starke, and the proposer.

Editorial Note. The other solutions were in general slight variations of the above, and most of the solvers stated that the same type of solution is given by Mascheroni. Morley mentioned that A and G are consecutive vertices of the inscribed square, the construction of which is Napoleon's problem. The proposer and Gelbart referred to two known compass constructions for bisecting, respectively, an arc of a circle and a straight line segment. The latter construction is given in the solution of 3327 [1929, 339] where four references are given to the literature of compass constructions. One of these is *La Geometria del Compasso*, di Lorenzo Mascheroni, Pavia, anno V della Repubblica Francese, 1797. The required construction is given in Problema 40, page 23. Also Problema 142, pp. 136–138 gives a construction for the center of a given circle. Gelbart remarks that a well known ruler and compass construction of the side of the inscribed pentagon is as follows: On the diameter AOD of the given circle let K be the mid-point of the radius OD: let H be the point on OA so that $KH=KG$: then GH is the length of the required side. A compass construction for G and H is then merely a matter of applying the two above compass constructions. This he does in his solution, and concludes with a remark which in substance is as follows: Any reader of Mascheroni must be aware of the fact that any construction which is possible with ruler and compass is also possible with compasses alone, and that one should endeavor to find more beautiful solutions than the one which he outlined. Bradley and La Fon state that the construction may also

be found in the second edition of the translation into French, *Géométrie du Compas*, by Carette, Paris, 1828, Problème 40, p. 50. The latter adds that it is given also on page 134 of the more recent book with the same title by A. Quemper de Lanascal, Librairie Scentifique, Albert Blanchard, Paris, 1925. Starke observed that in this problem as also in E100 [1935, 45] it is not explicitly stated that the center is given with the circle. If the circle is given without the center, the latter may be constructed by compasses alone as follows: Take a point N on the given circle as center and with a radius greater than one fourth the diameter of the given circle describe a circle (N) cutting the given circle in M and P. Find the symmetric N' of N with respect to MP, and describe a circle with N' as center and $N'N$ as radius cutting (N) in M' and P'. Find the symmetric O of N with respect to $M'P'$: then O is the required center.

3710 [1934, 582]. *Proposed by Harry Langman, Brooklyn, N.Y.*

If the C's represent binomial coefficients, show that

$$\begin{vmatrix} C_2^2 & C_3^3 & C_4^4 & \cdots & C_{n-1}^{n-1} & C_n^n & C_{n+1}^{n+1} \\ -(n-1) & C_2^3 & C_3^4 & \cdots & C_{n-2}^{n-1} & C_{n-1}^n & C_n^{n+1} \\ 0 & -(n-2) & C_2^4 & \cdots & C_{n-3}^{n-1} & C_{n-2}^n & C_{n-1}^{n+1} \\ 0 & 0 & -(n-3) & \cdots & C_{n-4}^{n-1} & C_{n-3}^n & C_{n-2}^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -2 & C_2^n & C_3^{n+1} \\ 0 & 0 & 0 & \cdots & 0 & -1 & C_2^{n+1} \end{vmatrix} = (n!)^2.$$

Solution by Frank Ayres, Jr., Dickinson College.

Let the given determinant be denoted by D_n. Then we may write

$$D_n = \begin{vmatrix} C_2^2 & C_3^3 & C_4^4 & \cdots & C_{n-2}^{n-2} & C_{n-1}^{n-1} & C_n^n & C_{n+1}^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -(n-1) & C_2^3 & C_3^4 & \cdots & C_{n-3}^{n-2} & C_{n-2}^{n-1} & C_{n-1}^n & C_n^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -(n-2) & C_2^4 & \cdots & C_{n-4}^{n-2} & C_{n-3}^{n-1} & C_{n-2}^n & C_{n-1}^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -(n-3) & \cdots & C_{n-5}^{n-2} & C_{n-4}^{n-1} & C_{n-3}^n & C_{n-2}^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -2 & C_2^n & C_3^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 & C_2^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -C_2^{n+1} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & -C_3^{n+1} & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & -C_4^{n+1} & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & -C_{n-1}^{n+1} & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & -C_n^{n+1} & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -C_{n+1}^{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

Adding the $(n+1+j)$th row to the $(n-j)$th row $(j=0, 1, \cdots, n-1)$, and expanding along the nth row, we have

$$
(1)\quad D_n = -\left|\begin{array}{ccccccccccc}
C_2^2 & C_3^3 & C_4^4 & \cdots C_{n-2}^{n-2} & C_{n-1}^{n-1} & C_n^n & 0 & 0 & 0\cdots 0 & 0 & 1 \\
-(n-2) & C_2^3 & C_3^4 & \cdots C_{n-3}^{n-2} & C_{n-2}^{n-1} & C_{n-1}^n & 0 & 0 & 0\cdots 0 & 1 & 0 \\
0 & -(n-3) & C_2^4 & \cdots C_{n-4}^{n-2} & C_{n-3}^{n-1} & C_{n-2}^n & 0 & 0 & 0\cdots 1 & 0 & 0 \\
0 & 0 & -(n-4) & \cdots C_{n-5}^{n-2} & C_{n-4}^{n-1} & C_{n-3}^n & 0 & 0 & 0\cdots 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots -2 & C_2^{n-1} & C_3^n & 0 & 0 & 1\cdots 0 & 0 & 0 \\
0 & 0 & 0 & \cdots 0 & -1 & C_2^n & 0 & 1 & 0\cdots 0 & 0 & 0 \\
0 & 0 & 0 & \cdots 0 & 0 & 1 & -C_2^{n+1} & 0 & 0\cdots 0 & 0 & 0 \\
0 & 0 & 0 & \cdots 0 & 1 & 0 & -C_3^{n+1} & 1 & 0\cdots 0 & 0 & 0 \\
0 & 0 & 0 & \cdots 1 & 0 & 0 & -C_4^{n+1} & 0 & 1\cdots 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \cdots 0 & 0 & 0 & -C_{n-1}^{n+1} & 0 & 0\cdots 1 & 0 & 0 \\
1 & 0 & 0 & \cdots 0 & 0 & 0 & -C_n^{n+1} & 0 & 0\cdots 0 & 1 & 0 \\
0 & 0 & 0 & \cdots 0 & 0 & 0 & -C_{n+1}^{n+1} & 0 & 0\cdots 0 & 0 & 1
\end{array}\right|
$$

Subtract the $(2n-1-j)$th row from the $(2n-2-j)$th row $(j=0, 1, 2, \cdots, n-3)$ in turn, and then add the $(n+1)$st row to the nth row. Finally, multiply the jth row by $(-1)^{n-j}$, $(j=1, 2, \cdots, n-1)$ and add them to the nth row. The nth row is now

$$(2)\qquad a_1, a_2, \cdots, a_{n-2}, a_{n-1}, -n^2, 0, 0, \cdots, 0, 0$$

whose elements a's, as we shall see later, need not be computed.

Adding the $(n+2+j)$th row to the $(n+1+j)$th row $(j=0, 1, \cdots, n-3)$ in turn, we obtain the determinant (1) except for the nth row which is given by (2). Expanding this determinant along the nth row, we have, letting Δ_{a_j} denote the minor of a_j,

$$
\begin{aligned}
D_n &= n^2 D_{n-1} + (-1)^n \sum_{j=1}^{n-1} (-1)^j a_j \Delta_{a_j} \\
&= n^2 D_{n-1}
\end{aligned}
$$

since $\Delta_{a_j}=0$, as can be seen by subtracting from the $(2n-2-j)$th row the $(j+1)$st row $(j=0, 1, 2, \cdots, n-2)$.

Since $D_2=2^2$, the proof by induction is complete.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

OUTLINE OF THE HISTORY OF MATHEMATICS, THIRD EDITION

Professor R. C. Archibald's *Outline of the History of Mathematics* was published in 1931 by The Society for the Promotion of Engineering Education. Since the Trustees of the Mathematical Association of America felt that a service would be rendered to teachers by making available a second edition, revised and enlarged, they published one in August 1934. This is now exhausted. But since there has been a steady demand for this publication, the Association decided to prepare a third edition, still further revised, and enlarged by four pages. Two of these pages give an account of the latest discoveries in connection with Babylonian mathematics. So many changes of importance are contained in this new edition that even owners of an earlier edition will probably desire to own this source of information nowhere else readily available. Copies may be obtained through the office of the Secretary-Treasurer at the rate of fifty cents per copy, postpaid; ten or more copies to the same address at forty cents each.

COWLES COMMISSION FOR RESEARCH IN ECONOMICS; PRELIMINARY ANNOUNCEMENT OF SEMINARS IN ECONOMICS AND STATISTICS

For the second year, advanced seminars in economics and mathematical statistics sponsored by the Cowles Commission for Research in Economics, which is affiliated with the Econometric Society, will be held in Colorado Springs. From July 6 to August 8, 1936, four lectures a week will be presented, each to be followed by a discussion period. No fees will be charged. Below is a preliminary list of subjects which will be treated in one or more sessions by the lecturers indicated.

I. *Economic Statistics*

1. Income and the Income Tax; Irving Fisher, Yale University.
2. Some Statistical Aspects of Keynes's Theory of Employment; W. L. Abbott, Colorado College.
3. Business Cycle Theory; Probably W. M. Persons, New York.
4. The Inadequacy of the Classical Theory of Foreign Trade; D. I. Vinogradoff, Westinghouse Electric Company.
5. The Statistical Basis of Capitalism; Carl Snyder, New York Federal Reserve Bank.
6. Scientific vs. Pseudo-Scientific Research in Economics; Joseph Mayer, Library of Congress.
7. Opportunities for the Use of Statistics in Industrial Research; Anson Hayes, American Rolling Mills.
8. New Indexes of Stock Market Action; Alfred Cowles 3rd, Cowles Commission.

9. Evidence of Structure in Stock Prices; W. F. C. Nelson, Cowles Commission.

10. Effects of Short Selling on Stock Prices; H. P. Hartkemeier, University of Missouri.

II. *Mathematical Statistics*

1. Tests of Significance; R. A. Fisher, University of London.

2. The Significance Tests for Periodogram Analyses; H. T. Davis, Indiana University.

3. The Significance of Means; E. L. Dodd, University of Texas.

4. Significance of Lag Correlation; C. F. Roos, Cowles Commission.

5. Tests of Significance of Index Numbers; T. H. Rawles, Colorado College.

6. The Significance of Analyses of Variance of Time Series; H. T. Davis, Indiana University.

7. Population Analysis; A. J. Lotka, Metropolitan Life Insurance Company.

8. Methods of Eliminating the Influence of Several Groups of Factors; Corrado Gini, Italian Central Institute for Statistics.

9. Sampling Theory; W. A. Shewhart, Bell Telephone Laboratories.

10. Demand and Supply Analysis: E. J. Working, University of Illinois.

The two seminars will be conducted concurrently. Facilities of the Cowles Commission laboratory in the way of computational and supervisory assistance will be made available to participants who are investigating problems within the sphere of interest of the Commission. All expecting to attend should notify the Cowles Commission as soon as possible.

A cash award of $1000.00 is offered by The Williams and Wilkins Company for the best manuscript on a science subject, presented before July 1, 1937. The publishers put no limitations on the subject-matter or manner of handling, and none on eligibility for the award. The manuscript must be in English and "of a sort calculated to appeal to the taste of the public at large." The desired length is given as 100,000 words.

Further details concerning the award may be had by addressing the publishers at Mt. Royal and Guilford Avenues, Baltimore, Md.

The constitution of the Circolo Matematico di Palermo, revised to bring it in accord with recent legislation, appears in full in No. 1, vol. 59 of the Rendiconti. The President and Vice-President are henceforth to be appointed by the Minister of Education (with the advice of the Society); the editorial board of the Rendiconti as well as all other officers of the Society are to be appointed by the President. The latter has full administrative powers, and only such business as he has placed on the agenda can be discussed in meetings of the Society.

The new constitution empowers the Minister of Education to further the expulsion from the Society of any member unworthy of belonging to it; although the final decision presumably rests with the President, the Society is expressly forbidden by the constitution to discuss or vote on such a matter. The by-laws of the Society must be approved by the Minister of Education.

The Society will lose much of its international character because of the proviso that at least two-thirds of the corresponding members must be Italian. At present less than one-third are so, and the new constitution provides that no non-Italians be admitted to membership until the desired ratio is attained.

Dr. G. A. Miller, professor emeritus of mathematics at the University of Illinois, reports that a new mathematical journal has appeared in Germany. Its title is Deutsche Mathematik. The first issue is dated January 1936. According to Professor Miller, this journal is an attempt to foster nationalism in mathematics instead of internationalism which has heretofore prevailed.

Professor N. A. Court of the University of Oklahoma gave a talk entitled "The Teaching of Geometry in the Secondary Schools in Retrospect and Prospect" before the Sixth Annual Teacher-Training Conference held March 13–14, 1936, at North Texas State Teachers College, Denton, Texas.

At the semi-centennial exercises at the University of Chattanooga on April 25, 1936, Professor Floyd Field of Georgia School of Technology was present as representative of the Mathematical Association of America.

Professor J. H. Kusner of the University of Florida has been elected secretary of the Florida Academy of Sciences.

At the annual meeting of the American Society of Mechanical Engineers, Professor Stephen Timoshenko of the University of Michigan was awarded the Worcester Reed Warner Medal.

Professor L. V. Ahlfors, of the University of Helsingfors, who has held a lectureship at Harvard University during the current academic year, has been promoted to an assistant professorship.

Professor R. C. Archibald of Brown University has been on leave of absence the second semester of the academic year.

Assistant Professor L. W. Cohen of the University of Kentucky has been promoted to an associate professorship.

Associate Professor N. A. Court of the University of Oklahoma has been promoted to a professorship.

Dr. P. G. Hoel of Rose Polytechnic Institute has been promoted to an assistant professorship.

Dr. I. O. Horsfall, president of Snow College, Ephraim, Utah, for the past three years, has been appointed director of the Extension Division of the University of Utah.

Assistant Professor O. W. Irvin of Brooklyn College has been promoted to an associate professorship.

Dr. Max Mason, who joined the Rockefeller Foundation as director for the natural sciences in 1928 and has been its president since 1929, has announced his retirement. He is planning to resume his mathematical work.

Professor G. M. Merriman, on leave of absence from the University of Cincinnati, has been at Harvard University for the second semester of this academic year.

Dr. J. A. Miller, research professor of astronomy at Swarthmore College and director of the Sproul Observatory, has retired.

At the University of Washington Professor R. E. Moritz, head of the department of mathematics since 1904, has been relieved of administrative duties, but will continue his teaching. Professor A. F. Carpenter succeeds Professor Moritz as executive officer of the department.

Dr. N. E. Rutt of Northwestern University has been promoted to an assistant professorship.

Mr. Garrett Birkhoff, of the Society of Fellows, has been appointed to an instructorship at Harvard University.

Dr. H. D. Larsen of the University of Wisconsin has been appointed to an instructorship in mathematics at the University of New Mexico.

W. A. Patterson of Ohio State University has been appointed instructor of mathematics at Fenn College, Cleveland, Ohio.

Dr. A. H. Taub of Princeton University has been appointed instructor at the University of Washington.

Dr. R. J. Aley, president emeritus of Butler University, died November 17, 1935. He was president of Butler University from 1921 to 1931, and had earlier been president of the University of Maine. He was for many years a member of the Mathematical Assciation.

Assistant Professor C. F. Bowles, of the South Dakota State School of Mines, died on February 7 after an illness of several weeks.

Professor B. H. Crenshaw of Alabama Polytechnic Institute died on November 25, 1935, at the age of sixty-eight years. He was a charter member of the Mathematical Association.

Assistant Professor Julia Dale of Duke University died on January 14, 1936.

Dr. E. M. Horsburgh, lecturer in technical mathematics and engineering dynamics at the University of Edinburgh, died December 28, 1935, after a prolonged illness. He had been a member of the Mathematical Association of America for fifteen years.

I. L. Miller, professor of mathematics at South Dakota State College since 1920, died February 22, 1936. He had been a member of the Mathematical Association since 1917.

Dr. J. M. Page, professor of mathematics and dean at the University of Virginia, died March 12, 1935, at the age of seventy-two years. He had been connected with the University of Virginia for thirty-eight years.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N. Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Summer Meeting, Harvard University, Aug. 31, 1936.

Twenty-first Annual Meeting, Duke University, Durham, N. C., Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, Normal, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, Topeka, March 14.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S. C., April 17–18.
SOUTHERN CALIFORNIA, Fullerton, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, Madison, May 9.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

As determined more recently by the Trustees, the prize is to be awarded for a noteworthy expository paper. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Through a gift by a former president of the Association the prize will be awarded hereafter every three years. The last award was in December, 1935.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 7, AUGUST-SEPTEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

AN ANNOUNCEMENT

The officers and trustees of the Mathematical Association of America announce the resignation of Professor Walter B. Carver as editor-in-chief of the American Mathematical Monthly, and the appointment of Professor E. J. Moulton as his successor, beginning with the issue for January 1937. All manuscripts submitted hereafter for publication should be sent to Professor Moulton at Northwestern University, Evanston, Illinois.

MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty-two persons have been elected to membership in the Association on applications duly certified:

V. P. BATHA, M.S.(Wisconsin) Asst. Prof., Carroll Coll., Waukesha, Wis.

I. W. BURR, M.S.(Chicago) Asst. Prof., Antioch College, Yellow Springs, Ohio

L. C. BYRAM, JR., Student, Univ. of New Mexico, Albuquerque, N.M.

E. BEATRICE CALLAHAN, A.M.(Wisconsin) Asso. Prof., St. Mary's Coll., Holy Cross, Ind.

E. A. CAMERON, A.M.(North Carolina) Instr., Univ. of North Carolina, Chapel Hill, N.C.

EDNA J. COFIELD, A.M.(Alabama) Teacher, West End High School, Birmingham, Ala.

L. S. DANCEY, A.M.(Illinois) Prof., Carroll Coll., Waukesha, Wis.

C. R. EASON, M.S.(Rutgers) Prof., Shaw Univ., Raleigh, N.C.

R. E. GASKELL, M.S.(Michigan) Teacher, High School, Brooklyn, Mich.

W. L. GRAVES, A.B.(Drury) Asst. Instr., Univ. of Pennsylvania, Philadelphia, Pa.

E. G. HARRELL, Ph.D.(Iowa) Head of Dept., Ouachita Coll., Arkadelphia, Ark.

CLAIRE A. HARRISON, A.B.(Northeastern T.C., Tahlequah, Okla.) Teacher, High School, Checotah, Okla.

M. A. HEASLET, Ph.D.(Stanford) Instr., San José State Coll., San José, Calif.

E. W. HELLMICH, Ph.D.(Columbia) Asso. Prof., N. Illinois State Teachers Coll., De Kalb, Ill.

Sister VINCENT DE PAUL KOLEY, M.S.(Notre Dame) Head of Dept., Mary Manse Coll., Toledo, Ohio

A. H. LARSEN, Ph.M.(Wisconsin) Asst. Prof., State Normal Univ., Normal, Ill.

A. E. MALLORY, Ph.D.(Peabody Coll.) Prof., Colorado State Coll. of Educ., Greeley, Colo.

JULIETTE E. MARSHALL, B.S.(N.J. State Coll. for Women) Grad. student, Univ. of Minnesota, Minneapolis, Minn.

CAROL V. McCAMMAN, A.M.(California) Research asst., Civics Research Inst., Washington, D.C.

E. P. NORTHROP, Ph.D.(Yale) Master in math., The Hotchkiss School, Lakeville, Conn.

E. A. OSSE, Design draftsman, retired, U.S. Navy Dept., Philadelphia, Pa.

JANE RUDOLPH, A.B.(Cincinnati) Grad. student, Univ. of Cincinnati, Cincinnati, Ohio

N. M. SHAH, M.A.(Cantab.) Prin., and Prof. of Math., M. T. B. College, Surat, India

C. F. STROBEL, A.B.(Buffalo) Asst., Univ. of Buffalo, Buffalo, N.Y.

HERBERT TATE, M.A.(Dublin) Asso. Prof., McGill Univ., Montreal, Canada

W. G. WARNOCK, Ph.D.(Illinois) Asst. Prof., State Teachers Coll., Hays, Kans.

E. B. WEDEL, A.B.(Bethel Coll.) Teacher, Jr. Coll., Holdenville, Okla.

J. V. WEHAUSEN, M.S.(Michigan) Univ. Fellow, Univ. of Michigan, Ann Arbor, Mich.

FERN WELKER, A.M.(Toledo) Instr., Univ. of Toledo, Toledo, Ohio

R. L. WESTHAFER, A.M.(Harvard) Instr., Hastings Coll., Hastings, Nebr.

C. R. WYLIE, JR., Ph.D.(Cornell) Instr., Ohio State Univ., Columbus, Ohio

The Trustees have voted to accept the invitation of Duke University to hold the Annual Meeting there December 31, 1936–January 1, 1937, in conjunction with the American Mathematical Society.

W. D. CAIRNS, *Secretary-Treasurer*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-fifth regular meeting of the Iowa Section was held at the University of Iowa, Iowa City, Iowa, on Friday and Saturday, April 3–4, 1936, in conjunction with the fiftieth regular meeting of the Iowa Academy of Science. The Section chairman, Professor Julia T. Colpitts, presided at both sections. At the request of the officers of the Iowa Academy of Science the paper on "Actuarial aspects of old age assistance" by Mr. L. A. Knowler of Iowa City was presented at the general meeting of the Iowa Academy of Science. The paper on "Mathematical considerations underlying the construction of geographic maps" by Professor N. B. Conkwright was an invited address of one hour's length.

The attendance was about forty-five, including the following twenty-two members of the Association: R. P. Baker, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, A. T. Craig, C. W. Emmons, Cornelius Gouwens, O. C. Kreider, Ruth O. Lane, R. B. McClenon, F. M. McGaw, J. V. McKelvey, E. E. Moots, J. F. Reilly, H. L. Rietz, Fred Robertson, W. J. Rusk, E. R. Smith, L. E. Ward, C. W. Wester, Roscoe Woods, C. C. Wylie.

On Friday evening the members of the Association joined with the members of the Iowa Academy of Science in the celebration of their fiftieth anniversary dinner. The officers of the Section elected for 1936–37 are as follows: Chairman, R. B. McClenon, Grinnell College; Vice-Chairman, L. E. Ward, State University of Iowa; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by their host, the State University of Iowa and especially the staff of the Department of Mathematics, was adopted at the business session. The following sixteen papers were read:

1. "On the summation of a certain series" by Professor L. E. Ward, University of Iowa.
2. "On the computation of a certain definite integral" by Professor L. E. Ward, University of Iowa.
3. "Tests of significance in change-over experiments" by Professor A. E. Brandt, Iowa State College, introduced by the Secretary.
4. "On the decomposition of $4(x^p-1)/(x-1)$ for a prime p, $100<p<200$" by Professor Cornelius Gouwens, Iowa State College.
5. "The iteration of one-dimensional collineations" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.
6. "Mathematical considerations underlying the construction of geographic maps" by Professor N. B. Conkwright, University of Iowa.

7. "Green's Function in a circular region" by Professor D. L. Holl, Iowa State College, introduced by the Secretary.

8. "Boundary value problems for some elliptic partial differential equations" by H. A. Luther, University of Iowa, introduced by Professor L. E. Ward.

9. "The orthogonal polynomials of Aitkin as applied to the method of least squares" by A. C. Olshen, University of Iowa, introduced by Professor H. L. Rietz.

10. "On the difference between two sample variances" by Professor A. T. Craig, University of Iowa.

11. "A direct expression of certain sets of numbers as four-squares" by Professor W. J. Rusk, Grinnell College.

12. "Space fillers in the plane" by Professor R. P. Baker, University of Iowa.

13. "An asymptotic expression for the approximate values of the roots of the Polynomials of Legendre" by Professor E. R. Smith, Iowa State College.

14. "Plane perspective reflections" by C. E. Snoke, Iowa State College, introduced by Professor E. R. Smith.

15. "Note on the Droz-Farney theorem" by Professor Roscoe Woods, University of Iowa.

16. "Actuarial aspects of old age assistance" by L. A. Knowler, University of Iowa, introduced by Professor H. L. Rietz.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Ward showed that with p an integer, positive, negative, or zero, the sum of the series

$$x - 2^p x^2 + 3^p x^3 - 4^p x^4 + \cdots$$

is $\theta^p x/(1+x)$, where θ is the operator xd/dx and where, for negative values of p, the inverse operator is a division by x followed by an integration with respect to x.

2. In this paper, Professor Ward derives an asymptotic series for the definite integral

$$\int_0^1 \frac{\sin^2 ax}{x^2}\, dx$$

and gives upper bounds for the error when any number of terms up to and including six are used. For $a>10$ the use of five terms gives a result in error by less than 0.00015.

3. In this paper Professor Brandt explained how in certain experiments individuals are frequently subjected to two tests or treatments. Differences between the results are the basis for a controlled comparison of the efficacy of the treatments. Since the comparisons thus obtained may not be independent of the time or of the order of presentation, two groups, either randomly selected or balanced, are treated or tested simultaneously but with the order of presenta-

tion in one group the reverse of that in the other. The arithmetical procedure known as the analysis of variance is presented as the proper statistical method for the analysis of data from such reversal or switch-back trials.

4. This paper supplements a paper on similar decompositions presented last year (see this MONTHLY, vol. 43, 1936, p. 283). Here Professor Gouwens displays the decomposition of $4(x^p-1)/(x-1)$ into the form $Y^2-(-1)^{(p-1)/2}pZ^2$, where Y and Z are polynomials in x and p is a prime, $100<p<200$. The results were presented in mimeographed form; some of these are still available on request.

5. Professor Allen develops a closed form for the ith iterate of $x_i=(\alpha+\beta x)/(\gamma+\delta x)$. The condition that a one dimensional collineation have a period i without being the identity, is

$$\sum_{2j+k+l=i-1} (\alpha\delta)^j\beta^k\gamma^l\binom{j+k}{j}\binom{j+l}{j} = 0.$$

6. This paper includes a brief account of the difficulties met with in an attempt to represent the earth's surface upon a plane, and a short history of cartography. A number of simple projections, including the stereographic and cylindrical projections, are examined. Other types of maps, including the polyconic, Albers equal area, and zenithal equal area projections, are discussed in greater detail. A part of Professor Conkwright's paper is devoted to a discussion of special purpose maps, including the Mercator projection and equal area maps.

7. Professor Holl constructed the Green's function $H(x, y; \xi, \eta)$ associated with the boundary value problem of determining a function $V(x, y)$ satisfying the equation $\nabla^4 V=f(x, y)$ within a circle and with the boundary conditions $V=\bar{V}(s)$ and $\nabla^2 V=g(s)$.

8. Mr. Luther discussed the problem of solving the Laplace and Poisson equations in two variables when the solution $u(x, y)$ must be such that u, $\partial u/\partial n$, $\partial u/\partial n-hu$ assume assigned values along a closed curve. The treatment in terms of integral equations, Green's functions, or by separation of variables was considered. The Green's function for the Laplace equation and the assignment $\partial u/\partial n$ along a circle was exhibited, and some mention was made of the extension of these methods to equations in the normal form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2a\frac{\partial u}{\partial x}+2b\frac{\partial u}{\partial y}+cu=f(x, y).$$

9. Mr. Olson pointed out that in fitting a curve to observed data by the method of least squares, the use of the Aitken form of the Tchebychef orthogonal polynomials will remove much of the criticism as to the laboriousness of this method. If instead of putting $P_k(x)=x^k$ in the graduating polynomial

$$u_x = a_0P_0(x)+a_1P_1(x)+\cdots+a_kP_k(x)+\cdots+a_rP_r(x),$$

$P_k(x)$ is set equal to $T_k(x)$ where $\sum T_r^2(x)=1, \sum T_r(x)T_s(x)=0$, $r>s$, upon the

application of the least square criterion, $T_r(x)$ appears as the rth difference of products of binomial coefficients. This form permits the fitting of curves of advancing degrees without affecting the coefficients of lower order terms. Also the summed square of residuals can be computed in a simple manner before one proceeds to the actual fitting.

10. Given two samples, each consisting of $2m+1$ independent values of a normally distributed variable, Professor Craig investigates the distribution of the difference between the sample variances and determines the probability that this difference will not exceed numerically an assigned multiple of the population variance.

11. Professor Rusk's paper was concerned with the direct expression of certain of the sets of numbers announced at last year's meeting (see this MONTHLY, vol. 42, p. 468). On examination of u_2, u_3, etc., where $u_n^2=x_n^2+y_n^2+z_n^2$, they were found in the case examined, to be series with first or second differences constant. From this fact the general terms were obtained. Then, u_n+y_n and u_n-y_n were found as two squares, and u_n as a four square. Also relations were obtained which enable one to express u_n in terms of u_1, u_2, etc.

12. In this paper, Professor Baker gave an enumeration of the "fixed" and "variable" forms of congruent polygons which fill the plane without gaps. This talk was illustrated with lantern slides.

13. Using an asymptotic expression, Professor Smith shows that a good approximation for the zeros of Legendre's polynomials is obtained from the expression:

$$\sin\sqrt{\lambda}\alpha - \frac{\lambda}{8}\sin\sqrt{\lambda}\alpha - \lambda^{3/2}\alpha\cos\sqrt{\lambda}\alpha$$

where $\lambda=1/n(n+1)$ and α is a zero of $\cos x$ or $\sin x$ accordingly as n is even or odd. For the two zeros nearest unity a better approximation is given by

$$1 + 2\beta\lambda + \tfrac{2}{3}(-\beta+\beta^2)\lambda^2,$$

where β is a zero of $J_0(2i\sqrt{\beta})$. Both of these formulas may be extended by the methods used so as to give more accurate results. For values of n not too small the zeros may be readily computed to five or more decimal places.

14. Mr. Snoke considers in this paper simple illustrative cases of plane perspective reflections, and then proceeds to the general case in which the mirror is considered as a rational curve of order n with an $(n-1)$ fold point at the center of perspectivity. A restriction from the general De Jonquières involutions is made in that the $2n$ simple fundamental points are collinear and coincident in pairs.

15. Professor Woods's paper is concerned with the application of a simple theorem in plane geometry to the nine-point circle and to a proof of the Droz-Farnay Theorem. This theorem says, "every pair of perpendicular lines, drawn through the orthocenter of a triangle, intercepts on the sides of the triangle three segments whose midpoints are collinear."

16. Mr. Knowler discussed some of the actuarial problems of old age assistance. In particular he pointed out the necessity for estimating the number of people in each of the various age groups as the present population is projected into the future. He also presented a brief summary of the operation of the old age assistance set-up in Iowa.

CORNELIUS GOUWENS, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-first annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 2, 1936, with an afternoon session, a dinner, and an evening session. Professor Jesse Pierce, Chairman of the Section, presided at these sessions.

Seventy-seven persons registered attendance, fifty-five of whom were members of the Association, namely: W. E. Anderson, Max Astrachan, A. H. Bailey, F. R. Bamforth, Grace M. Bareis, P. E. Baur, H. M. Beatty, H. A. Bender, M. G. Boyce, J. B. Brandeberry, R. S. Burington, I. W. Burr, W. D. Cairns, F. E. Carr, Rufus Crane, O. L. Dustheimer, C. W. Foard, T. M. Focke, B. C. Glover, N. A. Gilbert, Margaret E. Jones, H. K. Justice, E. M. Justin, M. W. Keller, L. C. Knight, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, C. J. McGee, Florentina Mathias, C. C. Morris, J. R. Overman, Jesse Pierce, H. S. Pollard, Tibor Radó, S. E. Rasor, C. E. Rhodes, R. F. Rinehart, Y. K. Roots, S. A. Rowland, Mary E. Sinclair, E. S. Smith, G. W. Spenceley, H. E. Stelson, C. F. Thomas, M. B. Tolar, C. C. Torrance, J. H. Weaver, Fern Welker, R. B. Wildermuth, F. B. Wiley, J. B. Winslow, Margaret Young-Woodbridge, C. R. Wylie, Jr., C. H. Yeaton.

The following officers were elected for the coming year: Chairman, J. H. Weaver, Ohio State University; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, C. O. Williamson, College of Wooster; Member of Program Committee, J. R. Musselman, Western Reserve University.

It is expected that the next meeting will be held on Thursday, April 1, 1937, at the Ohio State University.

The following papers were read:

1. "Solutions of systems of linear differential equations in the vicinity of singular points" by the Chairman of the Section, Professor Jesse Pierce, Heidelberg College.

2. "A remark on the function log x" by Professor Tibor Radó, Ohio State University.

3. "Apportionment of representatives in Congress" by Professor H. A. Bender, University of Akron.

4. "Calculation of potential about a wire ring" by Professor G. W. Spenceley, Miami University.

5. "An occurrence of Pell's equation in a study of regular polygons" by Professor W. D. Cairns, Oberlin College.

6. "Sufficient conditions for relative maxima and minima of a function of several variables" by Professor O. E. Brown, Case School of Applied Science.

7. "Teaching of indeterminate forms to a class beginning that subject" by Professor F. B. Wiley, Denison University.

8. "The presentation of quantum mechanics from an elementary standpoint" by Dr. C. C. Torrance, Case School of Applied Science.

9. "The valuation of a set of sums by simple interest" by Professor H. E. Stelson, Kent State University.

10. "Properties and applications of generalized determinants of Vandermonde" by H. Reingold, University of Cincinnati, introduced by Professor Barnett.

11. "*What* the mathematician explains" by Professor Mary E. Sinclair, Oberlin College.

12. "Getting the best from our best students" by Professor O. L. Dustheimer, Baldwin-Wallace College.

13. "An experiment in cooperative teaching" by Professor J. H. Weaver, Ohio State University.

14. "The teaching of geometry" by Professor Harold Fawcett, Ohio State University, introduced by Professor Weaver.

In the absence of Professor Brown, paper number 6 was read by Dean Focke of the Case School.

Abstracts of these papers follow:

1. Professor Pierce's paper will appear in an early issue of the Monthly.

2. In teaching Calculus to a class comprised of students above the average, the instructor might feel that he should treat some of the customary topics in such a way as to illustrate some of the fundamental ideas of higher mathematics. The treatment of log x, presented in this paper, is a sample of what Professor Radó was trying to do in this direction in his own teaching.

3. In 1920 Congress could not agree on a method for the apportionment of representatives. As a result, a given state had the same number of representatives from 1910 to 1930 irrespective of its change in population. The two methods most discussed for the apportionment of representatives are the Method of Major Fractions and the Method of Equal Proportions. In Professor Bender's paper a new method is proposed which is believed to be an improvement over the above two methods, and at the same time has the advantage of being very simple in its application.

4. Professor Spenceley shows in this paper that for graphing equi-potential surfaces, the classic expansion of V (potential) in terms of Legendrian coefficients is not nearly so effective as its computation by an easy elliptic integration.

5. In his paper on regular polygons at the St. Louis meeting of the Association, Professor Cairns found that the lengths of the sides of the five possible

11-gons in a unit circle, together with their negatives, are the roots of the equation

$$x^{10} - 11x^8 + 44x^6 - 77x^4 + 55x^2 - 11 = 0.$$

The attempt to factor this polynomial into two factors of the form

$$x^5 + ax^4\sqrt{11} + bx^3 + cx^2\sqrt{11} + dx + e\sqrt{11}$$

led to five equations, and from them, by addition, to Pell's equation $(b+d+1)^2-11(a+c+e)^2=1$. With the aid of the simplest solutions, $b+d+1=10$, $a+c+e=3$, he obtained values of the desired coefficients.

Similarly the polynomial $x^8-7x^6+14x^4-8x^2+1$ for the four regular 15-gons was factored in the form $x^4+ax^3\sqrt{15}+bx^2+cx\sqrt{15}+d$ through the Pell's equation $(b+d+1)^2-15(a+c)^2=1$ and in the form $x^4+ax^3\sqrt{3}+bx^2+cx\sqrt{3}+d$ through the Pell's equation $(b+d+1)^2-3(a+c)^2=1$, and similarly for other instances.

6. Sufficient conditions for relative maxima and minima of a function of several variables which require only the existence of first order derivatives were developed by Professor Brown in this paper. These conditions are easily restated for the case of a function of n independent variables. They fail to be necessary conditions only in the presence of functions not possessing continuous partial derivatives of the first order.

7. Professor Wiley commented on the lack of use of graphs in the teaching of indeterminate forms, as treated in the current texts in the calculus. He suggested that graphs of the functions, showing the vertical straight line branches at the points in question, will clarify, for the beginning student, the ideas involved. He suggested, likewise, their introduction in treating the limit $\Delta y/\Delta x$ as $\Delta x\to 0$.

8. In this paper, Dr. Torrance gave a brief indication of the recent progress made in developing an elementary exposition of quantum mechanics. He emphasized the fact that an elaborate training in classical mechanics was unnecessary before beginning the study of quantum mechanics.

9. There are numerous formulas which approximate simple interest in loans extending over fractional parts of years. Professor Stelson pointed out that the use of simple interest permits a variation in the sum due with a variation in the focal date.

10. In this paper Mr. Reingold discussed certain interesting determinants. Let a_{ij} $(i, j=1, 2, \cdots, n)$ be a square matrix, the determinant of which is a. Let $a_{rs}^{(m)}$ stand for the element in the rth row and the sth column of the mth power of the matrix, and let $a_{ij}^{(o)}$ denote the Kronecker δ. If $b_{ij}(i, j=1, 2 \cdots, n)$ is another square matrix, consider the n^2 traces given by

$$\sum_{i=1}^{n}\sum_{m=1}^{n} a_{im}^{(l)} b_{mi}^{(k)} \qquad (l = 1, 2, \cdots, n;\ k = 0, 1, \cdots, n-1).$$

The problem of proving the functional independence of these n^2 traces as func-

tions in the variables $a_{11}, a_{12}, \cdots, a_{nn}, b_{11}, \cdots, b_{nn}$ leads to the consideration of another determinant, called by Mr. Reingold a generalized determinant of Vandermonde. He then discussed some of the properties of this determinant, and gave a few applications.

11. The mathematician, through published research, treatise and textbook explains mathematics to different types of audiences. Professor Sinclair stresses the importance of bringing to the general public some proper concept of the amazing growth of modern mathematics. Publications, such as Mrs. Logsdon's "The Mathematician Explains," and Professor Bell's "Queen of the Sciences" help to break the layman's prevailing notion that mathematics is merely computing. Granted the necessity of explaining mathematics to the general public, what shall the mathematician explain?

12. Professor Dustheimer described a plan which has been introduced at Baldwin-Wallace College which provides a special incentive for students who have the ability and ambition to do more than the regular class requirements. All candidates for their highest rating must do some of this extra work. For example: one student worked out on his own initiative the square root of 2 to 214 decimal places, while two students recently proved that 2,305,843,008,139,952,128 is a perfect number.

13. An effort is being made at the Ohio State University to relieve the dissatisfaction caused by the fact that students in the college of engineering are not very proficient in applying the principles of mathematics to problems of science and engineering. The departments of mathematics, physics, mechanics, and electrical engineering are cooperating in this effort; and in particular, the departments of physics and mathematics are correlating the topics in the course in engineering physics and the topics in the course in calculus in such a way that the applications of the mathematical principles can be made in physics as soon as they are learned in mathematics. Professor Weaver discussed the plans which are under way to extend this effort to the advanced work in mathematics, physics, and engineering.

14. There is much evidence to support the statement that, as teachers of geometry, we are failing to develop in our pupils that quality of thought which we claim as the unique contribution of our subject. We profess to believe that through the study of demonstrative geometry the pupil will learn what it really means to prove something, that he will become acquainted with the nature of deductive thought and that he will gain power in thinking more clearly about the problems of modern life. How can we teach children what it really means to prove something if we fail to give them some appreciation of recent developments in the field of human thought? To know whether the world in which we live corresponds to Euclid's assumptions or to those of Lobachevski or Riemann is relatively unimportant; but it is very important for the pupil to realize that these assumptions are nothing more than agreements about an abstract space and describe what we have agreed shall constitute the nature of that space. All human thinking is carried on with reference to some set of assumptions and the

conclusions reached are the logical outcomes of the assumptions on which they depend and are true only to the degree that these assumptions are true. Professor Fawcett believes that demonstrative geometry is peculiarly qualified to cultivate this point of view. It cannot be fruitful of this result unless the teachers have a growing appreciation of its unique contribution to the development of reflective thought and are freed from the hampering influence of having to cover a given number of theorems for the purpose of meeting college entrance requirements.

RUFUS CRANE, *Secretary*

A PROOF OF THE LEBESGUE CONDITION FOR RIEMANN INTEGRABILITY

By A. B. BROWN, Columbia University

1. *Introduction.* The proof of Lebesgue's condition for Riemann integrability offered here is believed to be shorter than any appearing in the literature not based on the Lebesgue integral.* We do not use the Lebesgue integral, but develop in the paper the required part of the theory of Lebesgue measure. The treatment is given for the one-dimensional case only, but with the help of some theory of Jordan measure it can be extended to the case of multiple integrals.

Let $f(x)$ be given real and single-valued† *at each point of the interval J: $a \leqq x \leqq b$.*‡

DEFINITION.§ *Function $f(x)$ is integrable in the sense of Riemann over J if there exists a constant R such that, given $\epsilon > 0$, there corresponds a quantity $\delta > 0$ such that if J is divided into a set of non-overlapping*‖ *closed intervals of lengths $\Delta x_1, \cdots, \Delta x_n$ all less than δ, and x_k is taken anywhere in the kth interval, then*

$$\left| R - \sum_{k=1}^{n} f(x_k)\Delta x_k \right| < \epsilon. \tag{1}$$

2. *Preliminaries.* An open set on a finite segment of the x-axis is the sum of a denumerable number of open intervals, whose lengths are the terms of an infinite series which converges to a limit called the *Lebesgue measure* of the open set.¶ The *outer Lebesgue measure* of any set A on a finite segment of the x-axis

* The writer has been informed of the thesis of Pia Nalli, *Esposizione . . . delle diverse definizioni proposte per l'integrale definito . . .* , Palermo 1914, containing a simple proof, but the thesis is not available to the writer.

† The values are finite numbers, but $f(x)$ is not assumed to be bounded.

‡ It would require only slight changes to cover the case that f is defined only at the points of an everywhere dense subset of J.

§ The definition makes no reference to the so-called *improper integral.*

‖ Of course two intervals will have a common end-point in some cases.

¶ In number space of any finite number of dimensions, the Lebesgue measure of an open set equals the inner Jordan measure of the set.

is the greatest lower bound of the measures of all open sets containing A. A set of outer measure zero is said to be of *measure zero*. The sum of a denumerable number of sets of measure zero can be covered by open sets of measures ϵ, $\epsilon/2$, $\epsilon/4$, $\epsilon/8, \cdots$, for any $\epsilon>0$, and since $\epsilon+\epsilon/2+\cdots=2\epsilon$, it follows easily that the sum is itself of measure zero.

By the *oscillation* of $f(x)$ at x_0 is meant the limit, as δ approaches zero, of $\{$the least upper bound of $[|f(x_1)-f(x_2)|$ for all x_1, x_2 in J satisfying $|x_1-x_0|\leqq\delta$, $|x_2-x_0|\leqq\delta]\}$. Thus $f(x)$ is continuous at x_0 if and only if its oscillation at x_0 is zero.

3. *Lebesgue's Theorem.* We now state and prove the

THEOREM. *A necessary and sufficient condition that $f(x)$ be integrable in the sense of Riemann over J is that $f(x)$ be bounded in J and that the set, say T, of points of J at which f is discontinuous, be of Lebesgue measure zero.*

Suppose the integral exists. Since we can keep all the Δx_k's fixed and all the x_k's fixed except a particular one, say x_λ, it follows from (1) that $f(x)$ is bounded in the λth interval. Hence, it is bounded in all of J.

Now suppose merely that $f(x)$ is bounded in J, and that T is of positive outer measure. Then a constant $\alpha>0$ must exist so that if T_1 is the subset of T at each point of which the oscillation is $\geqq\alpha$, then T_1 is of positive outer measure, say m. For, we can classify the points of T into those for which the oscillation d satisfies respectively $1\leqq d$, $\frac{1}{2}\leqq d<1$, $\frac{1}{4}\leqq d<\frac{1}{2}, \cdots$, and if these were all of measure zero, then since they include all points of T, T would be of measure zero, a contradiction.

Now under any division of J into intervals of lengths $\Delta x_1, \cdots, \Delta x_n$, the intervals containing points of T_1 in their interiors are of total measure $\geqq m$, since the number of endpoints of the intervals is finite. Since each of these intervals contains an inner point at which the oscillation of $f(x)$ is at least α, it contains two points at which $f(x)$ differs by at least $\alpha/2$. Hence, we can form

$$\sum_{k=1}^{n} f(x_k)\Delta x_k \quad\text{and}\quad \sum_{k=1}^{n} f(x_k')\Delta x_k,$$

with x_k and x_k' in the kth interval, and the sums differing by at least $\alpha m/2$. Therefore, $f(x)$ is not integrable. We have thus proved the necessity of Lebesgue's conditions.*

Suppose conversely that $f(x)$ is bounded in J, say $|f(x)|\leqq M$, and that the set T is of measure zero.

LEMMA. *Given $\eta>0$, there corresponds $\delta>0$ such that if $S_1=\sum_{k=1}^{n_1} f(x_k)\Delta x_k$ and $S_2=\sum_{k=1}^{n_2}(x_k')\Delta' x_k$, where $\Delta x_k<\delta$ and $\Delta' x_k<\delta$, then $|S_1-S_2|<\eta$.*

Proof. Given $\eta>0$, T can be covered by an open set A of measure $<\eta/8M$. Let $B=J-J\cdot A$, so that B is closed. Let $\delta>0$ be a constant such that if x_0 is

* Up to this point the proof is more or less standard.

on B and x is in J with $|x-x_0|\leqq\delta$, then $|f(x)-f(x_0)|<\eta/8(b-a)$. That such a δ can be found follows, by use either of the Heine-Borel theorem or by the "pinching" process, from the fact that $f(x)$ as defined on J is continuous at each point of B.* Now let†

$$S_3 = \sum_{k=1}^{n_3} f(x_k'')\Delta'' x_k,$$

where $\Delta''x_1, \cdots, \Delta''x_{n_3}$ are the intervals into which J is divided by all the end points of the intervals for S_1 and of the intervals for S_2, and x_k'' is the coordinate of the mid-point of the kth interval of the new set. Then

$$|S_1 - S_3| \leqq \sum_{k=1}^{n_1} |f(x_k) - f(x_k''')|\,\Delta x_k,$$

where x_k''' is chosen from the x_j'''s in the same interval for S_1 as x_k, so as to maximize $|f(x_k)-f(x_k''')|$. Those closed intervals for S_1 containing a point of B contribute to this sum less than

$$\frac{\eta}{4(b-a)}\cdot(b-a) = \eta/4.$$

Those intervals Δx_k containing no point of B are on A and hence contribute less than $2M\cdot\eta/8M=\eta/4$ to the sum. Hence $|S_1-S_3|<\eta/4+\eta/4=\eta/2$. By similar proof $|S_2-S_3|<\eta/2$, and hence $|S_1-S_2|<\eta$, and the lemma is proved.

We draw two conclusions. First, if we let

$$\mathfrak{S}_p = \sum_{k=1}^{p} f(\xi_k)\cdot\frac{1}{p},$$

where $\xi_k=a+k(b-a)/p$, then the sequence $\mathfrak{S}_1, \mathfrak{S}_2, \cdots$, converges to a finite limit, say R. Secondly, given $\epsilon>0$, let us take $\delta>0$ to correspond to $\eta=\epsilon/2$ under the Lemma, and $N>(b-a)/\delta$ so large that $|R-\mathfrak{S}_N|<\epsilon/2$. Then for any subdivision of J into a number n of intervals of lengths $<\delta$, with x_k anywhere in the kth interval, taking $S_1=\sum_{k=1}^{n}f(x_k)\Delta x_k$ and $S_2=\mathfrak{S}_N$, we infer that $|S_1-\mathfrak{S}_N|=|S_1-S_2|<\eta=\epsilon/2$, so that $|R-S_1|<\epsilon$. Consequently, $f(x)$ is integrable in the sense of Riemann.

A STUDY OF GENERAL POLAR TANGENT CURVES

By W. T. STRATTON, Kansas State College

Introduction

If on the radius vector to a point P of a curve in polar coordinates we measure off a distance PQ equal to $\pm k$, where k is a constant, the locus of Q is called

* The property is a slight extension of a theorem regarding uniform continuity.

† In the proof for the case of multiple integrals, S_3 might be an infinite series.

a conchoid of the curve. Perhaps the best known conchoid is the bell-shaped curve known as the Conchoid of Nicomedes. It is the conchoid of the straight line $x=a$ and its equation in polar coordinates is $\rho=a \sec \theta \pm k$ and in Cartesian coordinates it is $(x-a)^2(x^2+y^2)=k^2x^2$. Another well-known conchoid is the conchoid of the circle $\rho=a \cos \theta$ which is usually written $\rho=a \cos \theta+k$. This is commonly called Pascal's Limacon. If the Cartesian equation of any curve is given, the equation of its conchoid may be obtained as follows: (1) express the given equation in polar coordinates, (2) replace ρ by $(\rho \pm k)$ and expand in terms of $(\rho \pm k)$, (3) collect the odd powers of k on one side of the equation and the even powers on the other side, (4) square both sides and then change back into Cartesian coordinates. The resulting equation is the equation of the conchoid.

Not a large number of conchoids have been studied in detail. Among those not mentioned above which have received special attention are those of the Archimedean spiral $\rho=a\theta$, the Fermat spiral $\rho=a\theta^{1/2}$, the Galilean spiral $\rho=a\theta^2$, the hyperbolic spiral $\rho=a/\theta$, and the logarithmic spiral $\rho=ae^{k\theta}$.

The conchoids of the foliate curves $\rho=a \cos p\theta/q$ have been studied by Dr. Moritz (*Cyclic-Harmonic Curves*, University of Washington Publication, June 1923) in relation to their general properties for the larger values of p and q. The conchoids of the tangent curves $\rho=a \tan p\theta/q$ have not been studied except for small values of p and q. The curve $\rho=a \tan \theta$ is often found in textbooks, also the strophoid $\rho=a \tan \theta/2$, but practically nothing has been done with the equation $\rho=a \tan p\theta/q+k$ where k does not equal zero and the values of p and q are comparatively large.

It is our purpose to study the curves given by the polar equation

$$\rho=a \tan p\theta/q+k$$

where a and k are constants, and p and q are integers prime to each other. We can easily show that in the study of the general properties of the curves we shall not lose in generality if we assume a and k both positive.

Cartesian Equations

We have assumed that p and q are integers prime to each other, that is, p/q rational. We wish now to show that the curves are algebraic in Cartesian coordinates, and that any straight line, θ equals a constant, will cut the curve in a finite number of points. In order to show that $\rho=a \tan p\theta/q+k$ can be represented by a rational algebraic equation in x and y we will make use of De Moivre's theorem using the identity

$$(\cos \theta + i \sin \theta)^p = (\cos p\theta/q + i \sin p\theta/q)^q.$$

From the equation $\rho=a \tan p\theta/q+k$ we get at once $\tan p\theta/q=(\rho-k)/a$, hence $\sin p\theta/q=(\rho-k)/\sqrt{a^2+(\rho-k)^2}$ and $\cos p\theta/q=a/\sqrt{a^2+(\rho-k)^2}$, also $\cos \theta=x/\rho$ and $\sin \theta=y/\rho$. Substituting in the identity above

$$\frac{(x+iy)^p}{\rho^p} = \frac{[a+i(\rho-k)]^q}{[a^2+(\rho-k)^2]^{q/2}}$$

or

$$[a^2 + (\rho - k)^2]^{q/2}(x + iy)^p = \rho^p[a + i(\rho - k)]^q.$$

Expanding and equating the real parts and the imaginary parts and remembering that the coefficients are the same as for the binomial expansion except that the even terms are missing in (1) and the odd terms in (2), we write

$$\begin{aligned}(1)\qquad &[a^2 + (\rho - k)^2]^{q/2}[x^p - {}_pC_2x^{p-2}y^2 + {}_pC_4x^{p-4}y^4 \cdots] \\ &\qquad = \rho^p[a^q - {}_qC_2a^{q-2}(\rho - k)^2 + {}_qC_4a^{q-4}(\rho - k)^4 \cdots]\end{aligned}$$

$$\begin{aligned}(2)\qquad &[a^2 + (\rho - k)^2]^{q/2}[{}_pC_1x^{p-1}y - {}_pC_3x^{p-3}y^3 + {}_pC_5x^{p-5}y^5 \cdots] \\ &\qquad = \rho^p[{}_qC_1a^{q-1}(\rho - k) - {}_qC_3a^{q-3}(\rho - k)^3 + {}_qC_5a^{q-5}(\rho - k)^5 - \cdots].\end{aligned}$$

If we eliminate the radical from these two equations we get another form,

$$\begin{aligned}(3)\qquad &[x^p - {}_pC_2x^{p-2}y^2 + {}_pC_4x^{p-4}y^4 \cdots][{}_qC_1a^{q-1}(\rho - k) - {}_qC_3a^{q-3}(\rho - k)^3 \cdots] \\ &\quad = [a^q - {}_qC_2a^{q-2}(\rho - k)^2 + {}_qC_4a^{q-4}(\rho - k)^4 \cdots] \\ &\qquad\qquad [{}_pC_1x^{p-1}y - {}_pC_3x^{p-3}y^3 + {}_pC_5x^{p-5}y^5 \cdots].\end{aligned}$$

When q is odd, equations (1) and (2) must be squared to get rid of the irrationality $[a^2+(\rho-k)^2]^{q/2}$. Moreover, when odd powers of ρ are present in (1), (2), or (3), they must be placed on one side and the equation squared to get rid of the irrationality $\rho=(x^2+y^2)^{1/2}$. In this way one obtains from (1), (2), and (3) respectively equations

$$F_1(x, y) = 0, \qquad F_2(x, y) = 0, \qquad F_3(x, y) = 0,$$

rational in x and y and of degree as follows:

F_2 is of degree $p+q$ when q is even and $k=0$,

F_1 and F_2 are of degree $4(p+q)$ when q is odd and $k\neq 0$,

F_1, F_2, and F_3 are of degree $2(p+q)$ in all other cases.

The Order of the Tangent Curves

Since all points of the curve $\rho=\tan p\theta/q+k$ satisfy all three of the equations $F_i(x, y)=0$, the Cartesian equation of the curve must be $F(x, y)=0$ where F is a common factor of F_1, F_2, and F_3.

When q is odd, the equation of the curve is $F(x, y)\equiv F_3(x, y)=0$, of degree $2(p+q)$. For $k=0$, F_1 and F_2 are the same as F_3; but for $k\neq 0$, $F(=F_3)$ is a factor of F_1 and F_2 which are of degree $4(p+q)$. F_1 and F_2, in this case, both contain factors which vanish for points not on our tangent curve.

When q is even, $k=0$, the Cartesian equation of the curve is $F(x, y)\equiv F_2(x, y)=0$, of degree $p+q$; F_2 in this case being a factor of F_1 and F_3 which are of degree $2(p+q)$. For q even, $k\neq 0$, the equation of the curve is $F(x, y)=0$, of degree $p+q$, where F is a common factor of F_1, F_2, and F_3, all of which are of degree $2(p+q)$.

To summarize, then

(a) If q is odd, k any value, the tangent curves are algebraic and of order $2(p+q)$ in x and y.

(b) If q is even, k any value, the tangent curves are algebraic and of order $(p+q)$ in x and y.

We note further that every tangent curve is of even order when q is odd and of odd order when q is even. It will be of interest to point out that the curves are of order $2(p+q)$ in ρ when q is odd and of order $(p+q)$ when q is even. The values of ρ are the distances from the origin of the points where the curve is met by a line through the origin making an angle θ with the initial line.

The Rationality of the Curves

We have just shown that the curves are algebraic and that their order may be expressed in terms of p and q. We now wish to show that the curves are rational.

An unicursal curve is one whose equation may be expressed rationally in terms of a single parameter. To show that the tangent curves $\rho = a \tan p\theta/q + k$ are rational we shall use the method given by Professor Moritz (See Annals of Mathematics, vol. 35, 1921, p. 37).

Substituting the value of ρ in $x = \rho \cos \theta$ and $y = \rho \sin \theta$ we get

$$x = (a \tan p\theta/q + k) \cos \theta, \qquad y = (a \tan p\theta/q + k) \sin \theta.$$

Now let $\phi = \theta/q$ or $\theta = q\phi$ and substitute in the equation above, then

$$x = (a \tan p\phi + k) \cos q\phi, \qquad y = (a \tan p\phi + k) \sin q\phi.$$

Since p and q are integers $\tan p\phi$, $\cos q\phi$, and $\sin q\phi$ may be expressed rationally in terms of $\tan \phi$, $\cos \phi$, and $\sin \phi$. We can simplify by making a further substitution $t = \tan \phi/2$, then $\cos \phi = (1-t^2)/(1+t^2)$, and $\sin \phi = 2t/(1+t^2)$. We thus get rational expressions for x and y in terms of t, that is

$$x = f(t), \qquad y = \psi(t).$$

Therefore all the curves $\rho = a \tan p\theta/q + k$ are unicursal.

We shall next present two numerical examples showing the application of the general formula for changing from polar to rectangular coordinates, noting particularly the degree of the equation in ρ, thus determining the number of times the curve is cut by a straight line through the origin, the degree in x and y, and the method of expressing the equation in terms of a single parameter. Let us take the equation

$$\rho = a \tan \theta/2 + k, \qquad p = 1, \ q = 2.$$

Then

$$[a^2 + (\rho - k)^2]y = \rho[2a(\rho - k)].$$

This is an equation of the third degree in ρ, hence a straight line through the origin will cut the curve in three points. See figure 1.

Expanding and writing the even powers of ρ on one side of the equation and the odd powers on the other, we get

$$(y - 2a)\rho^2 + (a^2 + k^2)y = 2(y - a)k\rho.$$

Squaring both sides and arranging in powers of ρ, it becomes

$$(y - 2a)^2\rho^4 + (2a^2y^2 - 4a^3y + 4ak^2y - 2k^2y^2 - 4a^2k^2)\rho^2 + (a^2 + k^2)^2y^2 = 0,$$

which is rational in x and y and is of the sixth degree. This equation may be written

$$\{(y-2a)(x^2+y^2)+(a^2-k^2)y+2kx\}\{(y-2a)(x^2+y^2)+(a^2-k^2)y-2kx\}=0,$$

and the first factor, equated to zero, is the desired Cartesian equation of the curve. When $k=0$ the two factors are the same, the equation in this case being

$$y^3 + x^2y - 2ax^2 - 2ay^2 + a^2y = 0.$$

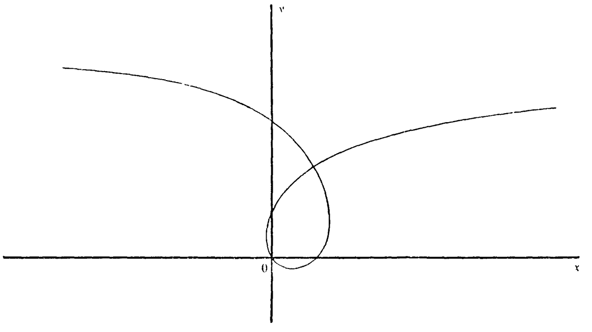


FIG. 1

To express the equation in terms of a single parameter we substitute $\rho = a \tan \theta/2 + k$ in $x = \rho \cos \theta$ and $y = \rho \sin \theta$, then

$$x = (a \tan \theta/2 + k) \cos \theta, \qquad y = (a \tan \theta/2 + k) \sin \theta.$$

Now let $\theta = 2\phi$

$$x = (a \tan \phi + k) \cos 2\phi, \qquad y = (a \tan \phi + k) \sin 2\phi.$$

Further, let $t = \tan \phi/2$, then $\cos \phi = (1-t^2)/(1+t^2)$ and $\sin \phi = 2t/(1+t^2)$. Making the proper substitutions

$$x = \frac{(2at + k - kt^2)(1 - 6t^2 + t^4)}{(1 - t^2)(1 + t^2)^2}, \qquad y = \frac{4t[2at + k(1 - t^2)]}{(1 + t^2)^2}.$$

Now if we let $k=0$, the expressions reduce to the comparatively simple forms

$$x = \frac{2at(1 - 6t^2 + t^4)}{(1 - t^2)(1 + t^2)^2}, \qquad y = \frac{8at^2}{(1 + t^2)^2}.$$

As a second example let us take

$$\rho = a \tan 3\theta/5 + k, \qquad p = 3, \ q = 5.$$

Applying the general formula (3) to change the equation to Cartesian coordinates, it reduces to

$$(x^3 - 3xy^2)[5a^4(\rho - k) - 10a^2(\rho - k)^3 + (\rho - k)^5]$$
$$= [a^5 - 10a^3(\rho - k)^2 + 5a(\rho - k)^4](3x^2y - y^3).$$

If, after collecting the even powers of ρ on one side of the equation and the odd powers on the other side, both sides are squared, the equation becomes rational in x and y and is of the sixteenth degree. If k is set equal to zero, the order of the equation remains the same, showing that the order is $2(p+q)$ without reference to the value of k. Figure 2 shows the curve for $k=0$. The equation can be expressed in terms of a single parameter by following the method used in the first example.

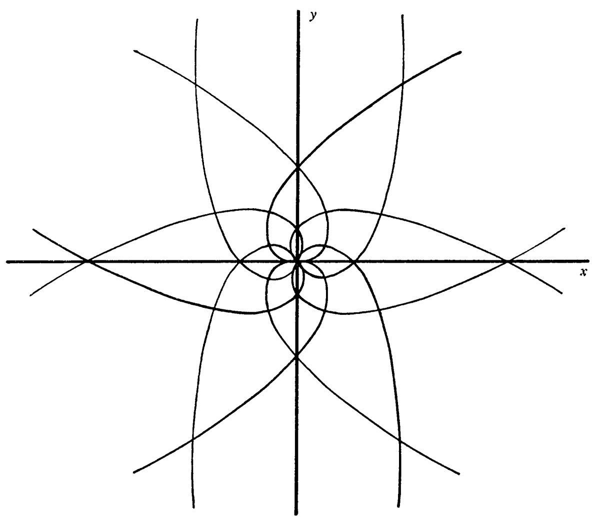


FIG. 2

Axes of Symmetry

With the equation expressed in Cartesian coordinates it is easy to determine whether or not the curves are symmetrical with respect to the x- and y-axes but in determining other axes of symmetry it is more convenient to use polar coordinates. In general if a curve $\rho = f(\theta)$ has an axis of symmetry it is necessary and sufficient that for some value of ϕ one or other of the equations

$$f(\phi + \theta') = f(\phi - \theta' + 2n\pi) \tag{4}$$

$$f(\phi + \theta') = -f[\phi - \theta' + (2n + 1)\pi], \tag{5}$$

where n is zero or some integer, shall be satisfied for all values of θ'. The various values of ϕ which satisfy these conditions give the directions of the axes of symmetry. Applied to $\rho = a \tan p\theta/q + k$ (4) becomes

$$\tan p(\phi - \theta' + 2n\pi)/q = \tan p(\phi + \theta')/q$$

and hence

$$p(\phi - \theta' + 2n\pi)/q = \mu\pi + p(\phi + \theta')/q, \qquad \mu = 0, 1, 2, 3, \cdots .$$

This can not be satisfied for all values of θ', hence there are no axes of symmetry arising from condition (4).

For the condition (5), we have

$$- a \tan p[\phi - \theta' + (2n + 1)\pi]/q - k = a \tan p(\phi + \theta')/q + k,$$
$$\tan p[\phi - \theta' + (2n + 1)\pi]/q + \tan p(\phi + \theta')/q = - 2k/a.$$

In order for this condition to be satisfied for all values of θ', k must be zero; hence, setting the above expression equal to zero, making use of the formula $\tan \alpha = \sin \alpha/\cos \alpha$, and reducing, we have

$$\frac{\sin \frac{p}{q}[\phi-\theta'+(2n+1)\pi]\cos\frac{p}{q}(\phi+\theta')+\cos\frac{p}{q}[\phi-\theta'+(2n+1)\pi]\sin\frac{p}{q}(\phi+\theta')}{\cos\frac{p}{q}[\phi-\theta'+(2n+1)\pi]\cos\frac{p}{q}(\phi+\theta')}=0,$$

or

$$\sin \frac{p}{q}[2\phi + (2n + 1)\pi] = 0.$$

This is independent of θ' and satisfies condition (5). Hence

$$\frac{p}{q}[2\phi + (2n + 1)\pi] = \mu\pi,$$
$$\phi = \frac{[\mu q - p(2n + 1)]\pi}{2p}.$$

We are interested only in the positive fractional values of the coefficients of π; hence, indicating $\mu q - p(2n+1)$ by λ, $\phi = \lambda\pi/2p$, where $\lambda = 0, 1, 2, 3, \cdots (2p-1)$ if q is odd and $\lambda = 1, 3, 5, \cdots (2p+1)$ if q is even.

There are $2p$ axes of symmetry if q is odd and p axes of symmetry if q is even. If q is odd the initial line is always an axis of symmetry. In every case k must be equal to zero.

Centro Symmetry

The necessary and sufficient condition for centro-symmetry is that one or other of the equations

(6) $$f(\theta) = f[\theta + (2n + 1)\pi]$$

(7) $$f(\theta) = - f(\theta + 2n\pi),$$

n equal to some integer, shall be satisfied for all values of θ. For $\rho = a \tan p\theta/q + k$, (6) takes the form

$$\tan p\theta/q = \tan p[\theta + (2n+1)\pi]/q,$$

from which

$$p[\theta + (2n+1)\pi]/q = \mu\pi + p\theta/q$$

and

$$\mu = p(2n+1)/q, \qquad \mu = 0, 1, 2, \cdots, p.$$

This condition is clearly satisfied if q is odd, since $2n+1$ is odd and μ may have any integral value whatever; but if q is an even number the condition will not be satisfied.

From condition (7) we get

$$a \tan p\theta/q = -a \tan p(\theta + 2n\pi)/q - 2k,$$

$$\tan p\theta/q + \tan p(\theta + 2n\pi)/q = -2k/a,$$

$$\frac{\sin \dfrac{p}{q}(2\theta + 4\mu\pi)}{\cos \dfrac{p}{q}(\theta + 2n\pi)\cos \dfrac{p}{q}\theta} = -\frac{2k}{a}.$$

This condition can not be satisfied for all values of θ even if k is zero, hence there is no centro-symmetry arising here. We conclude, then, that there is centro-symmetry if q is odd but none if q is even.

Polar Period

It is evident that ρ will assume the same series of values as $p\theta/q$ passes through the values from 2π to 4π as were assumed for the values from 0 to 2π; the same is true for from 4π to 6π and so on. If the ρ's are to be the same for θ as for $\theta + 2n\pi$, then

$$\tan p(\theta + 2n\pi)/q = \tan p\theta/q. \tag{8}$$

The least value of n which satisfies this equation for every value of θ determines the interval through which θ must vary before the curve begins to repeat. This interval is usually called the polar period of the curve. From (8) we get

$$p(\theta + 2n\pi)/q = \mu\pi + p\theta/q.$$

In order that this expression may be satisfied for all values of θ, we must have $2np/q = \mu$. The least value of n that satisfies this condition if q is even is $n = q/2$; hence we conclude that if q is even the polar period is $q\pi$. If q is odd then $n = q$ satisfies the condition and the polar period is $2q\pi$.

Polar Maxima and Minima

It is a well established fact that a necessary and sufficient condition for a maximum in polar coordinates is that

$$\frac{d\rho}{d\theta} = 0, \qquad \frac{\rho d^2\rho}{d\theta^2} < 0$$

and for a minimum point

$$\frac{d\rho}{d\theta} = 0, \qquad \frac{\rho d^2\rho}{d\theta^2} > 0, \quad \text{or} \quad \rho = 0.$$

Let us now examine $\rho = a \tan p\theta/q + k$ for maxima and minima.

$$\frac{d\rho}{d\theta} = \frac{ap}{q} \sec^2 \frac{p\theta}{q}.$$

But $\sec^2 p\theta/q$ can not be zero, therefore, there are no maxima or minima except when ρ equals zero. There are p of these values if q is even and $2p$ if q is odd. For if $\rho = a \tan p\theta/q + k = 0$ then $\tan p\theta/q = -k/a$, and $p\theta/q = \mu\pi + \tan^{-1}(-k/a)$, that is

$$\theta = q\mu\pi/p + q \tan^{-1}(-k/a)/p$$

where in general θ will have separate values if μ assumes the integral numbers $0, 1, 2, 3, 4 \cdots (p-1)$ and when q is odd the branches pass through the origin on opposite sides of the common tangent.

Double Points

If $\rho = a \tan p\theta/q + k$ is to have double points outside the origin, one of the two following conditions must be satisfied, (I) the ρ's equal and the θ's differ by an even multiple of π, (II) the ρ's equal and opposite in signs and the θ's differ by an odd multiple of π. We shall refer to the double points as of set I or set II according as condition (I) or (II) is satisfied.

Double Points of Set I. The condition for double points of set I may be stated by the following equations:

$$\theta - \theta' = 2\lambda\pi \tag{9}$$

$$a \tan p\theta/q + k = a \tan p\theta'/q + k. \tag{10}$$

Simplifying equation (10), we get

$$\tan p\theta/q = \tan p\theta'/q.$$

Substituting for θ' from (9)

$$\tan p\theta/q = \tan p(\theta - 2\lambda\pi)/q$$

or

$$\tan p\theta/q = \frac{\tan p\theta/q - \tan 2\lambda p\pi/q}{1 + \tan p\theta/q \tan 2\lambda p\pi/q},$$

from which

$$\tan p\theta/q + \tan^2 p\theta/q \tan 2\lambda p\pi/q = \tan p\theta/q - \tan 2\lambda p\pi/q,$$

$$\tan^2 p\theta/q = -1.$$

That is

$$\tan p\theta/q = \pm i.$$

Therefore, there are no real values of θ for which the ρ's are equal and the θ's differ by an even multiple of π, that is there are no double points of set I.

Double Points of set II. The condition for double points of set II may be stated as follows:

(11) $$\theta - \theta' = (2\lambda + 1)\pi,$$

(12) $$a \tan p\theta/q + k = -a \tan p\theta'/q - k.$$

Substituting the value of θ' from (11) into (12) we get

$$\tan p\theta/q + \tan p[\theta - (2\lambda + 1)\pi]/q = -2k/a.$$

Expanding the second expression, clearing of fractions, and collecting,

$$\tan \frac{p}{q}(2\lambda + 1)\pi \tan^2 \frac{p\theta}{q} + \left[2 + \frac{2k}{a} \tan \frac{p}{q}(2\lambda + 1)\pi\right] \tan \frac{p\theta}{q} + \left[\frac{2k}{a} - \tan \frac{p}{q}(2\lambda + 1)\pi\right] = 0.$$

Solving for $\tan p\theta/q$, the result may be written

$$\tan \frac{p\theta}{q} = -\frac{k}{a} - \cot \frac{p}{q}(2\lambda + 1)\pi \pm \sqrt{\cot^2 \frac{p}{q}(2\lambda + 1)\pi + \frac{a^2 + k^2}{a^2}},$$

$$\theta = \frac{\mu q\pi}{p} + \frac{q}{p} \tan^{-1}\left[-\frac{k}{a} - \cot \frac{p}{q}(2\lambda + 1)\pi \pm \sqrt{\cot^2 \frac{p}{q}(2\lambda + 1)\pi + \frac{a^2 + k^2}{a^2}}\right],$$

$$\theta' = \left[\frac{\mu q\pi}{p} - (2\lambda + 1)\pi\right] + \frac{q}{p} \tan^{-1}\left[-\frac{k}{a} - \cot \frac{p}{q}(2\lambda + 1)\pi \pm \sqrt{\cot^2 \frac{p}{q}(2\lambda + 1)\pi + \frac{a^2 + k^2}{a^2}}\right].$$

Now, since $\theta - \theta' = (2\lambda + 1)\pi$ and if θ and θ' are angles within the period, then

$$\theta - \theta' = (2\lambda + 1)\pi < 2q\pi,$$

that is $\lambda < q$. From the result just given $\theta = \mu q\pi/p$ plus or minus some angle which may be assumed to be positive if θ is to be always within the period 0 to $2q\pi$. Then $\mu q\pi/p < 2q\pi$ from which we see that $\mu < 2p$. The values of λ may be $0, 1, \cdots (q-1)$ and for μ, $0, 1, 2, \cdots (2p-1)$. We conclude, then, after taking into consideration the double signs in the values of θ and θ' that the maximum number of double points is $4pq$. However, after examining the values of θ and θ' we see that since $\cot(n\pi + \alpha) = \cot \alpha$ the value $4pq$ is twice as large as it should be. For let l be some integer between 0 and $q/2$, then

$$\theta_{l,\mu} = \frac{\mu q\pi}{p} + \frac{q}{p}\tan^{-1}\left[-\frac{k}{a} - \cot\frac{p}{q}(2l+1)\pi \pm \sqrt{\cot^2\frac{p}{q}(2l+1)\pi + \frac{a^2+k^2}{a^2}}\right],$$

$$\theta_{q/2+l,\mu} = \frac{\mu q\pi}{p} + \frac{q}{p}\tan^{-1}\left[-\frac{k}{a} - \cot\frac{p}{q}\left\{2\left(\frac{q}{2}+l\right)+1\right\}\pi \pm \sqrt{\cot^2\frac{p}{q}\left\{2\left(\frac{q}{2}+l\right)+1\right\}\pi + \frac{a^2+k^2}{a^2}}\right].$$

That is

$$\theta_{l,\mu} = \theta_{q/2+l,\mu}$$

or λ for distinct values of θ only, assumes the values from 0 to i where i is the integral part of the quotient $(q-1)/2$. We conclude, then, that the maximum number of double points in finite space outside the origin is $2pq$.

If we examine the values of θ we see that whenever $q = 2\lambda + 1$ the expression involving the cotangent becomes either infinite or 0, and $-\theta$ becomes either $\mu q\pi/p + q\pi/2p$ or $\mu q\pi/p$. If we substitute the value of θ into $\rho = a \tan p\theta/q + k$ then ρ is either infinite or k. There are, then, p double points at infinity and p at zero if k is zero. Now since $\lambda < q$ there can be only one value of λ for which $q = 2\lambda + 1$, so there are exactly $2p(q-1)$ double points when q is an odd number.

We shall next determine the number of double points if q is even. The period is $q\pi$, hence, λ is less than $(q-1)/2$, that is, λ takes the values $0, 1, 2, \cdots$ to an integral value less than $(q-1)/2$, and μ will assume the values $0, 1, 2, \cdots, (p-1)$, therefore, there are a maximum of $pq/2$ double points when q is even.

THEOREM: *The double points of set* II *lie* $2p$ *in number on* $(q-1)$ *circles if* q *is odd.*

Proof: Substitute the value of θ above into the general equation $\rho = a \tan p\theta/q + k$, then, after simplifying,

$$\rho = -a\cot\frac{p}{q}(2\lambda+1)\pi \pm \sqrt{\cot^2\frac{p}{q}(2\lambda+1)\pi + \frac{a^2+k^2}{a^2}}\cdot$$

The value of ρ is independent of μ; hence, for each μ there will be one value of ρ and if we remember that a negative ρ lies on the same circle as a positive ρ we see that there will be exactly $2p$ points on each circle. The number of circles depends upon the different values assumed by λ, but λ will range from 0 to $(q-1)$, and one set of values of ρ will be infinite so there will be exactly $(q-1)$ circles.

Similarly we can prove that the double points of set II lie p in number on $q/2$ circles if q is even.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE UNGENERATED SEVEN AS AN INDEX TO PYTHAGOREAN NUMBER THEORY

By GRACE MURRAY HOPPER, Vassar College

Of all the fanciful and philosophical attributes said to have been assigned to numbers by the Pythagorean theorists, the most enigmatic is the apotheosis of the number 7 as Athena, sprung full-armed from the head of Zeus. "For the unit being one and single and pure, begot the number 7, which has no mother but is born of itself alone, without taking any additional material whatever to aid him."* "On which account the Pythagoreans compare this number to the Goddess always virgin who was born without a Mother."† Similar statements in the writings of Cedrenus quoting Philolaus,‡ Macrobius,§ Plutarch,|| Ambrose,¶ Capella,** Isidore,†† and Clement of Alexandria‡‡ bear witness to the solidarity of the tradition.

To the modern mathematician the segregation of 7 from the members of the decad as the number which "neither produces nor is produced"§§ is entirely

* Philo, *On Who is the Heir of Divine Things*, xliii, in C. D. Young, *Works*, London, 1854, II, 136.

† Philo, *On the Allegories of the Sacred Law*, v; *Works*, I, 55.

‡ K. S. Guthrie, *Pythagorean Source Book and Library*, Platonist Press, 1919, 24.

§ Macrobius, *Commentarius in Somnium Scipionis*, I, 6; *Works*, ed. M. Nisard, Paris, 1868.

|| Plutarch, *Of Isis and Osiris*, X; *Miscellanies and Essays*, IV, ed. William Goodwin, 6th ed., Boston, 1898.

¶ Ambrose, *De Noe et Arca*, XII, 39; Migne P. L. XIV, 397–8.

** Martianus Capella, *De Nuptiis Philologiae et Mercurii*, VII, ed. Adolphus Dick, Leipzig, 1925.

†† Isidorus, *Liber Numerorum*, VIII, 34; Migne P. L. LXXXIII, 186.

‡‡ Clement of Alexandria, *Stromata*, VI, 16; *Ante Nicene Christian Library*, XII, ed. Roberts and Donaldson, Edinburgh, 1869.

§§ Philo, *A Treatise on the Account of the Creation of the World*, xxxiii, *Works*, I, 29.

meaningless. On the score of primality it differs in no respect from 3 and 5. If by generation is meant addition its claim to uniqueness is even less founded. Yet when we turn for illumination to the writers who repeat the tradition we are faced with a conspiracy of silence. Philo, in fact, is so far from offering an explanation of his assertion that he proceeds to generate 7 by addition in all possible ways: "The number 7 consists of 1 and 2 and 4 · · · for it (7) is divided first of all into the number 1 and the number 6; then into the 2 and the 5; and last of all into the 3 and the 4 · · · Every organic body has 3 dimensions, length, depth, and breadth; and 4 boundaries, the point, the line, the superficies, and the solid; and by these, when combined, the number 7 is made up."* Either Philo is exhibiting a quite unaccustomed reserve or else he is himself at a loss to justify the 'virginity' of 7.

Perhaps the clue lies in the categorical statement of Clement of Alexandria: "For neither by adding to another within 10 is 7 produced nor when added to any number within 10 does it make up any of them."† Here is the most definite statement of the problem. 'Generation' is explicitly defined as addition and still the isolation of 7 is maintained. Clearly some additive process other than the usual arithmetical addition is indicated. At the same time the failure of the Neo-pythagoreans to clarify the tradition which they have received intimates that the advance of mathematical knowledge had here obscured some more ancient mathematical procedure.

The necessity of using the works of Nichomachus, Porphyrius, Iamblichus, and Proclus, undoubtedly influenced by the Alexandrian schools, has presented to the mathematical historian the seemingly impossible task of sifting out from the later material that portion of the number theory directly attributable to Pythagoras and his immediate followers. Yet the internal evidence of the tradition of the virgin number, baffling as it seems to later writers, is perhaps in itself the key to the lost Pythagorean *modus operandi*. For it is possible to set up a completely consistent explanation of the Athena-like qualities of 7—and that by considering numbers only as geometric entities, spatial and concrete.

According to this principle the geometric figures and the numbers are divinely and inextricably linked. One, the monad, is not a number. It is the principle of 'sameness,' of stability, right, equality, light. In eternal opposition is the dyad, the principle of 'otherness,' mutability, diversity, inequality, darkness.‡ Itself not a number, it is the link between the monad and numbers.§ Three then is the first number in the Pythagorean sense, "for as rennet curdles flowing milk by its peculiar creative and active faculty, so the unifying force of the monad advancing upon the dyad, source of every movement and breaking

* Philo, ibid, xxxi–xxxii; *Works*, I, 27–9.

† Clement of Alexandria, *Stromata*, VI, 16.

‡ Porphyrius, "*Life of Pythagoras*," 38, 49, in Guthrie, 141, 145.

§ An arithmetic "proof" of this is offered by Proclus in his commentary on Euclid. For 1 the double is greater than the square, $1+1>1\times1$; for 2, $2+2=2\times2$; whereas for all successive numbers, $3+3<3\times3$, etc.

down, infixed a bound, and a *form*, that is, *number* upon the triad; for this is the beginning of actual number."* Three is the first number because number is limited by form and the triangle is the first plane figure.

This triad becomes the underlying base of all corporeal things. It is the "most original form of plane number"† and the "absolute principle of generation of begotten things."‡ In this connection it is of further importance to note the isolation of one specific triangle and that not the equilateral as might have been expected. Proclus quotes Philolaus as saying, "the right angled triangle is the beginning of all figures and of all qualities."[17] Philo observes that "the rectangular triangle, which is the beginning of all qualities consists of the numbers 3 and 4 and 5,"§ and is lost in admiration of "that holy number of 50, being the power of a rectangular triangle which is the foundation of the creation of the universe."‖

Smith¶ tells us that this triangle was known in ancient Egypt long before the time of Pythagoras. It is mentioned in a papyrus of the twelfth Dynasty (c. 2000 B.C.). According to Plutarch's conjecture, "the Egyptians called it the most perfect of triangles, because they likened the nature of the universe principally to that."**

Just as 3 defines the first plane figure, so to 4 is assigned the tetrahedron, the first of solid figures. Thus the first plane number 3 and the first solid number 4, the first odd or masculine number and the first even or feminine number, unite under the form of the right triangle to produce 5, the hypotenuse and 6, the area. So Plutarch: "Now in that triangle the perpendicular consists of 3 parts, the base of 4, and the subtense of 5, its square being equal in value with the squares of the two that contain it. We are therefore to take the perpendicular to represent the male property, the base female, and the subtense that which is produced by them both."†† In this manner 5 becomes the number of marriage, and 6, the area of this generative triangle is similarly related to the idea of generation.

The number 8 defines the cube, the second solid figure, while 9, the square of 3, is the first odd square, both familiar elementary forms. But 7 and 7 alone is unrelated to the elementary geometric forms and is ungenerated by the members of the cosmic triangle. It is indeed the number *par excellence* of the mistress of Athens.

---

* *Theologumena Arithmetica*, ed. Ast, 8, quoted in F. E. Robbins and L. C. Karpinski, "Studies in Greek Arithmetic" in *Nichomachus of Gerasa*, New York, 1926, 117. Italics are mine.

† Nichomachus of Gerasa, *Introduction to Arithmetic*, II, 7, 4; trans. M. L. D'Ooge, New York, 1926, 239.

‡ Proclus quoting Philolaus, *Ad Euclid Elementa*, I, 38 in Guthrie, 20.

§ Philo, ibid, xxxii; *Works*, I, 28.

‖ Philo, *On the Life of Moses*, iii, 4; Works, III, 90.

¶ David Eugene Smith, *History of Mathematics*, II, 288.

** Plutarch, op. cit. lvi, 56; *Miscellanies*, 115.

†† Ibid.

The very circumstance that 7 is the only member of the decad considered ungenerated argues that the early Pythagorean concept of generation was entirely geometric in its nature. The Neo-pythagorean inability to explain the virginity of 7 is paralleled by a similar confusion in their attempts to justify 5 and 6 as the marriage numbers. The common solution considers them as the sum and product of 2 and 3 but the use of the dyad as the feminine principle amounts to a denial of their recurring definition of 3 as the first number.* Furthermore it was entirely contrary to their philosophical tradition to give the feminine unstable dyad precedence over the first masculine positive principle. The confusion of these old and new traditions is most evident in Plutarch's closing statement: "Now in that triangle the perpendicular consists of 3 parts, the base of 4, and the subtense of 5 · · · . We are therefore to take the perpendicular to represent the male property, the base female, and the subtense that which is produced by them both · · · . For the number 3 is the first odd and perfect number, and the number 4 is a square, having for its sides the even number 2. The number 5 also in some respects resembles the father and in some again the mother, being made up of 3 and 2."† In short, the Neo-pythagorean dilemma appears to have arisen from their inability to comprehend the number 4 as a first principle; that is, to have lost sight of the purely geometric basis of number.

We may assume that this geometric number theory was known at least until the fourth century B.C. since it coincides with Plato's derivation of the marriage number in the *Republic* from the 3, 4, 5 right triangle.‡ The section, "But the number of a human creature is the first number in which root and square increases, having received 3 distances and 4 limits of elements that make both like and unlike and wax and wane, render all things commensurable and rational with one another," is interpreted by Adam§ and Young‖ alike as $3^3+4^3+5^3=216$, relating the parts of the cosmic triangle. The subsequent computation of 12960000 is detailed by Adam as

$$(3 \times 4 \times 5)^4 = (3600)^2 = 4800 \times 2700.$$

The first term is directly derived from the generative triangle. The second and last are elementary geometric forms, a square, "the one equal an equal number of times," and "a hundred cubes of 3." But the fact that 4800 is not so simply derived accounts for Plato's tortuous derivation of this term from 5, an element

* Iamblichus, *Life of Pythagoras*, xxviii, in Guthrie, 71; Augustine, *City of God*, XI, 31; ed. Marcus Dods, Edinburgh, 1871, I, 475; *Theologumena Arithmetica*, Ast, 8, cited by Robbins and Karpinski, 117.

† Plutarch, ibid.

‡ *Republic*, 546, B, C, D.

§ James Adam, *The Republic of Plato*, Cambridge, 1902, II, 264–306.

‖ Grace Chisholm Young, *On the Solution of a Pair of Simultaneous Diophantine Equations connected with the Nuptial Number of Plato*, *Proceedings of London Math. Soc.*, ser. 2, xxiii, 27–44 (1925).

of the triangle, "the one side of 100 squares rising from the rational diameters of 5 diminished by 1 each, or if from irrational diameters, by 2."

A Note on Partial Fractions

By L. S. Johnston, University of Detroit

In manuscript notes left by the late Rear Admiral John P. Merrell, United States Navy, Head of the Department of Mathematics, United States Naval Academy in the 1890's and later (about 1905–1908) President of the Naval War College, the writer discovers the following method of resolving into its partial fractions the proper fraction

$$\frac{f(x)}{(x^2 + ax + b)(x^2 + cx + d)}$$

where the denominator is not separated into linear factors. While it does not appear that the method possesses any advantage on the score of brevity, it is at least somewhat different from the more conventional methods. The method will be illustrated by a particular example rather than proved, though the proof is not difficult.

Consider the equation

$$\frac{x^3 - 8x^2 - 10x - 30}{(x^2 + x + 3)(x^2 + 2x + 5)} = \frac{Ax + B}{x^2 + x + 3} + \frac{Cx + D}{x^2 + 2x + 5} \tag{1}$$

or

$$x^2(x - 8) - 10x - 30 = (Ax + B)(x^2 + 2x + 5) + (Cx + D)(x^2 + x + 3). \tag{2}$$

Replacing x^2 by $-2x-5$, but not disturbing x itself, we have

$$(-2x - 5)(x - 8) - 10x - 30 = (Cx + D)(-x - 2),$$

or

$$-2x^2 + x + 10 = -Cx^2 - Dx - 2Cx - 2D.$$

Again replacing x^2 by $-2x-5$ without disturbing x itself, we have

$$5x + 20 = -Dx + 5C - 2D.$$

Equating coefficients of like powers of x, we have $D=-5$, $C=2$. Similarly we might have replaced, in (2), x^2 by $-x-3$ and carried through the operation in exactly the same manner, finding $A=-1$, $B=-3$.

In every case, then, the method consists in writing the analogue of (2) in such a way as to display the left member as the sum of linear functions of x multiplied by integral powers of x^2, and then replacing x^2 by the linear function of x which will make any given factor of the denominator vanish. This replace-

ment of x^2 by the proper linear function of x continues so long as the resulting equation contains powers of x higher than the first. Eventually such substitutions will reduce the equation to linear form, at which time we equate coefficients of like powers of x.

The method is perfectly general for any proper fraction the denominator of which consists of quadratic factors none of which are repeated.

As Admiral Merrell remarks in his note, the reader will readily note that the method is equivalent to resolving the denominator into complex linear factors and expanding in the usual manner, and then equating coefficients of real and imaginary terms in the resulting equation.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Tables of the Higher Mathematical Functions. Computed and compiled under the direction of H. T. Davis. Bloomington, Indiana, The Principia Press, 1935. xiv+391 pages. $6.50.

Interpolation and Approximation by Rational Functions in the Complex Domain. By J. L. Walsh. American Mathematical Society Colloquium Publications, volume xx. New York, 1935. x+382 pages. $5.00.

Mathematics of Modern Engineering. By R. E. Doherty and E. G. Keller. Volume I. New York, John Wiley and Sons, 1936. xxii+314 pages. $3.50.

Gewachsene Raumlehre. By B. Petermann and Karl Hagge. Freiburg, Herder & Co., 1935. x+164 pages, with plates. 4.40 marks.

The Revolution in Physics. By Ernest Zimmer. New York, Harcourt Brace and Company, 1936. xvi+240 pages. $3.75.

Back to Newton. By Georges de Bothezat. New York, G. E. Stechert & Co., 1936. viii+152 pages. $2.50.

Advanced Algebra. By P. H. Graham and F. W. John. New York, Prentice-Hall, Inc., 1936. xvi+262 pages. $1.85.

(The previous edition of this work was reviewed by H. M. Hosford in this MONTHLY, 1931, page 109.)

College Algebra. By A. M. Harding and G. W. Mullins. Revised Edition. New York, The Macmillan Company, 1936. viii+379 pages. $2.25.

(The previous edition of this work was reviewed by E. L. Mickelson in this MONTHLY, 1928, page 311.)

Commercial Algebra, College Course. By B. H. Crenshaw, T. M. Simpson, and Z. H. Pirenian. New York, Prentice-Hall, Inc., 1935. x+174+20 pages. $2.00.

(This text comprises Part I of the same authors' *Mathematics of Finance,*

reviewed by H. E. Bushey in this MONTHLY, 1932, page 540. A chapter on Statistics has been added.)

An Introduction to Projective Geometry. By L. N. G. Filon. London, Edward Arnold & Co., 1935. Fourth Edition. xviii+407 pages.

Plane and Spherical Trigonometry. By H. L. Rietz, J. F. Reilly, and Roscoe Woods. Revised Edition. New York, The Macmillan Company, 1936. xii+168+72+xiii pages. $2.00.

Plane and Spherical Trigonometry. By A. L. Nelson and K. W. Folley. New York, Harper and Brothers, 1936. xiv+186+92 pages.

Business Mathematics. By C. L. Richtmeyer and J. W. Foust. New York, McGraw-Hill Book Company, 1936. xii+250 pages. $2.00.

Text Book of Spherical Trigonometry. By Pramatha Nath Mitra. University of Calcutta, 1935. xxii+164 pages.

Die Magischen Quadrate. By Alb. Schaefer. Leipzig, B. G. Teubner, 1935. Paper, 36 pages. Rm. 1.60.

REVIEWS

Differential Equations for Electrical Engineers. By Philip Franklin. New York, John Wiley and Sons, 1933. vii+300 pages. $2.75.

This book is unlike any other text on differential equations now in the market. The first seventy-nine pages, i.e. all of Chapters I and II, deal with complex numbers, average values, and Fourier Series. Toward the end of the book pages 211 through 280, namely Chapters VII and VIII, deal with the theory of analytic functions of both real and complex variables, of their expansions in series, and of operations with these series, also with the theory of the convergence of Fourier Series.

Linear ordinary differential equations with constant coefficients are well treated in Chapter III, along with an excellent discussion of special equations of this type that arise in electrical engineering. Chapter IV discusses partial derivatives and partial differential equations. The heading of Chapter V speaks for itself, namely "The Physical Meaning of Certain Partial Differential Equations." Chapter VI deals with the solutions of partial differential equations that satisfy given boundary conditions.

It is easy to see that the text is "the outgrowth of a course given for over ten years to electrical engineering students in their junior year at the Massachusetts Institute of Technology," as the preface states. All the difficulties usually encountered by students are discussed in this book. This text could follow a first course in the calculus, but it would go better after a semester course in elementary differential equations. It could be studied without the help of a teacher because it is full of illustrative examples, the discussion of theory is quite thorough, there are numerous hints and discussions given with the problems, and the answers are given for most of the problems. A good bibliography is included, and the index is excellent. The book is beautifully set up and printed, with numerous very good figures. The lists of exercises are so spaced that the

book divides naturally into a sufficient number of lessons for a class meeting three times a week during a semester.

The discussion of complex numbers begins with the topics treated in college algebra but continues on to such more advanced topics as conformal mapping, derivatives of complex variables, the Cauchy-Riemann differential equations, Cauchy's integral theorem, Taylor's series. Chapter IV opens with an elementary discussion of derivatives and differentials, change of variables, but follows this with a discussion of the formation of partial differential equations by the elimination of an arbitrary function and the solution of certain useful types of partial differential equations. The geometrical significance of the abovementioned topics as well as of curvilinear coordinates is well brought out. The development of certain differential equations from the physical theory is very thoroughly discussed. Even a student who knows little or no physics could follow this discussion. The treatment of solutions satisfying certain boundary conditions is especially valuable for students of electrical engineering. In fact, in every way this book is admirably suited to its purpose of introducing electrical engineers to differential equations that arise in their field.

A. D. CAMPBELL

Editor's Note. The tardiness in the appearance of the above review is due to circumstances over which neither the editor nor the reviewer had any control.

Plane Trigonometry. By H. L. Rietz, J. F. Reilly, and Roscoe Woods. New York, The Macmillan Company, 1935. xii+142+72+x pages. $2.20.

In the preface to this textbook for college freshmen the authors announce that it was their purpose to write a textbook based on "the cardinal principle of introducing only one important new idea at a time." This principle has indeed determined, at least in part, the arrangement of material; and since the most appreciable departures of this textbook from traditional practice are to be found in this arrangement, it is appropriate that this review should begin with that consideration.

The authors start with the definitions of the functions of an acute angle. In accordance with their cardinal principle they do not introduce all six functions at once, but in two sets of three each. After a chapter on the immediate consequences of these definitions, there follows a more or less lengthy chapter on the solutions of right triangles. There is a considerable number of applied problems in this chapter, requiring the use of three-place tables without interpolation; the various notions, such as angle of depression, and the technique of computation are explained with unusual care.

The definitions of the functions of a general angle come next. This is done by the usual method of introducing coordinates and restricting attention to points P for which the "distance" is positive. There is nothing unusual about the chapter containing these definitions; except, perhaps, the fact that two methods of reducing to functions of acute angles—by taking away the largest

multiple of 90° and by finding geometrically the acute angle with the horizontal —are both given, the reader being supposed to take his choice.

After a chapter on identities, there comes a chapter on line values and graphs. The authors discuss the line representation of all six functions. The discussion of graphs is unusually full and there are, in the exercises, questions about amplitude, etc., which are intended to be answered from the graphs themselves. Strangely enough, however, the graphs are plotted from tables; and the possibility of plotting them by a purely geometric process is not even mentioned.

One of the peculiar features of the book is the insertion at this point, where neither addition theorems nor logarithms have been mentioned, of a preliminary chapter on oblique triangles. Here the law of sines, law of cosines, and Heron's formula for areas are proved. The exercises comprise numerical problems using four place tables, and an unusual variety of theoretical problems requiring geometrical proofs and algebraic manipulations of the fundamental laws. Some of these problems require thought.

The subject of logarithms is not introduced until after the addition formulas and identities depending on them have been disposed of. A reasonable amount of attention is paid to questions of technique in computation, accuracy of results, etc. There is no treatment of problems such as $(3.6185)^{2.9746}$, nor of equations involving the unknown in the exponent. The solution of triangles by means of logarithms forms a separate chapter immediately following; this chapter contains a section on checking, using Newton's formulas.

This chapter on the logarithmic solution of triangles seems to form the culmination of the authors' plan. The topics radian measure, inverse functions, and trigonometric equations are not placed in the framework, but are stuck on at the end like postscripts. These chapters contain the usual material for a brief course.

As to the tables, the authors state in their preface that it is their belief that students are confused by the presence of tables they do not use. Hence they say they omit all tables not necessary for the immediate purpose. Many tables useful for collateral purposes are indeed omitted; yet the value of π is given to thirty decimal places.

The really important questions about a textbook are, in the reviewer's opinion, the following:* first, is it logically sound, or does it lend the weight of its authority to slovenliness, thereby hampering both teacher and student; second, does it contain the material essential to give the student a mastery of the subject when he is through; and third, does it contain adequate problem material for practice. It is now the reviewer's task to deal with these questions; and it will be convenient to take them in reverse order.

The question of problem material requires no lengthy comment. The col-

* Those who have had the privilege of studying under Professor W. F. Osgood will recall his insistence on "accuracy and adequacy" as primary criteria for judging a piece of mathematical work.

lection of problems seems adequate, both as to number and variety. As the authors state in their preface, there are about twice as many problems as can be used in a standard course, and among these are some really difficult ones for the benefit of the best students. There are books with a greater fund of problems, but this is enough to allow considerable selection. Some special comments on the problems have already been made. Answers are given to odd numbered problems.

The question of adequacy is partially answered by the outline of topics at the beginning of this review. From the point of view of the average student, it may be said that the ground is well covered indeed. But from the point of view of the superior student, the one who is not satisfied with a bare minimum, but wishes to obtain a thorough mastery, there are points which may be criticized. The reviewer hopes that he may, without being misunderstood, call attention to one such point which is particularly troublesome.

This concerns the definition of functions of a general angle. As already stated, the authors follow the traditional procedure; they use a coordinate system and restrict attention to points P such that the "distance" is positive. Now there are situations later on where this somewhat arbitrary restriction causes trouble; and some books exhibit inconsistencies in critical places (e.g.—the line representation of the tangent). These inconsistencies can be avoided in Trigonometry itself by suitable subterfuges, as in the present book; nevertheless, there are situations later on when such subterfuges seem unnatural. For example, how should one discuss the graph of the limaçon

$$r = 1 - 2 \cos \theta$$

if negative values of r are excluded? Moreover, in analytic geometry, where one or more coordinate systems are already in use, it is confusing to have to introduce a new one for each angle that has to be discussed. For these reasons it is suggested that so extensive a treatment of plane trigonometry as this—an outline is given for a course of 45 lessons—should contain somewhere a reference to the definition of the trigonometric functions as ratios of directed line segments, and the related topic of projections, as in the Trigonometry by Bôcher and Gaylord (Holt and Co., 1914), p. 33 ff. Admittedly, this is a difficult topic, and one which perhaps cannot be taught except to able students; on the other hand, the technique of the directed line segment is a powerful tool, and one whose mastery is well worth the trouble it takes such a student to achieve it.

How to introduce such topics into a book in such a manner as to be useful to the superior student without confusing the more stupid ones is a problem whose solution is not clear. The present book follows the current American tradition of glorifying mediocrity by omitting such topics altogether. The problem is thus evaded; but it can scarcely be called solved.

We come now to the all-important question of accuracy. In this regard, if we except the chapter on inverse functions, the book deserves praise. It is distinctly better than most American textbooks. The reviewer has examined

several tricky places and has found an error in only one of them; this is on page 80, where the proof of the formula for tan $(x/2)$ is invalid.

On the other hand, the chapter on inverse functions is rankly illogical. Thus on page 133 the authors derive what essentially amounts to the equation

$$\text{arc tan}\left(\frac{2\tan x}{1-\tan^2 x}\right) = 2x$$

This is a truly amazing result; because the left hand side should be a periodic function of period π, while the right hand side is an aperiodic linear function. The source of the confusion is this: the expression $y = \text{arc tan } x$ (arc tan x is not defined in any other connection) is defined to mean "y is an angle whose tangent is x." Thus the $=$ sign is used for the relation of membership in a class, a relation which is neither reflexive nor symmetric nor transitive; whereas the statement mentioned is treated in many places as an equation. In fairness to the authors it must be emphasized that this situation is not a specific fault of their textbook; it is a fault of American* textbooks in general. It is, however, a situation which seems to the reviewer intolerable. As a remedy, he would suggest that multiple valued inverse functions should be banished from elementary trigonometry altogether; the student has no use for them until he comes to the theory of functions of a complex variable. In case one wishes to express the fact that y is an angle whose tangent is x, one can use the equation "$\tan y = x$." If multiple valued inverse functions are thought desirable, the equality sign should be used for equality and nothing else; while the relationship of the preceding sentence may be written "y is an arc tan x" or—if one *must* have a symbol—"$y \in \text{arc tan } x$."

These then are the facts concerning the book. The final judgment on its merits as a textbook the reviewer must leave to the reader to make for himself. The reviewer has never taught from the book; and his a priori opinion would be of no value to those responsible for choosing texts. But the reader should bear in mind that, in the foregoing, the book has been criticized against an ideal standard. In most of the respects in which an adverse criticism has been made almost any other existing book will fall down, too. All in all it is a comparatively good book, and, except for the chapter on inverse functions, it can be recommended as such to any teacher who finds its order of presentation acceptable.

H. B. CURRY

Editor's Note. Since this review was written, a revised edition of this text has appeared; this is listed under New Books Received. The chapters which appeared in the book reviewed are unchanged, but three chapters on spherical trigonometry have been added.

---

* The reviewer has no information in regard to foreign textbooks.

MATHEMATICS CLUBS

EDITED BY F. W. OWENS and HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 463 East Foster Ave., State College, Pa.

A DIRECTORY OF MATHEMATICS CLUBS IN COLLEGES AND UNIVERSITIES OF THE UNITED STATES AND CANADA

In Volume 25, pages 458–460 of this MONTHLY appears a compilation of mathematics clubs in the United States and Canada made by Professor R. C. Archibald. Of the thirty-eight clubs listed, twenty-four had been in existence before January 1, 1916 when this MONTHLY became the official organ of the Mathematical Association of America. Of those twenty-four, eighteen will be found in the present list, either under the same name or that of a logical successor. Three others have been absorbed in general science clubs.

Though expressed in varying forms each of the 128 clubs in the present list has for its purpose: (1) the broadening and intensifying of the students' interest in and knowledge of mathematics through discussion of problems and topics beyond the scope of the classroom; (2) the development of friendly relations among students and teachers of mathematics. With clubs in Junior colleges, four year colleges and universities the relative stress on mathematics and social life shifts with the school. Every club has a serious program while none entirely eliminates the social side, whether through an annual banquet or picnic, or a cup of tea preceding the serious work of the program.

In this list no effort is made to discuss the work of the club. Programs of all have appeared or will soon appear in this Department. Communications addressed to a club, care of the Mathematics Department of the school it represents, will bring further information to those interested.

The date of organization is given immediately after the name of the club. Unless otherwise specified each club here listed may be considered to have:

(1) a purpose as expressed in the preceding paragraphs,
(2) as members—faculty, undergraduate and (where graduate work is offered) graduate students,
(3) definite qualifications as to scholarship and mathematical training,
(4) management by students with faculty cooperation,
(5) monthly meetings with programs planned to broaden the mathematical outlook of the members,
(6) social features.

In a later issue will appear the story of several mathematics clubs with similar ideals but not concerned with undergraduate problems.

Pi Mu Epsilon

Founded 1903. Incorporated 1914. An academic fraternity in institutions of university grade. Its first and primary aim is scholarship for individual members in all subjects, particularly in

mathematics; secondly, advancement of the science of mathematics; lastly, mutual and personal advancement of its members.

Each of its thirty-three active chapters decides the eligibility for membership subject to these restrictions: members of mathematical faculty; graduate students with a major or minor in mathematics; undergraduate students with a major or minor in mathematics who are in the upper half of their class in general scholarship and in the upper third in mathematical scholarship.

Further details may be obtained from any of the National Officers:

Director-General, G. C. Evans, University of California, Berkeley, California.

Vice Director-General, Professor W. E. Milne, Oregon State College, Corvallis, Oregon.

Secretary-General, J. S. Gold, Bucknell University, Lewisburg, Pennsylvania.

Members of Council: W. C. Brenke, University of Nebraska;
Alan Campbell, Syracuse University;
D. N. Lehmer, University of California;
F. W. Owens, The Pennsylvania State College.

Kappa Mu Epsilon

Founded 1931. A professional mathematics fraternity for four year colleges. Its purposes are (1) to develop an appreciation of mathematics and an interest in it; (2) to provide a society for achievement in this field; (3) to foster fraternal relationship among its devotees.

Each of its thirteen active chapters decides the eligibility for membership subject to these restrictions: students majoring or minoring in mathematics who have taken sufficient courses in mathematics to ensure their permanent interest and who are in upper half of student body in general scholarship; faculty members of mathematics and related departments.

Further details may be obtained from any of the National Officers:

President Pythagoras, Professor J. A. G. Shirk, Kansas State Teachers' College, Pittsburg, Kansas;

Vice President Euclid, Dr. O. J. Peterson, Emporia, Kansas;

Secretary Diophantus, Professor C. D. Smith, State College Mississippi;

Treasurer Newton, Professor L. E. Pummill, Springfield, Missouri;

Historian Hypatia, Miss E. Marie Hove, Wayne, Nebraska.

Intercollegiate Mathematical Association of Milwaukee

Founded 1934. A grouping of undergraduate mathematics clubs of Milwaukee, Wisconsin.

Alabama State Teachers College, Florence, Alabama.

Kappa Mu Epsilon, Alabama Beta. May 20, 1935. Membership 20.

Albion College, Albion, Michigan.

Albion College Mathematics Club. January, 1911. Mathematics major, three semesters of College Mathematics, average B in all mathematics.

Athens College, Athens, Alabama.

Kappa Mu Epsilon, Alabama Alpha. May 5, 1935. Semi-monthly meetings. Beautifying Rivers Hall where mathematics is taught. Making mathematical models.

Ball State Teachers College, Muncie, Indiana.

The Mathematics Club of Ball State Teachers College. Fall, 1928. Open to all with interest in mathematics. Membership 38.

Boston University, Boston, Massachusetts.

The Mathematics Club of Boston University. 1919. Anyone interested in higher mathematics. Membership 24. Gives annual prize for best paper read at meeting by undergraduate.

Bowdoin College, Brunswick, Maine.

Mathematics Club of Bowdoin College. 1920. Student management. Must have completed first semester's work of Sophomore year. Membership 39.

Brooklyn College, Brooklyn, New York.

Pi Mu Epsilon of Brooklyn College. 1933. Semi-monthly.

Brown University, Providence, Rhode Island.

The Mathematics Club of Brown University. February 1915. Sophomores, juniors and seniors and freshmen with grade A in first semester mathematics. Membership 80. Printed programs issued in fall.

Bucknell University, Lewisburg, Pennsylvania.

Pi Mu Epsilon, Beta of Pennsylvania.

Butler University, Indianapolis, Indiana.

The Mathematics Club of Butler University. January, 1923. Present or former students in mathematics and sponsor. Membership 20.

Carleton College, Northfield, Minnesota.

Mathematics Club. 1929. Irregular informal discussion meetings. The Greek idea of intellectual discussion. All interested. Membership 10.

Case School of Applied Science, Cleveland, Ohio.

The Case Mathematical Club. March 14, 1930. Meetings every three weeks. Open to faculty and students interested. Membership 51. Prize of $35 awarded at Case Commencement for best student paper at a meeting of the Club.

Columbia University, New York City.

Mathematics Club of Columbia College. 1910. Informal, semi-monthly meetings.

Connecticut College for Women, New London, Connecticut.

The Mathematics Club of Connecticut College. 1916, the first department club in the College. All interested in Mathematics. No formal membership. Average attendance 22. In 1934–35 purchased a set of models for use in mathematics classes.

Cooper Union Institute of Technology, New York City.

The Mathematics Club of Cooper Union. 1917. Open to all students. Offers a prize for excellence in first year mathematics. Membership over 200.

Denison University, Granville, Ohio.

Denison Mathematics Club. November 1914, successor to club organized 1910. Semi-monthly. Any student, subject to election. Membership 22.

DePauw University, Greencastle, Indiana.

Naperian Club. December 6, 1924. Integral Calculus and average B in all mathematics taken.

Duke University, Durham, North Carolina.

Pi Mu Epsilon, of Duke University. 1931. Membership 55. Prize for best paper read at meeting by undergraduate.

Eastern Illinois State Teachers College, Charleston, Illinois.

Kappa Mu Epsilon, Illinois Beta Chapter. April 11, 1935. Membership 46. Made a survey

by questionnaire of conditions and trends in the mathematics departments of Eastern Illinois high schools.

The Mathematics Club of Eastern Illinois State Teachers College. About 1925. Semi-monthly. All college students welcomed. Membership 38.

Elmira College for Women, Elmira, New York.

Chi Upsilon Zeta. 1928. Any student who has taken or is taking a course in mathematics.

Extension Division of the University of Wisconsin, Milwaukee, Wisconsin.

Junior Mathematics Club of Extension Division of the University of Wisconsin. October 12, 1929. All interested in Mathematics. Member of Intercollegiate Mathematics Association of Milwaukee. Euler prize for best paper by student. Held Mathematical Exhibit for high school mathematics clubs.

The George Washington University, Washington, D.C.

Mathematics Club of the George Washington University. October 17, 1933. Semi-monthly. Completion of course in differential calculus and active interest in mathematics or its applications.

Goucher College, Baltimore, Maryland.

The Mathematics Club of Goucher College. November 1913. Juniors and seniors majoring in mathematics. Sophomores are sometimes guests. Membership 28. Special contest by dividing undergraduates into four teams for year; score being kept on correct solution of 50 problems, elegance of solution, success in oral presentation.

Harris Teachers College, St. Louis, Missouri.

Hypatia. Spring, 1925. At least a second major in mathematics.

Harvard University, Cambridge, Massachusetts.

Harvard Mathematical Club. December 14, 1904. Successor of club organized 1898. Students who have taken any course above Freshman mathematics. Membership 31. Endowment fund affords annual Robert Fletcher Rogers prizes, $35 and $15 respectively for best and next best presentations at Club. Makes annual report on state of mathematics at Harvard. Exchanges a speaker with Yale Mathematics Club.

Haverford College, Haverford, Pennsylvania.

Mathematics Club of Haverford College. September 1933. All interested. Membership 25.

Hunter College, New York City.

Pi Mu Epsilon of Hunter College. May 25, 1925.

Mathematics Club of Hunter College. 1910. Semi-monthly. Faculty and students.

Freshman Mathematics Club of Hunter College. 1911. Semi-monthly.

Illinois State Normal University, Normal, Illinois.

Kappa Mu Epsilon, Illinois Alpha. January 26, 1933. Membership 32.

Indiana University, Bloomington, Indiana.

Euclidean Circle. September 30, 1907, reorganized from a combined physics and mathematics club which had existed since 1886, the oldest department club at Indiana University. Student management. All students with at least ten hours mathematics in university.

Iowa State College, Ames, Iowa.

Pi Mu Epsilon of Iowa State College. Sponsors a mathematics club for junior college students.

Iowa State Teachers College, Cedar Falls, Iowa.

Kappa Mu Epsilon, Iowa Alpha. May 1931. Successor of club founded December 1909. Stresses individual research. Membership 25.

Kansas State College of Agriculture and Applied Sciences, Manhattan, Kansas.

Mathematics Club. October 1934. All students in mathematics are eligible. Membership 51. Pi Mu Epsilon of Kansas State College, May 23, 1935. Is sponsor of the Mathematics Club.

Kansas State Teachers College, Pittsburg, Kansas.

Kappa Mu Epsilon, Kansas Alpha. January 30, 1932, organized as successor of a club founded in 1916. Its open meetings are still held under name, Math Club. Membership 41.

Kansas State Teachers College, Emporia, Kansas.

Kappa Mu Epsilon, Kansas Beta. May 12, 1934.

Lafayette College, Easton, Pennsylvania.

William S. Hall Mathematics Club. October 1933. Six meetings a year. Those who have evidenced an interest in mathematics and who welcome an opportunity to further their knowledge of it.

Lehigh University, Bethlehem, Pennsylvania.

Pi Mu Epsilon of Lehigh University. May 21, 1929. Sponsored an open lecture by J. L. Coolidge of Harvard.

Newtonian Society. Second semester, 1928. All freshmen with grade A in mathematics. Membership 51.

Louisiana State University, Baton Rouge, Louisiana.

Louisiana State University Mathematics Club. January 8, 1933. All interested in mathematics or its applications. Issues monthly Math Club News. Holds meetings of problem solving groups. Membership 28.

Los Angeles Junior College, Los Angeles, California.

Mathematics Club. September 1932. Semi-monthly meetings. Making a study of mathe matics clubs in Junior Colleges. Membership 28.

Marquette University, Milwaukee, Wisconsin.

Pi Mu Epsilon of Marquette University. February 1933. Member Intercollegiate Mathematical Association of Milwaukee. Membership 60. Issues a monthly mimeographed bulletin containing "a record of business, and abstract of paper presented at previous meeting and information concerning the next meeting." Annually conducts the Frumveller Competition, a competitive examination for properly prepared high school seniors.

Milwaukee-Downer College, Milwaukee, Wisconsin.

The Mathematics Club of Milwaukee Downer College. 1932. All willing to contribute to value of the programs. Membership 18. Operate the Used Book Exchange. Contribute to Dean Pieters Memorial. Member Intercollegiate Mathematical Association of Milwaukee.

Milwaukee State Teachers College, Milwaukee, Wisconsin.

Mathematics Club of State Teachers College. Member Intercollegiate Mathematical Association of Milwaukee.

Mississippi State Agricultural and Mechanical College, State College, Mississippi.

Kappa Mu Epsilon, Mississippi Beta. December 14, 1932. Irregularly called meetings. Mem-

bership 32. Display of more than sixty drawings in color for a mathematical exhibit. Issues mimeographed annual bulletin.

Mississippi State Teachers College for Women, Columbus, Mississippi.

Kappa Mu Epsilon, Mississippi Alpha. May 30, 1932. Membership 20.

Mount Mary College, Milwaukee, Wisconsin.

Mount Mary Mathematics Club. October 1934. All students of mathematics. Graduates may be associate members. Member Intercollegiate Mathematical Association of Milwaukee. Membership 11.

New Jersey College for Women, New Brunswick, New Jersey.

The Mathematics Club of New Jersey College for Women. 1924. Fortnightly meetings. Students who have taken or are taking calculus.

New Jersey State Teachers College, Montclair, New Jersey.

Sigma Phi Mu. 1930. Semi-monthly meetings. Those students who acquit themselves with distinction in their mathematics courses.

New York State College for Teachers, Albany, New York.

Mathematics Club of New York State College for Teachers. 1931. Semi-monthly meetings. Student management. Faculty and students who have completed one semester of analytics. Maintains a regular Puzzle Box and uses debates to arouse interest.

Northeastern Teachers College, Tahlequah, Oklahoma.

Kappa Mu Epsilon, Oklahoma Beta. April 18, 1931. Semi-monthly. Membership 35. Annually issues mimeographed booklet, the "Exponent," containing program, news of members, skits, etc. Conducts contest annually in connection with scholastic meet, the prize being tuition for one year.

Northeastern University, Boston, Massachusetts.

The Northeastern Mathematics Society. September 1931. Student management. Engineering students who have completed at least one and one-half years of school work with average not less than C in mathematics. Membership 32.

North Texas Teachers College, Denton, Texas.

The Mathematics Club of North Texas Teachers College. 1932. Mathematics majors with average of B in mathematics and C in all work.

Northwestern University, Evanston, Illinois.

Mathematics Club of Northwestern University. January 1916. Faculty, graduate students, juniors and seniors majoring in mathematics. Those with a minor in mathematics are eligible for associate membership.

Oberlin College, Oberlin, Ohio.

The Mathematics Club of Oberlin College. 1894, by Professor Frederick Anderegg. Semi-monthly meetings. All interested in mathematics. Membership 30. Prizes offered for best solution of problems.

Ohio State University, Columbus, Ohio.

Pi Mu Epsilon of Ohio State University. 1919. Semi-monthly meetings. Organized and sponsors the Undergraduate Club. Membership 50.

The Undergraduate Mathematics Club of Ohio State University. January 1936. Semi-monthly meetings. Student management. Membership 40.

Ohio Wesleyan University, Delaware, Ohio.

Pi Mu Epsilon of Ohio Wesleyan. April 1927, successor of an older club. Membership 15.

Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma.

The A and M Mathematics Club. 1935.

Oregon State College, Corvallis, Oregon.

Pi Mu Epsilon of Oregon. May 19, 1931. Successor of Mathematics Club founded October 1916. This chapter is divided into two sections, one at Eugene, the other at Corvallis. They hold some joint meetings. The chapter cooperated in the Willamette Valley Students' Scientific Conference. Membership 40.

Oshkosh State Teachers College, Oshkosh, Wisconsin.

The Mathematics Club. February 1931. At least one year of College mathematics. Award made for best paper presented. Membership 31.

The Pennsylvania State College, State College, Pennsylvania.

Pi Mu Epsilon of The Pennsylvania State College. May 21, 1930. Membership 80. Holds annual examination in mathematics up to and including the differential calculus, open to all sophomores. Three prizes, $15, $10, $5, awarded.

Pennsylvania Women's College, Pittsburgh, Pennsylvania.

Beta Chi. 1930. Major in mathematics. Membership 16.

Rutgers University, New Brunswick, New Jersey.

The Mathematics Club of Rutgers University. 1913. All students above freshman year whose major interest is mathematics.

St. Lawrence University, Canton, New York.

Alpha Mu Gamma. 1915. All interested in mathematics. Membership 36.
Pi Mu Epsilon of St. Lawrence University. May 29, 1935. Membership 12. Cooperates with Alpha Mu Gamma.

St. Norbert College, West DePere, Wisconsin.

The St. Norbert College Mathematical Society. 1913. Students with a knowledge of calculus.

St. Xavier College, Chicago, Illinois.

The Mathematics Club of St. Xavier College. 1931. Semi-monthly meetings. Student management. Students with at least two quarters of mathematics. Membership 25. Holds an annual open meeting.

Smith College, Northampton, Massachusetts.

Mathematics Club of Smith College. October 30, 1899. Meetings every three weeks. Students majoring in mathematics and those with at least one semester of Calculus with grade B.

Southern Methodist University, Dallas, Texas.

Sigma Delta Rho. 1925. Semi-monthly meetings. Honorary fraternity which requires B+ average in twelve hours mathematics, including integral calculus and a B average in all subjects.

Southwest Missouri Teachers College, Springfield, Missouri.

Kappa Mu Epsilon, Missouri Alpha, May 20, 1932.

State College of Washington, Pullman, Washington.

Pi Mu Epsilon. May 25, 1931. Membership 30. Sponsors Newtonian Society. Entertained Yakima Junior College Math Club.

Newtonian Society, October 21, 1919. Admits freshmen and all interested in mathematics. Membership 35. Holds many joint meetings with Pi Mu Epsilon.

State Teachers College of Nebraska, Wayne, Nebraska.

Kappa Mu Epsilon, Nebraska Alpha. January 17, 1933.

The State University of Iowa, Iowa City, Iowa.

Undergraduate Mathematics Club of the State University of Iowa. 1918. Anyone interested. Membership 25.

Syracuse University, Syracuse, New York.

Pi Mu Epsilon of Syracuse University. 1903. This is the Alpha Chapter of the National Pi Mu Epsilon. Membership 90.

Mathematics Club of Syracuse University. February 5, 1933. Any undergraduate who is taking (or has taken) differential calculus and any graduate student in mathematics.

University of Alabama, University, Albama.

Pi Mu Epsilon of University of Alabama, 1919. Successor to the Mathematics and Physics Club founded November, 1916. Membership 42.

University of Alberta, Edmonton, Alberta.

Mathematics Club of University of Alberta.

University of Arkansas, Fayetteville, Arkansas.

Pi Mu Epsilon of the University of Arkansas. 1931, succeeding a mathematics club organized in 1919. Membership 40. Awards $10 annually for best mathematics paper at meeting of club

University of British Columbia, Vancouver, British Columbia.

The Mathematics Club of the University of British Columbia. 1926. Semi-monthly. Third and fourth year students majoring or honoring in Mathematics.

University of Buffalo, Buffalo, New York.

Mathematics Club. 1929. All students of mathematics. Membership 30. Presents Wilfred H. Sherk Memorial prize in mathematics annually for the best paper on pure or applied mathematics submitted to the faculty.

University of California, Berkeley, California.

Pi Mu Epsilon, Beta of California. February 1930. Meetings every three weeks. Membership 50.

University of California at Los Angeles, Los Angeles, California.

Pi Mu Epsilon, Alpha of California. 1925. Presents annual prize for best paper in a special calculus examination.

Mathematics Club of U.C.L.A. 1925. Faculty, all students who are taking or have taken calculus, with secondary membership for all interested. Only primary members hold office.

University of Chicago, Chicago, Illinois.

Junior Mathematical Club of the University of Chicago. 1906. Semi-monthly meetings. Graduate student management. All students, graduate and undergraduate, specializing in mathematics or mathematical astronomy. Membership 45. Meetings open to all who care to attend.

University of Cincinnati, Cincinnati, Ohio.

Mathematics Club of University of Cincinnati. 1931. Semi-monthly meetings. Graduate and undergraduate and honor students in mathematics. Undergraduates majoring in mathematics and others, who are qualified, in chemistry, physics and engineering. Student management.

University of Colorado, Boulder, Colorado.

Mathematics Club. October 1915. Semi-monthly meetings. Anyone interested in mathematics. Holds many meetings open to all students of the university. Membership 40.
Pi Mu Epsilon of the University of Colorado. May 4, 1936.

University of Florida, Gainesville, Florida.

Informal Seminar. Semi-monthly. Mostly faculty speakers. Undergraduate members.

University of Georgia, Athens, Georgia.

Pi Mu Epsilon of the University of Georgia. March 2, 1934. Membership 30.

University of Illinois, Urbana, Illinois.

Pi Mu Epsilon of University of Illinois. 1930. Successor of an Undergraduate Club, organized in 1899. Semi-monthly meetings. Membership 35.
Students Mathematical Round Table. About 1920. Graduate students and upper class students in mathematics. Takes especial interest in current mathematical literature. Membership 30.

University of Kansas, Lawrence, Kansas.

Pi Mu Epsilon of University of Kansas. November 30, 1928. Irregular meeting time. Membership 24.
Mathematics Club of the University of Kansas. 1911. Members elected by the faculty from those students who have completed calculus. Semi-monthly meetings. Membership 50. Participates in colloquium with chemistry and physics groups.

University of Kentucky, Lexington, Kentucky.

Pi Mu Epsilon of the University of Kentucky. December 18, 1926. Annually presents the mathematics department a collection of books of mathematical literature. Membership 32.
White Mathematics Club. Named in 1914, the club having been formed about 1907. Majors or minors in mathematics or with definite interest in the subject. Since 1931 under student management. Membership 25.

University of Michigan, Ann Arbor, Michigan.

Junior Mathematical Society of University of Michigan. October 1930. Any student interested in mathematics. Membership 25.

University of Missouri, Columbia, Missouri.

Pi Mu Epsilon, Alpha of Missouri. December 4, 1923. Successor of Journal Club organized 1905. Semi-monthly meetings. Annually award $10 each, sophomore or junior in calculus, and freshman or sophomore in analytic geometry. Winner chosen by competitive examinations.

University of Montana, Missoula, Montana.

Pi Mu Epsilon of University of Montana. 1921, successor of a mathematics club founded March 1918. Semi-monthly meetings. Membership 14.

University of Nebraska, Lincoln, Nebraska.

Pi Mu Epsilon of the University of Nebraska, October, 1928, successor of club founded October, 1915. Annual prizes offered in calculus and in analytics. Membership 35.

University of Nevada, Reno, Nevada.

Mathematics Club of the University of Nevada. September 1935. All interested in mathematics and enrolled in mathematical course. Membership 30.

University of New Mexico, Albuquerque, New Mexico.

Kappa Mu Epsilon, New Mexico Alpha. March 28, 1935. Membership 55.

University of Oklahoma, Norman, Oklahoma.

Pi Mu Epsilon of the University of Oklahoma. 1931. Semi-monthly meetings. Two annual prizes given for papers at meetings, one for "best example of an advanced discussion which necessitated simple treatment," the other for "best example of a more elementary discussion." Membership 41.

University of Oregon, Eugene, Oregon.

Pi Mu Epsilon of Oregon. May 19, 1931. Successor of Mathematics Club founded October, 1916. This chapter is divided into two sections, one at Eugene, the other at Corvallis. They hold some joint metings. The chapter cooperated in the Willamette Valley Students' Scientific Conference. Membership 40.

University of Pennsylvania, Philadelphia, Pennsylvania.

Pi Mu Epsilon of University of Pennsylvania. 1920. Awards annually two prizes, $12.50 each, for best papers presented by undergraduates and two prizes, $5 each, for best solutions of problems proposed. Membership 75.

University of Rochester, Rochester, New York.

Mathematics Club. 1935, after three years of informal meetings.

University of Saskatchewan, Saskatoon, Saskatchewan.

Shuttleworth Mathematical Society, successor of Mathematical Society organized November 1916.

University of Toledo, Toledo, Ohio.

Delta x. 1929. All who have taken or are taking calculus. Membership 73. Pi Mu Epsilon of University of Toledo, February 7, 1936.

University of Toronto, Toronto, Ontario.

Mathematical and Physical Society of the University of Toronto. 1882. Membership 162.

University of Virginia, Charlottesville, Virginia.

Echols Mathematics Club. October 13, 1931. Student management. Semi-monthly meetings. Faculty, graduate students, and undergraduates who show a marked proficiency in the field. Membership 19.

University of Washington, Seattle, Washington.

Pi Mu Epsilon of the University of Washington. March 5, 1932. Successor of Junior Mathe-

matics Club organized 1929. Semi-monthly meetings.
Zeta Mu Tau. 1927. All interested in mathematics.

Vanderbilt University and Peabody College, Nashville, Tennessee.

The Mathematics Club of Vanderbilt University and Peabody College. 1932. Students who have taken or are taking the calculus in either institution. Faculty and graduate students in both institutions. Meetings held alternately on the two campuses. Membership 25.

Washington and Jefferson College, Washington, Pennsylvania.

Phi Chi Mu. February 19, 1920. Juniors and seniors doing outstanding work. No man is active member until he has presented a paper. Faculty may be honorary members. Active membership 15.

Washington Square College of New York University, New York City.

Pi Mu Epsilon of Washington Square College. October 28, 1933. Four meetings each year. Holds annual Pi Mu Epsilon Interscholastic Mathematics Contest. In 1935 there were 395 contestants from 108 schools. Medals are awarded to individual winners and cups to school winners in the four sections. Membership 50.
Mathematics Club of Washington Square College. September 1928. Weekly meetings. Any student in good standing. Membership 35. Publishes mimeographed magazine, Math X, which is sold to students. System of free coaching classes in mathematics, conducted by student members.

Washington University, St. Louis, Missouri.

Pi Mu Epsilon of Washington University. June 4, 1925. Sponsors open lecture by visiting mathematician. Membership 50.

Wayne University, Detroit, Michigan.

The Mathematics Club of Wayne University. 1925. All students and faculty interested in mathematics. Membership 35.

Wellesley College, Wellesley, Massachusetts.

The Wellesley College Mathematics Club. Spring, 1921. Student management exclusively. Juniors and Seniors taking mathematics, sophomores making grade II in mathematics. Conducts various contests. Current contest is for mathematics song.

Wesleyan University, Middletown, Connecticut.

Van Vleck Mathematics Club. All taking advanced courses in Mathematics. Membership 20.

Westminster College, New Wilmington, Pennsylvania.

Delta Nabla. 1929. Semi-monthly meetings. Majors in mathematics, who have completed three semesters with average B and are unanimously approved by active members. Membership 20. Makes annual award to the most outstanding mathematics student in the freshman class.

William and Mary College, Williamsburg, Virginia.

Euclid Club. November 20, 1931. Majors and minors in mathematics who have completed nine semester hours with average grade of 85% and at least one mathematics course with grade above 90%. Membership 49.

Women's College of Delaware, Newark, Delaware.

Women's College Mathematics Club. October 1925. Membership 35.

Women's College of the University of North Carolina, Greensboro, North Carolina.

Square Circle. 1926. Those with major or minor in mathematics.

Yakima Junior College, Yakima, Washington.

Mathematics Club. 1932. Any student interested in phases of mathematics not offered in the classroom. Weekly meetings. Makes annual visits to the State College and the University of Washington. Membership 10.

Yale University, New Haven, Connecticut.

The Yale Undergraduate Mathematics Club, April, 1934. Bi-weekly meetings. Managed by students. Prizes awarded for best solution of set of problems proposed by the faculty. *Bulletin*, containing original work of undergraduates is published. Exchanges one speaker annually with Harvard Mathematical Club. Average attendance 25.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 225. *Proposed by A. Gloden, l'Athênê de Luxembourg.*

Determine a pentagonal number in the decimal system with digit pattern $aabb$, and show that the solution is unique. [Note: a pentagonal number is of the form $n(3n-1)/2$, with n a positive integer.]

E 226. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

If AD and BC are the diagonals of the square $ABCD$, and M is any point on its circumscribed circle, show that the points in which MA and MD cut BC are concyclic with the points in which MB and MC cut AD.

E 227. *Proposed by V. Thébault, Le Mans, France.*

Obtain the general solutions in integers for the pair of simultaneous equations:

$$(2n+1)x+1 = y^2$$
$$2nx+1 = z^2$$

and show that each of the numbers, $(n+1/2)x+1$ and $nx+1$, is equal to the sum of the squares of two consecutive integers.

SOLUTIONS

E 189. *Proposed by V. Thébault, Le Mans, France.*

What is the smallest positive integer base which may be used for a system of enumeration, such that in this system there exists a perfect square written with five digits, all the same?

Solution by J. A. Ward, University of Wisconsin

Let each digit be a. Let the base be b. Then $1<b$ and $a<b$. Hence a and b are integral solutions of the equation,

$$ab^4 + ab^3 + ab^2 + ab + a = n^2,$$

which may be written in the form,

$$a(b^5 - 1)/(b - 1) = n^2.$$

The smallest value possible for b is 2. Then $a=1$ and the left member is 31, which is not a square.

The next smallest value of b is 3. Then the left member is $a(242/2)$, which is a square if $a=1$. Therefore the smallest base is 3 and the solution is 11111 in the ternary system, or $121=11^2$ in the decimal system.

Editorial Note. Are there any other solutions?

Also solved by W. E. Buker, J. E. Burnam, Fred Discepoli, Wm. Douglas, O. E. Eggert, Daniel Finkel, Bernard Greenspan, Joseph Milkman, D. K. Pease, Walter Penney, C. F. Pinska, J. Rosenbaum, E. P. Starke, C. W. Trigg, M. J. Turner, Simon Vatriquant and the proposer.

E 190. *Proposed by Fred Discepoli, New York City.*

If a and n are positive integers greater than the positive integer b, then if $a^n+b^n=c^n$, c can never be an integer.

Solution by Joseph Milkman, Brooklyn College

Since $a^n+b^n=c^n$, therefore $a<c$. Furthermore, $(a+1)^n=a^n+na^{n-1}+\cdots$, and $b<n$, and $b^{n-1}<a^{n-1}$. Therefore $a^n+b^n<(a+1)^n$, or $c^n<(a+1)^n$. Consequently, $a<c<a+1$, and since c thus lies between two consecutive integers, c cannot be an integer.

Also solved by J. Rosenbaum (who points out that b need not be an integer), E. P. Starke, Simon Vatriquant and the proposer.

E 191. *Proposed by E. P. Starke, Rutgers University.*

Three points on a central conic are joined to form a triangle. The tangents at these points form a second triangle. Prove that the center of the conic and the centroids of the triangles either all coincide or are all distinct.

Solution by C. E. Springer, University of Oklahoma

Let the points A, B and C on the conic $x^2/a^2+y^2/b^2=1$ have coordinates

$(a \cos \alpha, b \sin \alpha)$, $(a \cos \beta, b \sin \beta)$ and $(a \cos \gamma, b \sin \gamma)$ respectively. The coordinates of the point of intersection A' of the tangents to the conic at points B and C are given by

$$\left(a \cos \frac{\beta+\gamma}{2} \Big/ \cos \frac{\beta-\gamma}{2},\ b \sin \frac{\beta+\gamma}{2} \Big/ \sin \frac{\beta-\gamma}{2}\right).$$

The coordinates of the other vertices, B' and C', of the triangle formed by the tangents, can be written by cyclic permutation of the eccentric angles. The centroid G of triangle ABC has coordinates

$$\left(\frac{a}{3} \sum \cos \alpha,\ \frac{b}{3} \sum \sin \alpha\right)$$

and the centroid G' $(\bar{x}, \bar{y})$ of the triangle $A'B'C'$ has coordinates given by

$$\left(\frac{a}{3} \sum \frac{\cos \frac{\beta+\gamma}{2}}{\cos \frac{\beta-\gamma}{2}},\ \frac{b}{3} \sum \frac{\sin \frac{\beta+\gamma}{2}}{\sin \frac{\beta-\gamma}{2}}\right).$$

Now

$$\frac{3\bar{x}}{a} = \sum \frac{\sin \frac{\beta-\gamma}{2} \cos \frac{\beta+\gamma}{2}}{\sin \frac{\beta-\gamma}{2} \cos \frac{\beta-\gamma}{2}} = \sum \frac{\sin \beta - \sin \gamma}{\sin (\beta - \gamma)}$$

and

$$\frac{3\bar{y}}{b} = \sum \frac{\sin \frac{\beta-\gamma}{2} \sin \frac{\beta+\gamma}{2}}{\sin \frac{\beta-\gamma}{2} \cos \frac{\beta-\gamma}{2}} = - \sum \frac{\cos \beta - \cos \gamma}{\sin (\beta - \gamma)}$$

In general, for given values of α, β and γ, the coordinates of G' do not vanish, and do not equal those of G. But if G coincides with the center O of the conic, then $\Sigma \cos \alpha = 0$ and $\Sigma \sin \alpha = 0$. Multiply the first of these two equations by $\sin \gamma$ and the second by $\cos \gamma$ and subtract to find that $\sin (\gamma-\alpha)+\sin(\gamma-\beta) = 0$. Thus $\sin (\beta-\gamma) = \sin (\gamma-\alpha) = \sin (\alpha-\beta)$, and the coordinates of G' are given by

$$\frac{3\bar{x}}{a} \sin (\alpha - \beta) = \sum (\sin \beta - \sin \gamma) \equiv 0$$

and

$$\frac{3\bar{y}}{b}\sin(\alpha-\beta)=\sum(\cos\beta-\cos\gamma)\equiv 0.$$

Hence the points G, G' and O coincide.

Also proved by Fred Discepoli, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3792. *Proposed by Frank Morley, Johns Hopkins University.*

A square is divided into n^2 unit squares, like a chess-board. Any two horizontal lines and any two vertical lines form a rectangle. We count a square as a rectangle. Each rectangle has a breadth b, less than or equal to its length. There is one rectangle of breadth n, namely the original square. Prove that there are 2^3 rectangles of breadth $n-1$, 3^3 of breadth $n-2, \cdots, n^3$ of breadth 1.

Deduce the formula

$$1^3+2^3+\cdots+n^3=[n(n+1)/2]^2.$$

3793. *Proposed by V. Thébault, Le Mans, France.*

Three parallels drawn from the mid-points of the sides BC, CA, AB of a triangle cut again the nine-point circle in α, β, γ. (a) Show that the circles with centers α, β, γ passing respectively through the feet A', B', C' of the altitudes AA', BB', CC' of the given triangle cut each other in pairs in three points of a straight line Δ and also pass through a point Q. (b) When the direction of the parallels varies, prove that the line Δ passes through a fixed point and that the point Q describes the nine-point circle of ABC. (c) Generalize the theorem by replacing the nine-point circle by the pedal circle of a point D in the plane of the triangle ABC and the points A', B', C' by the orthogonal projections of D on BC, CA, AB.

3794. *Proposed by R. S. Underwood, Texas Tech. College, Lubbock, Tex.*

Prove that the series

$$\sum_{n=1}^{\infty}\left|\frac{\sin n\theta}{n}\right|$$

is divergent unless θ is an integral multiple of π.

3795. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let P be a point on the circumcircle of a triangle ABC, A_0, B_0, C_0 the points where AP, BP, CP meet BC, CA, AB, A_1, B_1, C_1 the points dividing AA_0, BB_0, CC_0 in the same ratio so that

$$AA_1 : A_1A_0 = BB_1 : B_1B_0 = CC_1 : C_1C_0 = k,$$

and finally A_2, B_2, C_2 the points where the altitudes meet the perpendiculars at P to AP, BP, CP. Prove that the ratio of the areas $A_1B_1C_1$ and ABC equals

$$\frac{1-k}{(1+k)^2} + \frac{k^2}{4(1+k)^2} \frac{AA_2 \cdot BB_2 \cdot CC_2}{R^3},$$

R being the circumradius.

SOLUTIONS

3711 [1934, 634]. *Proposed by J. H. M. Wedderburn, Princeton University.*

If $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is a given polynomial, and $f_r(x) = a_0x^r + a_1x^{r-1} + \cdots + a_r$, express the product f_rf_s in the form

$$\sum_{i=0}^{r+s} \alpha_i f_i. \tag{1}$$

See Weber's *Algebra*, vol. 1, §§74, 78.

Solution by Ellis R. Ott, University of Buffalo

We assume $a_0 \neq 0$, r and s to be integers, $r \geqq s$, and write

$$\begin{aligned} f_r(x) &= a_0x^r + a_1x^{r-1} + \cdots + a_sx^{r-s} + \cdots + a_r, \\ f_s(x) &= a_0x^s + a_1x^{s-1} + \cdots + a_s. \end{aligned} \tag{2}$$

In equating the coefficients of the terms of the product f_rf_s to the coefficients of (1), there are, rather obviously, three general cases to consider. If we represent the exponent of a general term of the product by m, the three cases are: (a) $r+s \geqq m \geqq r$, (b) $r > m \geqq s$, and (c) $s > m \geqq 0$. After equating coefficients of x^{r+s-k} in each member of the desired equality, we shall have in case (a)

$$a_k(\alpha_{r+s} - a_0) + a_{k-1}(\alpha_{r+s-1} - a_1) + \cdots + a_0(\alpha_{r+s-k} - a_k) = 0,$$
$$k = 0, 1, \cdots, s.$$

Since the determinant of this system of $(s+1)$ equations is a_0^{s+1}, we must have

$$\alpha_{r+s-k} = a_k, \qquad 0 \leqq k \leqq s. \tag{3}$$

If we use the values (3) in case (b) there are certain cancellations, and we have left the set of equations

$$a_{k-1}\alpha_{r-1} + a_{k-2}\alpha_{r-2} + \cdots + a_0\alpha_{r-k} = 0, \qquad k = 1, 2, \cdots, r-s,$$

and hence

$$\alpha_{r-k} = 0, \qquad k = 1, 2, \cdots, r - s. \tag{4}$$

Finally for case (c) after using the values in (3) and (4) we have

$$a_{k-1}(\alpha_{s-1} + a_{r+1}) + a_{k-2}(\alpha_{s-2} + a_{r+2}) + \cdots + a_0(\alpha_{s-k} + a_{r+k}) = 0, \qquad k = 1, 2, \cdots, s,$$

and hence

$$\alpha_{s-k} = - a_{r+k}, \qquad k = 1, 2, \cdots, s. \tag{5}$$

These results then give the desired identity

$$f_r f_s = - \sum_{i=0}^{s-1} a_{r+s-i} f_i + \sum_{i=r}^{r+s} a_{r+s-i} f_i.$$

It is to be noted that if $r+s>n$, such that $r+s=n+k'$, $\alpha_0=\alpha_1= \cdots =\alpha_{k'-1}=0$, since for $m>n$, $a_m=0$.

Solved also by the proposer.

Editorial Note. The desired relation may be obtained by grouping the terms of the product $f_r f_s$ and introducing such terms as are needed to complete a polynomial f_i so that they cancel. The general term of the product is

$$a_{r-i} b_{s+i-m} x^m, \qquad 0 \leqq i \leqq r, \qquad i \leqq m \leqq s + i, \tag{1}$$

where b denotes a coefficient of f_s which will be replaced later by a. For a fixed i, we have

$$a_{r-i} \sum_{m=i}^{s+i} b_{s+i-m} x^m = a_{r-i} f_{s+i} - a_{r-i} \sum_{j=1}^{i} b_{s+j} x^{i-j}, \tag{2}$$

where on the right the sum is replaced by zero if $i=0$. Summing both sides of (2) we have

$$\begin{aligned} f_r f_s &= \sum_{i=0}^{r} a_{r-i} f_{s+i} - \sum_{i=1}^{r} a_{r-i} \sum_{j=1}^{i} b_{s+j} x^{i-j}, \\ &= \sum_{i=0}^{r} a_{r-i} f_{s+i} - \sum_{j=1}^{r} b_{s+j} \sum_{i=j}^{r} a_{r-i} x^{i-j}, \\ f_r f_s &= \sum_{i=0}^{r} a_{r-i} f_{s+i} - \sum_{i=1}^{r} a_{s+i} f_{r-i}. \end{aligned} \tag{3}$$

If $r>s$ there is a cancellation of

$$\sum_{i=1}^{r-s} a_{s+i} f_{r-i},$$

and we get

$$f_r f_s = \sum_{i=r-s}^{r} a_{r-i} f_{s+i} - \sum_{i=r-s+1}^{r} a_{s+i} f_{r-i}.$$

The proposer proved the last result in (3) of this note by induction, and wrote it in the form

$$\sum \zeta_{rst} a_t f_{r+s-t},$$

where, if δ_{ij} is the Kronecker delta,

$$\zeta_{rst} = 1 - \sum_{\alpha=1}^{t} \delta_{r,\alpha-1} - \sum_{\alpha=1}^{t} \delta_{s,\alpha-1}.$$

The proposer also made the following two remarks: If the a's are non-commutative quantities but are all commutative with x, the last result in (3) still holds true if we write the product in the subtracted sum as $f_{r-i}a_{s+i}$. The theorem is capable also of the following extension. If

$$u_r(x) = x(x+1)\cdots(x+r-1),$$

$$f_r(x) = a_0u_r(x) + a_1u_{r-1}(x) + \cdots + a_r,$$

then

$$f_r(x)f_s(x+r) = \sum \zeta_{rst} a_t f_{r+s-t}(x+t).$$

3712 [1934, 634]. *Proposed by J. R. Musselman, Western Reserve.*

The necessary and sufficient condition that the six points of intersection of two parallel equilateral triangles, sides produced if necessary, lie on an equilateral hyperbola is that the sides of the two triangles be equal.

Solution by Robert C. Yates, University of Maryland

Let the vertices of the two equilateral triangles be 1, w, w^2; and $b+a$, $b+wa$, $b+w^2a$; where a and b are complex and w is an imaginary cube root of unity. If the triangles are to be parallel then $a=\bar{a}$, that is, real. (The bar here indicates a conjugate). The two triangles may be expressed as cubics which intersect in nine points and since they are in perspective, three of the intersecting points lie on the line at infinity, the other six on a conic.

Forming the pencil of cubics (a linear combination of the products of sides) we have:

$$k(wx+\bar{x}+w^2)(w^2x+\bar{x}+w)(x+\bar{x}+1)$$

$$=(b-aw^2+\bar{b}w-x-w\bar{x})(b+\bar{b}-a-x-\bar{x})(b+w^2\bar{b}-aw-x-w^2\bar{x}),$$

which, for $k=-1$, reduces to the conic in question:

$$bx^2+(a-1)x\bar{x}+\bar{b}\bar{x}^2+L=0,$$

where L represents the collection of linear terms.

If this conic is to have perpendicular asymptotes, the term in $x\bar{x}$ must be missing. Therefore, $a=1$ and the triangles are congruent. The last statement

may be verified by rewriting the real equation of an equilateral hyperbola in self-conjugate form.

Solved also by J. W. Clawson, E. P. Starke, and the proposer.

Editorial Note. The solutions by the proposer and Starke found the rectangular coordinates of the six points of intersection and then formed the equation of the conic passing through them. This was also done by Clawson, but homogeneous coordinates were used with one triangle as the base of the system.

In the solution above it appears that ordinary rectangular coordinates would not complicate the solution unduly. If the coordinates of the vertices of one triangle are $(-s, 0)$, $(s, 0)$, $(0, \sqrt{3}\,s)$, the equation of the sides is easily found to be

$$f(x, y, s) = y^3 - 3x^2y - 2\sqrt{3}\,sy^2 + 3s^2y = 0, \tag{1}$$

which is homogeneous in x, y, s. If we replace in (1) x, y, s by $x+h$, $y+k$, $s+p$, we have the equation of the sides of a second parallel triangle, where $2(s+p)$ is the length of the equal sides, and $-h$, $-k$ are the coordinates of the midpoint of the side parallel to the x-axis. In particular if $p=0$ the second triangle is merely a translation of the first. If $p=-2s$, the second triangle is a translation of the first followed by a rotation of 180° in its plane. There are then two cases of similitude according as $s+p$ is positive or negative. The difference of the left sides of the two equations gives the equation of the conic through the six finite points of intersection of the sides of the two triangles. It suffices to obtain the terms of the second degree in x and y; for the conic is required to be a rectangular hyperbola and hence this part of the equation set equal to zero must give two real straight lines parallel to the orthogonal asymptotes. We pick out this part from the first three terms of the development

$$\begin{aligned} f(x+h, y+k, s+p) &- f(x, y, s) \\ &= hf_x(x, y, s) + kf_y(x, y, s) + pf_s(x, y, s) + \cdots, \end{aligned} \tag{2}$$

and we readily find

$$-3kx^2 + (3k - 2\sqrt{3}\,p)y^2 - 6hxy. \tag{3}$$

The necessary and sufficient condition that (3) equated to zero gives two perpendicular straight lines is that $p=0$. Hence the second triangle is a translation of the first. The rectangular hyperbola obviously degenerates if two corresponding sides lie in a straight line, and this can happen in three ways. The only other way in which it degenerates is much more interesting and perhaps not so obvious. If the center of the second triangle lies on the circle concentric with the circumcircle of the first and with a radius equal to the length of a side, the hyperbola reduces to two straight lines at right angles. These results follow from the discriminant of the quadratic form, which is of the fifth degree. The discussion would be incomplete without examination of the case $p=-2s$, which is the one where the second triangle is obtained by rotating the first in its plane through 180° and then translating the rotated triangle. Here the conic may be of the type of the ellipse, hyperbola, or parabola according to the final position of the

second triangle. If the center of the second triangle lies within the circumcircle of the first, it is an ellipse; if outside, an hyperbola; if on the circumcircle, it is a pair of parallel straight lines. The only other case of degeneration of the conic is when two parallel sides of the two triangles lie in the same straight line. This last case is obvious: that these are all the cases follows as before.

The above analysis is just as easily carried out for a triangle of any form whose sides are cut by three straight lines parallel to its sides and forming a triangle. If the conic through the points of intersection is to be an equilateral hyperbola, we must have

$$k(3 + m_1m_2) + p(m_1 - m_2) - h(m_1 + m_2) = 0,$$

where m_1, m_2 are the slopes of the sides of the first triangle through $(s, 0)$, $(-s, 0)$, respectively. If the conic is an equilateral hyperbola for every position of the second triangle, we must have

$$m_2 = - m_1 = \sqrt{3}, \text{ and } p = 0.$$

There are certain facts which are obvious in a synthetic manner in the general as in the special case. The six points of intersection taken in a suitable order are the vertices of a hexagon whose opposite sides meet on the line at infinity. Hence a conic passes through these six points. The three straight lines joining the mid-points of these opposite parallel chords meet in the center of the conic. If the conic is a rectangular hyperbola, any one of these chords and its conjugate diameter must be equally inclined in opposite directions to an asymptote. It may be possible to obtain a simple synthetic solution of this problem with the aid of some such properties.

3715 [1934, 635]. *Proposed by Harriet B. Herbert, New York City.*

In the sequence of square, cube, tesseract, continued to n dimensions, what angles are formed between diagonals for each figure?

Solution by N. Anning, University of Michigan

Lines in n-space joining two of the points $(\pm 1, \pm 1, \cdots, \pm 1)$ to the origin make an angle whose cosine is $(n-2k)/n$, $k=0, 1, 2, \cdots, n$. A count of the appearances of angles for the various k's gives the total $=2^n$ as one should expect.

Solved also by E. P. Starke and the proposer.

3718 [1935, 48]. *Proposed by Frank Morley, Johns Hopkins University.*

Show that the ellipse through the points given by the complex numbers a, b, c and with center $(a+b+c)/3$ has semi-axes whose lengths are

$$|a + \omega^2 b + \omega c|/3 \pm |a + \omega b + \omega^2 c|/3,$$

where $\omega = (-1+i\sqrt{3})/2$.

I. *Solution by M. G. Boyce, Western Reserve University*

Let p and q be any two constant complex numbers and t a variable parameter of modulus unity. Then in the Argand plane the locus of z defined by

$$z = pt + q/t \tag{1}$$

is an ellipse with its center at the origin. For, elimination of t from z and $\bar{z}$ leaves a second degree equation in z and $\bar{z}$ with no first degree terms, and $|z|$ is finite. The maximum and minimum values of $|z|$ are $|p|+|q|$ and $||p|-|q||$, respectively, which are then the lengths of the semi-axes. It might be noted also that the direction of the major axis is that of the bisector of the angle between the vectors p and q at the origin. The ellipse with its axes of above lengths and directions but with center at r has the equation

$$z = pt + q/t + r. \tag{2}$$

Now for the problem under consideration, the equation

$$3z = (a + \omega^2 b + \omega c)t + (a + \omega b + \omega^2 c)/t + a + b + c \tag{3}$$

is that of the ellipse through a, b, c, since z takes these values when $t=1, \omega, \omega^2$, respectively. It is readily seen that the center and semi-axes are as stated in the problem.

A ruler-compass construction for the axes of the ellipse may be devised from the properties discussed by utilizing the construction for $a+\omega^2 b+\omega c$ and $a+\omega b+\omega^2 c$ given by Frank and F. V. Morley in *Inversive Geometry*, page 206.

II. *Solution by Otto J. Ramler, Catholic University of America*

Let z_1, z_2, z_3, be the complex numbers representing the strokes from the center of the ellipse to the three given points; then if $g=(a+b+c)/3$

$$a = g + z_1, \qquad b = g + z_2, \qquad c = g + z_3.$$

Since $1+\omega+\omega^2=0$, the expression for the lengths of the axes is unaltered by replacing a, b, c, respectively by z_1, z_2, z_3. Moreover since the centroid of the triangle formed by the given points is also the center of the ellipse, the ellipse is the Steiner ellipse and the eccentric angles of the three points differ by $2\pi/3$.

Representing the lengths of the semi-axes by A and B, $A>B>0$, and the inclination of the major axis to the axis of reals by α, we may write

$$\begin{aligned}
2e^{-i\alpha}z_1 &= (A + B)t + (A - B)t^{-1} \\
2e^{-i\alpha}z_2 &= (A + B)\omega t + (A - B)\omega^2 t^{-1} \\
2e^{-i\alpha}z_3 &= (A + B)\omega^2 t + (A - B)\omega t^{-1},
\end{aligned}$$

where $t=e^{i\phi}$ and ϕ is the eccentric angle of the point represented by z_1.

Multiplying these equations by $1, \omega^2, \omega$ respectively, and adding the results, we have $A+B=2|z_1+\omega^2 z_2+\omega z_3|/3$ and similarly $A-B=2|z_1+\omega z_2+\omega^2 z_3|/3$.

Solving for A and B we obtain the desired results.

Solved also by E. P. Starke.

Editorial Note. Starke's solution used the fact that the two absolute values in the problem are unaltered by the linear transformation of the complex plane $z=g+rz'$, $|r|=1$; and, after such a transformation, the origin may be taken at the center of the ellipse and the x-axis along the major axis. This solution differs from the other two in using the single equation of the ellipse in x, y instead of the parametric equations. The brevity of solution I is made possible by the fortunate choice of values for p and q in the equation (1) after showing that the latter equation is that of an ellipse. The equation (1) admits a simple geometric interpretation as an ellipse, and this interpretation leads at once to solution II. If p and q are represented by the vectors OP and OQ of lengths P and Q, $P>Q$; and if OA is the bisector of the angle between OP and OQ, the equation says that OP and OQ are rotated through the same angle but in opposite directions from OA. Then z is the end point of the sum of the rotated vectors. Hence a construction for the points z is obtained by drawing any vector from O and marking on it the points S, P, R in this order from O so that OP has the fixed length P and $SP=PR$ have the lengths Q. The parallel to OA through S and the perpendicular to OA through R meet in a z. For SzP is an isosceles triangle and OP and zP are equally inclined in opposite directions to OA and they have the lengths P and Q. Hence z describes an ellipse with semi-major axis of length $P+Q$ along OA, and with semi-minor axis of length $P-Q$ along the perpendicular OB at O to OA. Thus OR and OS describe the major and minor auxiliary circles. Also, if Pz is produced to cut OA and OB at M and N, then MN has the constant length $2P$ while Mz and zN have the lengths $P-Q$ and $P+Q$. Then as the line MN of fixed length slides on OA and OB, the point z fixed on MN describes the ellipse. This is the trammel construction. It is clear that the ellipse is unchanged if p and q in equation (1) are replaced by a pair of rotated values $pe^{i\gamma}$ and $qe^{-i\gamma}$ without changing the position of the axis of reals for t, or we may also change the position of this axis. The ellipse is the orthogonal projection of a circle with a diameter along OA, center at O and radius equal to $P+Q$. For simplicity take the real axis for t along OA. Since the centroid of z_1, z_2, z_3 is at O, the centroid of the corresponding points R_1, R_2, R_3 on the projecting circle is also at O, and hence the triangle $R_1R_2R_3$ is equilateral. Thus $z_1z_2z_3$ is one of the triangles of maximum area inscribed in the ellipse, whereas the triangle formed by the tangents to the ellipse at z_1, z_2, z_3 is one of the triangles of minimum area circumscribing the ellipse as is obvious by considering the corresponding triangle of the circle. A related theorem is given in the solution of 3565 [1933, 372]. It then follows that the corresponding points t_1, t_2, t_3 on the unit circle form an equilateral triangle, and we may take for their values 1, ω, ω^2. We then have

$$(1) \qquad z_1 = p + q, \qquad z_2 = p\omega + q\omega^2, \qquad z_3 = p\omega^2 + q\omega,$$

and we easily obtain as in II the values of p and q and the solution of the problem.

This suggests certain related theorems in regard to the general cubic equation with real or complex coefficients, but, for simplicity, we suppose that the coefficient of the third power of the unknown is unity. A well known method for the formal solution of such a cubic is to remove the second power of the unknown by a linear transformation reducing the cubic to the form

$$z^3 - 3Az + B = 0. \tag{2}$$

Set $z = \eta + \eta'$ in this equation and then set $\eta\eta' = A$. We obtain for η the quadratic equation in η^3

$$\eta^6 + B\eta^3 + A^3 = 0, \tag{3}$$

and we find for the roots of the cubic three expressions like (1) above in which p and q are replaced by η and η'. We shall show that η and η' are the complex numbers for two points such that the segment of a line joining them is normal at its mid-point to an ellipse described below. It will also be shown that, if Z_1, Z_2, Z_3 are the roots of the original cubic equation, the roots of its first derived equation are the foci of the ellipse tangent to the sides of the triangle $Z_1Z_2Z_3$ at their mid-points and with the ellipse center at the centroid of the triangle.

Let Z_1', Z_2', Z_3' be the mid-points of the sides of $Z_1Z_2Z_3$, which we shall assume to be an actual triangle. A conic may be passed through the three mid-points tangent to Z_1Z_2 and Z_1Z_3 at Z_3' and Z_2', respectively. The polar of Z_1 is $Z_2'Z_3'$, which is parallel to Z_3Z_2. Since the chord $Z_2'Z_3'$ is bisected by Z_1Z_1', the latter is a diameter conjugate to the chord, and Z_2Z_3 must also be tangent to the conic at Z_1'. Hence the medians of $Z_1Z_2Z_3$ are diameters and the centroid of the triangle is the center of the conic. The two triangles which are symmetric to $Z_1Z_2Z_3$ and $Z_1'Z_2'Z_3'$ with respect to the common centroid are similarly situated with respect to the conic. Hence the conic is an ellipse inscribed in a hexagon with parallel and equal opposite sides and the points of tangency are the mid-points of the sides of the hexagon. Let a and b be the lengths of the semi-major and semi-minor axes, and let α be the inclination of the major axis to the axis of reals. The transformation

$$Z = g + e^{i\alpha}z, \qquad 3g = Z_1 + Z_2 + Z_3, \tag{4}$$

makes the centroid the new origin O and the major axis the axis of reals for z. This transformation removes the second power of the unknown from the cubic equation, and, as appears later, makes the coefficient of the first power real and not positive. The complex numbers for the inscribed hexagon are now in order z_1', $-z_3'$, z_2', $-z_1'$, z_3', $-z_2'$. The intersection U of the two tangents $(-z_1)(-z_2)$ and z_1z_3 gives OU as the direction of the diameter conjugate to the diameter $-z_1'z_1'$. The altitude of the equilateral triangle with side OU gives the length of the semi-diameter Oz_1'' along OU. A construction for the lengths and positions of the principal axes will be contained in what follows. If θ is the eccentric angle for z_1', we have

$$z_1' = a\cos\theta + ib\sin\theta = \tfrac{1}{2}[(a+b)t + (a-b)t^{-1}], \qquad t = e^{i\theta}. \tag{5}$$

The eccentric angle for z_1'' is $\pi/2+\theta$, so that

$$z_1'' = -a\sin\theta + ib\cos\theta. \tag{6}$$

Rotate Oz_1'' through the angle $\pi/2$ to the position Oz_1'''; then the latter is parallel to the normal at $-z_1'$ and we have

$$z_1''' = iz_1'' = -b\cos\theta - ia\sin\theta = \tfrac{1}{2}[-(a+b)t + (a-b)t^{-1}]. \tag{7}$$

Lay off on the normal at $-z_1'$ lengths equal to that of Oz_1''' in opposite directions from $-z_1'$ thus giving two points η and η' on this normal. Then

$$\begin{aligned} \eta &= -z_1' + z_1''' = -(a+b)t, \\ \eta' &= -z_1' - z_1''' = -(a-b)t^{-1}, \qquad z_1 = -2z_1' = \eta + \eta'. \end{aligned} \tag{8}$$

It is obvious from the above equations for η and η' that the major axis bisects the angle between the vectors for these two complex numbers, and that $2a = |\eta| + |\eta'|$, $2b = |\eta| - |\eta'|$. By replacing θ by $\theta + 2\pi/3$ and $\theta - 2\pi/3$, we have with the above result

$$\begin{aligned} z_1 &= \eta + \eta', & 0 &= z_1 + z_2 + z_3, \\ z_2 &= \omega\eta + \omega^2\eta', & 3\eta &= z_1 + \omega^2 z_2 + \omega z_3, \\ z_3 &= \omega^2\eta + \omega\eta', & 3\eta' &= z_1 + \omega z_2 + \omega^2 z_3. \end{aligned} \tag{9}$$

A simple computation gives

$$\begin{aligned} z_2z_3 &= \eta^2 + \eta'^2 - \eta\eta', \qquad z_3z_1 = \omega^2\eta^2 + \omega\eta'^2 - \eta\eta', \\ z_1z_2 &= \omega\eta^2 + \omega^2\eta'^2 - \eta\eta', \quad z_1z_2z_3 = \eta^3 + \eta'^3, \\ z_2z_3 &+ z_3z_1 + z_1z_2 = -3\eta\eta' = -3(a^2 - b^2) = -3a^2e^2. \end{aligned} \tag{10}$$

The transformed cubic with the roots z_1, z_2, z_3 is then

$$z^3 - 3a^2e^2z - (\eta^3 + \eta'^3) = 0, \tag{11}$$

and it is obvious that the roots of its first derived equation are the foci of the ellipse. It also follows that the ellipse through z_1, z_2, z_3 with its center at the centroid of the triangle $z_1z_2z_3$ and with the semi-axes $2a$ and $2b$ is defined by the equation

$$z = \eta t + \eta' t^{-1}, \qquad t = e^{i\theta},$$

and we have again the equation of solution I.

In the solution of 3565, referred to above, are references to the construction of the lengths and positions of the principal axes. A construction is also given in the note to the solution of 3650 [1935, 117] which is quite convenient for the present data, using the diameter $-z_3'\ O\ z_3'$ and taking for the axis of the affine transformation the straight line through $-z_1'$ and z_2'.

Returning to the first part of this note, we see that points z on the ellipse

may be constructed by first prolonging OP to OP' and OQ to OQ' so that $OP'=2OP$ and $OQ'=2OQ$. Draw $P'Q'$ and the parallel to OQ through P cutting $P'Q'$ in z. Then $P'Q'$ is the normal to the ellipse at z. This admits a simple proof independent of the above.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

It is with great regret that the officers of the Mathematical Association of America announce the death of Mrs. Mary Hegeler Carus of LaSalle, Illinois, on June 27, 1936. Her continued interest in the projects of the Association was shown chiefly through her generosity in assisting in the publication and the distribution of the Carus Monographs, both personally and through the medium of the Open Court Publishing Company.

The officers of the Mathematical Association hope that the members will have in mind the plan whereby, under a business arrangement between the Association and both the *Annals of Mathematics* and the *Duke Mathematical Journal*, individual members of the Association are entitled to subscribe for each of these publications at one-half the regular price. Subscriptions or renewals should be sent directly to the office of each magazine, with the statement that one is a member of the Association.

A Florida Academy of Science has been founded by members of the University of Florida, the Florida State College for Women, and the University of Miami. A charter was applied for and granted in February. Professor J. H. Kusner of the University of Florida was elected Secretary of the organization.

The second John Howard Appleton Lecture of Brown University for the year 1935–1936 was delivered on April 6, by Professor Arthur Haas of the University of Vienna. His subject was "Fundamental ideas of modern physics."

Associate Professor Nola L. Anderson, of the Sophie Newcomb College of Tulane University, has been promoted to a professorship.

Assistant Professor F. R. Bamforth of Ohio State University has been promoted to an associate professorship.

Dr. J. J. Gergen of the University of Rochester has been appointed an associate professor at Duke University.

Assistant Professor Lincoln LaPaz of Ohio State University has been promoted to an associate professorship.

Assistant Professor A. A. Shaw of the University of Arizona has been promoted to an associate professorship.

Dr. Marie J. Weiss of Sophie Newcomb College has been appointed an assistant professor at Vassar College.

The following changes have been announced in the mathematics department at the University of Pennsylvania:

Professor F. H. Safford has retired at the close of the academic year 1935–36. Assistant Professors P. A. Caris and J. A. Shohat have been promoted to associate professorships. Visiting Professor H. Rademacher has been appointed an assistant professor, and Dr. J. A. Clarkson has been appointed instructor.

Dr. F. C. Touton, vice president of the University of Southern California, died June 1, 1936 at the age of 55. He was a charter member of the Association and had just been elected chairman of the Southern California Section.

The following seventy-eight doctorates with mathematics or mathematical physics as major subject were conferred during 1935 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

L. A. Alden, California Institute of Technology, June, major in mathematical physics, *The effect of configuration interaction on intensity anomalies.*

Max Astrachan, Brown, June, *Studies in the summability of Fourier series by Nörlund means.*

A. S. Avakian, Massachusetts Institute of Technology, June, minor in physics, *Almost periodic functions and the vibrating membrane.*

R. P. Bailey, Pennsylvania, February, *Convergence of sequences of positive linear functional operations.*

Gertrude Blanch, Cornell, September, *Properties of the Veneroni transformation in S_4.*

C. E. Buell, Washington University (St. Louis), May, minor in physics, *An investigation on the location of the zeros of Jacobi polynomials and polynomials related thereto.*

E. J. Camp, Chicago, June, *The motion of an unsymmetric top.*

E. W. Cannon, Johns Hopkins, June, *On complex orthogonal four-rowed matrices with applications to Lorentz matrices and semi-vectors.*

J. W. Cell, Illinois, June, *Second order linear differential equations invariant with respect to linear fractional substitution groups of finite order.*

L. H. Chambers, Cornell, June, minor in physics, *On* (2, 2) *planar correspondences.*

W. R. Church, Yale, June, *Theory of finite distributive free structures.*

B. G. Clark, Illinois, February, minor in astronomy, *The configuration of six points of the plane.*

J. A. Cooley, Illinois, June, minor in physics, *Certain symmetric surfaces and curves invariant in De Jonquières transformations.*

A. P. Cowgill, Nebraska, June, minor in civil engineering, *On the summability of a certain class of series of Jacobi polynomials.*

J. H. Curtiss, Harvard, June, (1). *Interpolation in transforms of the roots of unity*; (2). *The Jacobi interpolation series on the lemniscate of convergence.*

Wayne Dancer, Michigan, June, *Concerning symmetrical cut sets.*

E. R. Elliott, Duke, June, minor in applied mathematics, *A mixed boundary-value problem for the heat equation.*

W. H. Erskine, Johns Hopkins, June, *On the Lagrangians of the new field theory.*

G. M. Ewing, Missouri, June, *A contribution to the theory of discontinuous solutions in the calculus of variations.*

M. J. Fish, Massachusetts Institute of Technology, June, minor in physics, *A particular boundary-value problem.*

M. M. Flood, Princeton, April, *Division by non-singular matric polynomials.*

A. H. Fox, Yale, June, *Differential equations with continuous spectra.*

Jeannette Fox, Yale, June, *Existence of a Euclid algorithm in quadratic fields.*

G. A. Garrett, Rice Institute, June, *Necessary and sufficient conditions for potentials of single and double layers.*

K. S. Ghent, Chicago, August, *On sums of values of a polynomial multiplied by constants.*

E. H. Hanson, Ohio State, December, *A theorem of Denjoy, Young, and Saks. The tau limit.*

F. S. Harper, State University of Iowa, August, minors in applied mathematics and physics, *An actuarial study of infant mortality.*

E. G. Harrell, State University of Iowa, August, minor in applied mathematics, *On the topology of a two-parameter, non-metric, and non-separable space.*

E. A. Hedberg, Missouri, August, *Rings and modular systems.*

Fritz Herzog, Columbia, May, *Systems of algebraic mixed differential equations.*

Sister Laetitia Hill, Catholic University, June, minors in physics and mechanics, *The quadrilaterals in- and circumscribed to the rational unicuspidal quartic curve.*

Ingomar Hostetter, Washington, August, minor in physics, *A new solution of the simplest problem of the calculus of variations by vector methods.*

G. B. Huff, Illinois, June, minor in physics, *Discontinuous groups associated with the Cremona groups.*

P. M. Hummel, Ohio State, August, *Continued fractions and matrices.*

M. Gweneth Humphreys, Chicago, August, *On the Waring problem with polynomial summands.*

J. T. Hurt, Rice Institute, June, *On the uniformly bounded turning of level curves of the Green's function.*

W. L. Hutchings, California (Berkeley), May, *On a certain canonical form of a system of two linear, homogeneous differential equations of the second order, with applications to the theory of ruled surfaces.*

E. D. Jenkins, Ohio State, March, *The composition of quadratic forms.*

A. E. Johns, Toronto, June, minor in quantum mechanics, *On the bilinear transformation in the real plane.*

F. B. Jones, Texas, June, minor in physical chemistry, *Concerning R. L. Moore's axiom* 5_1.

W. J. Kirkham, Indiana, June, major in applied mathematics, minor in physics, *The normal structure of pentotomic molecules of the type* Y_4.

Walter Leighton, Jr., Harvard, June, *The minimizing of a singular quadratic functional.*

Norman Levinson, Massachusetts Institute of Technology, June, minor in electrical engineering, *On the non-vanishing of a function.*

H. M. MacNeille, Harvard, June, *Extensions of partially ordered sets.*

Henry Malin, Massachusetts Institute of Technology, June, minor in physics, *On gap theorems.*

J. L. Maulbetsch, Michigan, February, major in engineering mechanics, *Direction methods of the calculus of variations with applications to the problem of torsion, and thermal stresses in plates.*

R. A. Merrill, Wisconsin, October, major in physics, *Spectral multiplets belonging to configurations of the type* $d^k ms\ d^k msns$.

R. J. Michel, Missouri, August, *A Galois theory in a reducible ring.*

Harriet F. Montague, Cornell, June, minor in logic, *Certain non-involutorial Cremona transformations of hyperspace.*

F. J. Murray, Columbia, June, *The theory of linear transformations between Hilbert spaces and the application of this theory to partial differential equations.*

G. D. Nichols, Nebraska, August, minors in physics and philosophy, *The arithmetized series expansions for certain pseudo-periodic functions with some number-theoretic applications.*

E. N. Oberg, Minnesota, June, minor in physics, *Approximate solutions of integral equations.*

Arthur Ollivier, State University of Iowa, August, minor in applied mathematics, *On certain mathematical developments underlying an analysis of general death rates.*

E. M. J. Pease, Massachusetts Institute of Technology, June, minor in electrical engineering, *On the relationship between simple distillation and true boiling-point curves.*

I. E. Perlin, Chicago, December, *Sufficient conditions for a minimum in the problem of Lagrange with isoperimetric conditions.*

A. E. Pitcher, Harvard, June, *Certain invariants of closed extremals.*

A. R. Poole, California Institute of Technology, June, minor in physics, *Finite ova.*

R. M. Robinson, California (Berkeley), May, *Some results in the theory of schlicht functions.*

Rafael Sanchez-Diaz, California (Berkeley), May, *A new method of factorization supplementary to the factor stencils.*

S. T. Sanders, Jr., State University of Iowa, August, minor in applied mathematics, (1). *Derived sets and their complement*; (2). *The characterization of certain set-valued set-functions.*

Henry Scheffe, Wisconsin, June, *The asymptotic solutions on certain linear differential equations in which the coefficient of the parameter may have a zero.*

C. G. Schilling, Illinois, June, minor in physics, *Some geometric applications of algebraic correspondences.*

Sister Mary Leontius Schulte, Michigan, February, *Additions in arithmetic,* 1483–1700, *to the sources of Cajori's "History of mathematical notations" and Topfke's "Geschichte der Elementarmathematik."*

R. Q. Seale, Stanford, June, minor in physics, *A simple proof of Minkowski's theorem on the product of two linear forms.*

Ching Lai Shen, Michigan, June, *Fundamentals of the theory of inverse sampling.*

Yu-cheng Shen, Harvard, February, *On interpolation with approximation to an analytic function by rational functions with preassigned poles.*

Ruth C. Stauffer, Bryn Mawr, June, *The construction of a normal basis in a separable normal extension field.*

A. H. Taub, Princeton, June, *Quantum equations in cosmological spaces.*

Mary M. Taylor (Mrs. E. R. Speer), Pittsburgh, June, *Reciprocals of certain curves and surfaces with respect to a space cubic curve.*

G. B. Van Schaack, Harvard, June, *Topics in the theory of critical points.*

H. E. Vaughan, Michigan, June, *Local Betti numbers.*

E. E. Walden, Illinois, February, minor in physics, *Algebraic surfaces invariant under the symmetric* G_{120} *with special reference to quintics and sextics.*

J. F. Wardwell, Johns Hopkins, June, *Continuous transformations preserving all topological properties.*

M. S. Webster, Pennsylvania, June, *On the zeros of Jacobi polynomials, with applications.*

C. P. Wells, Iowa State, August, minors in mathematical physics and physics, *Calculation of the ionization potential of lithium.*

L. R. Wilcox, Chicago, June, *Pairs of surfaces in five-dimensional space.*

L. A. Wolfe, Wisconsin, October, *Similarity of matrices in which the elements are real quaternions.*

Margarete C. Wolf, Wisconsin, October, *Symmetric functions of matrices.*

The following doctorate was conferred in 1934 but not included in the list in the preceding volume of this MONTHLY (vol. 42, p. 460).

Solomon Kullback, George Washington University, June, major in theoretical statistics, *An application of characteristic functions to the distribution problem in statistics.*

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N. Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Annual Meeting, Duke University, Durham, N. C., Dec. 31, 1936–Jan. 1, 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, Normal, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, Topeka, March 14.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov. 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S. C., April 17–18.
SOUTHERN CALIFORNIA, Fullerton, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, Madison, May 9.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 8, OCTOBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty-eight persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

E. F. BECKENBACH, Ph.D.(Rice) Instr., Rice Inst., Houston, Texas

A. H. BLACK, Ph.D.(Cornell) Prof., Lebanon Valley Coll., Anneville, Pa.

A. W. BOLDYREFF, Ph.D.(Michigan) Asso. Prof., Univ. of Arizona, Tucson, Ariz.

J. W. BRANSON, M.S.(Purdue) Prof., Head of Dept., New Mexico State Coll. of A. and M.A., State College, N.M.

FOSTER BROOKS, Ph.D.(Ohio State) Instr., Kent State Coll., Kent, Ohio

JOHN CHESNA, Tool and machine designer, Rochester, N.Y.

VELMA L. CLOYD, A.M.(Colorado Teachers Coll.) Math. Training Teacher, State Teachers Coll., Johnson City, Tenn.

W. A. DWYER, M.S.(Creighton) Instr., Creighton Univ., Omaha, Nebr.

Sister M. EMELINE, M.S.(Michigan) Instr., Catholic Jr. Coll., Grand Rapids, Mich.

E. J. FINAN, Ph.D.(Ohio State) Instr., Catholic Univ. of America, Washington, D.C.

J. S. FRAME, Ph.D.(Harvard) Instr., Brown Univ., Providence, R.I.

A. M. GIGNILLIAT, A.M.(Georgia) Instr., Armstrong Jr. Coll., Savannah, Ga.

JULIUS GOLDMAN, B.S., A.B.(Detroit Inst. of Tech.) Instr., Detroit Inst. of Tech., Detroit, Mich.

E. A. HAZLEWOOD, A.M.(Cornell) Asst. Prof., New Mexico State Coll. of A. and M.A., State College, N.M.

R. L. HUGHES, student, Bridgeport Engineering Inst.; Clerk, Genl. Elec. Co., Bridgeport, Conn.

RALPH HULL, Ph.D.(Chicago) Instr., Univ. of Michigan, Ann Arbor, Mich.

W. E. JONES, A.B.(Maryville Coll.) Instr., Maryville Coll., Maryville, Tenn.; grad. student, 1936–37, Univ. of Chicago.

A. I. KATSH, A.M.(New York Univ.) Instr., Div. of Genl. Educ., New York Univ., New York, N.Y.

A. R. KIRBY, A.M.(Columbia) Instr., Notre Dame Coll. of Staten Island, Staten Island, N.Y.

A. A. KOCH, M.S.(Calif. Inst. of Tech.) Sr. Engineering Aide, Corps of Engineers, U. S. Army, Los Angeles, Calif.

T. J. LESLIE, M.S.(Alabama) Prof., Erskine Coll., Due West, S.C.

J. M. MALONE, A.M.(Birmingham-Southern) Asso. Prof., Educ., Birmingham-Southern Coll., Birmingham, Ala.

F. E. MARRIN, A. M.(Creighton) Instr., Creighton Univ., Omaha, Nebr.

A. J. MILES, Sc.D.(Michigan) Prof., Math. and Physics, Shurtleff Coll., Alton, Ill.

K. W. MILLER, M.S.(Union Coll.) Dir. of Research, Utilities Research Commission, Inc., Chicago, Ill.

J. F. MILOS, A.M.(Columbia) Teacher, Memorial High School, Pelham, N.Y.

E. E. NASH, C.E.(Rensselaer) Instr., Rensselaer Poly. Inst., Troy, N.Y.

G. W. PATTERSON, III, A.M.(Columbia) Teacher, High School, Vergennes, Vt.

R. D. PERRY, Ph.D.(Peabody) Prof., W. Ky. State Teachers Coll., Bowling Green, Ky.

V. V. QUILLIAM, A.B.(Univ. of Calif. at L.A.) Cost Accountant, Adohr Milk Farms, Los Angeles, Calif.

J. W. ROSS, A.M.(Texas) Asst. Prof., A. and M. Coll. of Texas, College Station, Texas

HELEN G. RUSSELL, Ph.D.(Radcliffe) Asst. Prof., Wellesley Coll., Wellesley, Mass.

J. L. A. SAUNDERS, A.M.(North Carolina) Asst. Prof., The Citadel, Charleston, S.C.

W. H. SEARS, JR., A.M.(Harvard) Instr., Georgia School of Tech., Atlanta, Ga.

F. C. SMITH, Ph.D.(Michigan) Instr., Coll. of St. Francis, Joliet, Ill.

B. J. SOUTH, M.S.(Notre Dame) Instr., Univ. of Notre Dame, Notre Dame, Ind.

H. O. TAYLOR, Ph.D.(Cornell) Prof., Chm. of Dept., Math., Physics and Astr., Wheaton Coll., Wheaton, Ill.

B. J. TOPEL, A.M.(Harvard), S.T.B.(Catholic Univ.) Prof., Carroll Coll., Helena, Mont.

To Institutional Membership

THE UNIVERSITY OF SOUTH DAKOTA, Vermillion, S. D.

W. D. CAIRNS, *Secretary-Treasurer*

THE TWENTY-SECOND ANNUAL MEETING OF THE KANSAS SECTION

The twenty-second annual meeting of the Kansas Section was held at Topeka, in the high-school building, on Saturday, March 14, 1936. The morning session was held jointly with the Kansas Association of Mathematics Teachers. After a social hour at luncheon together, the two organizations met in separate sessions in the afternoon. Professor W. T. Stratton, Kansas State College, Manhattan, presided at the joint meeting, as well as at the meeting of the Kansas Section.

The attendance was ninety-five, including the following thirty-one members of the Association: C. H. Ashton, R. W. Babcock, Wealthy Babcock, Lois E. Bell, M. T. Bird, Florence L. Black, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, A. J. Hoare, M. Gweneth Humphreys, Emma Hyde. W. C. Janes, H. E. Jordan, C. F. Lewis, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza A. Mossman, O. J. Peterson, C. B. Read, B. L. Remick, J. A. G. Shirk, G. W. Smith, R. G. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, W. G. Warnock, J. J. Wheeler, A. E. White.

At the joint business meeting, it was voted to accept the invitation to meet in Wichita next year, probably in March. The University of Wichita, Friends University, and the High Schools of Wichita joined in the invitation. At the business meeting of the Section the following officers were elected for the coming year: Chairman, R. G. Smith, Kansas State Teachers College, Pittsburg; Vice-Chairman, W. G. Warnock, Kansas State College, Hays; Secretary, Lucy T. Dougherty, Junior College, Kansas City, Kansas.

The following six papers were read:

1. "The number e" by Professor U. G. Mitchell, University of Kansas.
2. "Line configurations and group properties" by Professor W. G. Warnock, Kansas State College, Hays.
3. "A certain paradoxical property of conditionally convergent series" by Professor C. B. Tucker, Kansas State Teachers College, Emporia.
4. "A depleted Fourier series" by Professor C. F. Lewis, Kansas State College, Manhattan.
5. "On the Waring problem" by Dr. M. Gweneth Humphreys, Mount St. Scholastica College.
6. "On a classification of integral functions" by Professor M. T. Bird, Southwestern College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The paper presented by Professor Mitchell on "The number e" is printed in full in the current volume (1936) of Osiris, pp. 476–496.

2. After a brief introduction to the development of line coordinates, Professor Warnock subjected the fundamental identity to the operations of an imprimitive group of degree six and order 48. The number of lines of the system was extended to 384 by the introduction of seven new types, all obtained by placing conditions upon the first identity. The points and planes of intersections of the complete system were classified as to types and sets. The last part of the paper reconsidered the results already derived from the viewpoint of the group theory. It demonstrated the great saving of time and labor in this work by the applications of the properties of sub-groups and co-sets.

3. Professor Tucker presented a brief discussion of the limit for the sums of the sub-series of a conditionally convergent series. The demonstration of a series for which this limit is one was followed by certain generalizations and a résumé of the various possibilities which exist. The basis for this discussion is a paper by Knopp (Math. Zeitschrift, vol. 25).

4. The paper by Professor Lewis was a consideration of a depleted Fourier series, that is, a series with certain terms missing. As an example, an expression was found for the value of the complete series

$$\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots,$$

and linear functions of x were found which were equal to the depleted series

$$\frac{\cos x}{1^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \frac{\cos 11x}{11^2} + \cdots$$

within certain definite intervals. While this is a special problem general results may be obtained.

5. The famous Waring conjecture or problem is that every positive integer is a sum of nine cubes, 19 fourth powers, and so on; and that every positive integer of the proper form is a sum of a finite number of values of a polynomial. Dr. Humphreys presented a survey of the most significant work that has been done on the problem both for powers and for polynomial summands. The results and brief outlines of methods for powers were given for the theorems obtained by algebra alone, for the asymptotic theorems obtained by Hardy and Littlewood, James, and Winogradoff by analysis, and for the universal theorems obtained by Dickson using algebraic methods along with the asymptotic theorems. This discussion was followed by a description of the progress that has been made on the problem for polynomial summands.

6. As an introduction to the classification of integral functions, Professor Bird reviewed the more important topics related to such a study. In particular the notion of the superior limit was discussed and related to the discussion of power series and analytic functions. A classification of integral functions, which covered as a special case the classification of functions of exponential type, was

presented. This classification represented the joint contribution of Professor R. D. Carmichael of the University of Illinois, Doctor W. T. Martin of the Institute for Advanced Study, and the speaker.

LUCY T. DOUGHERTY, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The Annual Meeting of the Texas Section of the Mathematical Association of America was held at the Agricultural and Mechanical College of Texas, College Station, Texas, on April 11, 1936. All sessions were presided over by the Chairman, Professor F. W. Sparks of the Texas Technological College.

Among the forty-one persons attending the meeting were the following twenty-three members of the Association: J. H. Binney, L. W. Blau, A. A. Blumberg, E. O. Box, H. E. Bray, Alice C. Dean, Nat Edmonson, H. J. Ettlinger, L. R. Ford, E. H. Hanson, C. M. Howard E. C. Kennedy, A. A. McSweeny, E. D. Mouzon, Jr., W. L. Porter, J. W. Querry, J. E. Redden, W. A. Rees, O. M. Rogers, H. C. Searcy, C. R. Sherer, F. W. Sparks, F. E. Ulrich.

The Texas Section met jointly with the Texas Section of the Society for the Promotion of Engineering Education to hear and discuss a paper read by Dr. L. W. Blau of the Humble Oil and Refining Company on the mathematical training of engineering students. The discussion of this paper was led by Professor H. E. Bray of The Rice Institute. This joint session was presided over by Professor Sparks.

At the business session following the presentation of the papers the following officers were elected for the coming year: Chairman, L. R. Ford, The Rice Institute; Vice-Chairman, J. H. Binney, Agricultural and Mechanical College of Texas. It was also voted that the incoming chairman appoint a committee having the power to make all arrangements for the 1937 meeting.

The following papers were read:

1. "The necessity for quantum mechanics" by Dr. L. J. B. Lacoste, University of Texas, introduced by Professor Ettlinger.
2. "Linear fractional transformations in two complex variables" by J. W. Hahn, Rice Institute, introduced by Professor Bray.
3. "Mathematics for the engineer" by Dr. L. W. Blau, Director of Geophysical Research, Humble Oil and Refining Company.
4. "Matrix functions defined by differential equations" by J. O. Mayhugh, University of Texas, introduced by Professor Ettlinger.
5. "Properties of lemniscates" by J. W. Green, Rice Institute, introduced by Professor Bray.
6. "The sum of a nil-potent matrix and the identical matrix" by Dr. G. B. Huff, Southern Methodist University, introduced by Professor Mouzon.
7. "Jordan curves" by A. A. Aucoin, Rice Institute, introduced by Professor Bray.

Abstracts, numbered to correspond to the list of papers, follow:

1. An experiment in physics, the results of which could not be predicted or explained by classical mechanics, was described by Dr. Lacoste.

2. In this paper, Mr. Hahn extended to two complex variables certain results on linear fractional transformations obtained for the unit circle with one complex variable.

3. The suggestion that teachers of mathematics make determined efforts to secure a better understanding on the part of the students by correlating the work with situations arising in practical life was made by Dr. Blau. He emphasized the importance of stressing the geometrical and physical aspects of mathematics. Engineering students do not need more courses in mathematics; they are lacking in an understanding of the principles involved, and frequently do not acquire a working knowledge of the subject.

4. Mr. Mayhugh defined power series and polynomial functions of a square matrix as the analogues of the corresponding functions of a real or a complex variable. He outlined a method for giving a finite representation of a function of a matrix as given by Professor R. W. Barnard. A system of simultaneous linear differential equations was expressed as a matrix differential equation and its solution obtained as a function of a matrix. The general non-homogeneous linear matrix differential equation of the first order was solved, and an application made to the general Heaviside problem.

5. Mr. Green discussed the problem of approximating simple closed analytic curves by lemniscates and the problem of passing a lemniscate through a given number of fixed points, a lemniscate being the locus of a point which moves so that the product of its distances from n fixed points is constant.

6. In several places in the literature explicit numerical matrices appear which have the following property: the kth power of the given matrix can be expressed as a matrix whose elements are polynomials in the exponent k. It develops that a matrix can have this property if and only if it is the sum of a nil-potent matrix and the identical matrix. Dr. Huff exhibited explicit formulae for the coefficients in the polynomials when this condition is satisfied, and made application of the results to certain problems in Cremona geometry.

7. Mr. Aucoin's paper was a critical study of a proof of the Theorem of Jordan on plane curves by Kerekjarto.

NAT EDMONSON, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The fourteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of South Carolina, Columbia, S. C., on Friday and Saturday, April 17–18, 1936. Sessions were held in the afternoon and evening of the 17th and on the morning of the 18th. The chairman of the Section, Professor F. W. Kokomoor presided, except Friday evening and part of Saturday morning when the Section was divided into subgroups according to the nature of the papers presented.

There were in attendance about one hundred seventy persons from thirty-one institutions including the following fifty members of the Association: D. H. Ballou, Walter Bartky, Helen Barton, W. S. Beckwith, E. T. Browne, C. W. Bruce, Iris Callaway, E. A. Cameron, T. C. Carson, B. G. Clark, Velma L. Cloyd, J. B. Coleman, H. M. Cox, Forrest Cumming, B. F. Dostal, L. A. Dye, W. W. Elliott, A. M. Gignilliat, C. L. Hair, R. A. Hefner, Archibald Henderson, P. R. Hill, J. A. Hyden, J. B. Jackson, Rosa L. Jackson, F. W. Kokomoor, G. B. Lang, J. W. Lasley, Jr., T. J. Leslie, F. A. Lewis, S. W. McInnis, J. S. Morrel, C. F. Myers, Jr., W. P. Ott, W. V. Parker, Alice A. Peck, C. J. Ramage, W. W. Rankin, Caroline M. Reaves, B. P. Reinisch, H. A. Robinson, J. A. Saunders, T. M. Simpson, F. H. Steen, Ruth W. Stokes, Cora Strong, T. L. Wade, Jr., W. W. Weber, W. L. Williams, F. L. Wren.

On the evening of the 17th a dinner was held in honor of the visiting speaker, Dr. Walter Bartky of University of Chicago. At this time Professor J. B. Coleman presided.

At the business session on the 18th the following officers were chosen for 1936–37: Chairman, W. W. Rankin, Duke University; Vice Chairman, J. B. Jackson, University of South Carolina; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The next meeting was scheduled for April 1937 at Vanderbilt University and Peabody College. A resolution was passed relative to the loss sustained by the Section in the passing of Professors B. H. Crenshaw and Julia Dale.

The following twenty-four papers were read:

1. "The nature of mathematics" by Professor W. W. Rankin, Duke University.

2. "Signatures for Nasik magic squares" by Dr. R. A. Hefner, Georgia School of Technology.

3. "Trigonometric approximation in the mean" by E. S. Quade, University of Florida, introduced by the Secretary.

4. "An analysis of Literary Digest polls" by P. R. Hill, University of Georgia.

5. "On the derivation and reduction of algebraic concomitants by means of tensor algebra" by Professor T. L. Wade, Mercer University.

6. "Identities implying desmic tetrahedra" by Dr. F. A. Lewis, University of Alabama.

7. "A degenerate case of the Weddle surface and its corresponding Kummer surface" by Dr. J. A. Hyden, Vanderbilt University.

8. "New slants on relativity" by Professor Archibald Henderson, University of North Carolina.

9. "Mathematical analysis of some Lissajous figures" by F. H. Mitchell, University of Alabama, introduced by the Secretary.

10. "The expanding universe—pro and con" by Dr. Walter Bartky, University of Chicago.

11. "Approximate solution of an equation" by Dr. F. H. Steen, Georgia School of Technology.

12. "Some determinants whose elements are zero and one" by Dr. J. A. Greenwood, Duke University, introduced by the Secretary.

13. "Some notes on the Georgia educational and vocational guidance test in mathematics" by H. M. Cox, Secretary to Examiners, University System of Georgia.

14. "A note on computing complex roots of rational integral equations" by Dr. J. S. Morrell, Vanderbilt University.

15. "Differential equations of projective transformations" by Dr. V. A. Hoyle, University of North Carolina, introduced by the Secretary.

16. "Systems of linear differential equations" by Dr. Walter Bartky, University of Chicago.

17. "Quadratic factors of symbolic forms" by Professor Ruth W. Stokes, Mitchell College.

18. "Laplace integrals and the theta null functions" by Dr. D. H. Ballou, Georgia School of Technology.

19. "An involutorial transformation of order 29" by Dr. L. A. Dye, The Citadel.

20. "Concerning preparedness of freshmen for college algebra" by Professor J. B. Coleman, University of South Carolina.

21. "Mathematics in the general college" by Professor F. W. Kokomoor, University of Florida.

22. "Some remarks on osculants to plane curves" by E. A. Cameron, University of North Carolina.

23. "A note on a problem of regions" by Professor H. A. Robinson, Agnes Scott College.

24. "The Pascal configuration determined by the vertices of two maximum triangles inscribed in an ellipse" by Dr. B. G. Clark, University of Alabama.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Rankin gave a historical paper on the nature of mathematics due to the postulational method from the time of Zeno to the present.

2. Dr. Hefner gave general methods of constructing the signatures of certain Nasik magic squares.

3. In this paper Mr. Quade showed relations which exist between the closeness of the approximation, the degree of the trigonometrical polynomial and the strength of the generalized Lipschitz condition satisfied by a function.

4. Mr. Hill dealt with the study of the Literary Digest polls and their accuracy in forecasting.

5. Illustrations of the construction of tensor algebra of algebraic invariants and concomitants were given by Professor Wade. Tensor and symbolic methods were compared.

6. In his paper, Dr. Lewis considered identities implying desmic systems in

which the coordinates of the vertices and the coefficients in the equations of the planes are zero or nth roots of unity.

7. Dr. Hyden discussed cases of Weddle surfaces and their corresponding Kummer surfaces in which the six base points of the generating quadric surfaces coincide in pairs.

8. Professor Henderson gave new geometric interpretations to the Lorentz transformation with derivations of the invariability of the Riemannian space-time "interval" for inertial systems. The second part of the paper was his contribution to the new relativity theory recently advanced by Professor Page of Yale.

9. The resolution of Lissajous figures into their component waves was considered as a problem of expressing a plane curve in a given parametric form. Dr. Mitchell illustrated methods for transforming certain Cartesian equations into this form.

10. Dr. Bartky gave in an illustrated lecture observations that lead him to state spiral nebulae or external galaxies reveal that, excluding a tendency to form individual groups, these galaxies are uniformly distributed in space. Furthermore the spectral lines of these spirals show in general a displacement toward the red in amounts proportional to distances. If these red shifts are interpreted as due to a motion of recession, then the universe of galaxies is expanding. Seemingly confirming this hypothesis is the theoretical deduction of the relativists: namely, a static universe of constant density is unstable. There is the possibility that a red shift may be due to other causes than motion. The question arises as to whether or not an expansion in which the unit of measurement expands should be detectable.

11. Dr. Steen derived a method, related to Whittaker's, for obtaining in determinant form, without iteration, an approximation to the roots of an algebraic or transcendental equation.

12. Dr. Greenwood gave a reduction formula involving a certain auxiliary determinant for the evaluation of some determinants whose elements are zero and one.

13. Based upon the very high correlations found in testing freshmen in thirty-one Georgia colleges, Mr. Cox gave suggestions for future procedure in vocational guidance testing in mathematics.

14. Dr. Morrell reviewed the procedure for computing complex roots of a rational equation with real coefficients. Each root of the resulting eliminant was shown to be the arithmetic mean of a pair of roots of the original equation.

15. Dr. Hoyle discussed partial differential equations satisfying the general linear fractional transformation in n variables.

16. Associated with a scalar function of a scalar variable there is in general a polynomial in a square matrix having properties analogous to the scalar function. Dr. Bartky showed that these associated polynomials simplify the theory of systems of linear differential equations with constant coefficients, periodic coefficients, and coefficients involving singularities.

17. The problem of finding the conditions that a given form be divisible by a quadratic form was considered. Professor Stokes gave a particular case in which the quadratic involved four variables.

18. Through a double Laplace integral representation of the cotangent, tangent, cosecant and secant, Dr. Ballou obtained linear transformations of the four theta null functions.

19. Dr. Dye's transformation was generated by establishing a (1, 1) correspondence between the generators of a ruled quartic surface having a space cubic as a double curve and the surfaces of a pencil of general quadrics.

20. Professor Coleman gave the results of a study of the preparedness of freshmen entering University of South Carolina.

21. Professor Kokomoor told of the progress his university is making with certain new courses in freshman mathematics.

22. At every point of a plane curve there are ordinarily determined certain osculants such as the osculating conics. Another closely connected curve is Transon's parabola. Mr. Cameron pointed out some relations among the elements of these curves.

23. Professor Robinson derived formulas for the number of n-cells formed by any number of cutting elements where every cutting element cuts every other cutting element a given number of times.

24. Two triangles of maximum area inscribed in an ellipse were shown to be triply perspective. Dr. Clark gave some interesting cases in which certain points coincided.

H. A. ROBINSON, *Secretary*

THE THIRTEENTH ANNUAL MEETING OF THE NEBRASKA SECTION

The thirteenth annual meeting of the Nebraska Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Nebraska Academy of Sciences at the Nebraska Wesleyan University, Lincoln, Nebraska, Friday afternoon, May 8, 1936. Professor J. M. Howie of Nebraska Wesleyan University was chairman.

The attendance was twenty-nine including the following fourteen members of the Association: A. K. Bettinger, W. C. Brenke, C. C. Camp, A. L. Candy, J. A. Daum, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, J. M. Howie, R. M. McDill, Sigurd Mundhjeld, J. D. Novak, T. A. Pierce, Lulu L. Runge.

Officers for the ensuing year were elected as follows: Chairman, C. C. Camp, University of Nebraska; Secretary-Treasurer, T. A. Pierce, University of Nebraska; Member of Executive Committee, J. M. Earl, Omaha University. A committee consisting of the present and past chairmen and selected members was formed to stimulate interest in high school mathematics.

The following program was presented:

1. "The correspondence between ruled surfaces in R_3 and curves on a hyperquadric in R_5" by J. D. Novak, University of Nebraska.

2. "Approximate factors" by Professor H. C. Feemster, York College.

3. "A problem in approximate integration" by Professor C. C. Camp, University of Nebraska.

4. "Echoes from St. Louis" by Professor A. R. Congdon, University of Nebraska, introduced by the Secretary.

5. "The group of a knot" by Dr. Anna A. Stafford, University of Nebraska.

6. "The place of mathematics in a teachers college" by Professor C. A. Huck, Peru State Teachers College, introduced by the Secretary.

7. "Jacobi and Laguerre Polynomials with negative parameters" by Dr. M. S. Webster, University of Nebraska, introduced by the Secretary.

8. "A method for evaluating π" by Morris Dansky, Creighton University, introduced by Professor Bettinger.

Abstracts of some of the papers follow:

1. By means of Klein's correspondence between the lines of ordinary space and the points on a hyperquadric in a linear space of five dimensions, problems concerning ruled surfaces can be transformed into problems concerning curves on the hyperquadric. Mr. Novak demonstrated the use of this method in studying the flecnode sequence of ruled surfaces defined by a given ruled surface. Some special cases of the termination and of the cyclic possibilities of this sequence were presented.

2. In this paper Professor Feemster defined approximate factors, gave some examples illustrating the definitions, and stated a few theorems and proofs.

3. Professor Camp showed that the integral $\int_0^1(1+x^2)^{-1}dx$ may be used advantageously to calculate the value of π by employing Simpson's rule with the form of remainder series previously given by the author.

5. Dr. Stafford traced the history of the knot problem from its inception as a mathematical curiosity, through the work of Tait on vortex atoms, to the developments of Dehn and Alexander. She presented some of the elementary properties of free groups, as well as the connection between the knot problem and the topology of a three dimensional manifold.

6. A teachers college is a college from which students go out to teach in the public and private schools of the land. For a number of years complaints have been registered to the effect that mathematics has lost its grip on the students of today. This loss can be charged against the lack of inspired teachers. The question is not, "Where is mathematics?" so much as "Where are the mathematics teachers?" Professor Huck believes that mathematics should have a prominent place in every teachers college, should rank in importance with other major subjects, and that the mathematics people should put forth an organized effort to require a certain amount of training in mathematics for every teacher in the American public and private school system.

7. In this paper Dr. Webster gave several interesting relations involving Jacobi and Laguerre polynomials, some of which are new. He included a detailed

study of the number of the real zeros of these polynomials when the parameters are allowed to be negative.

8. In this paper Mr. Dansky showed that when a needle of length a is thrown at random upon a circle of radius a, the ratio of the number of throws when the needle lies completely within the circle to the number of throws when at least one end lies within the circle is a linear function of π. Thus, after a ratio has been obtained from a number of throws, one can derive an approximation for π by solving this equation.

T. A. PIERCE, *Secretary*

NOTE ON THE BEHAVIOR OF A POLYNOMIAL AT INFINITY

By J. L. WALSH, Harvard University

It is the object of the present note to establish the following theorem and corollary concerning the location in the complex plane of the roots of a polynomial:

THEOREM. *Let the polynomial*

$$p(z) \equiv (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

be given. For $|A|$ *sufficiently large, the roots* z *of the equation* $p(z) = A$ *lie in arbitrarily small neighborhoods of the roots of* $(z-\bar{\alpha})^n = A$, *where* $\bar{\alpha} = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n$ *is the center of gravity of the roots of* $p(z)$. *That is to say, let* $\epsilon > 0$ *be given; there exists* R *depending on* ϵ *such that* $|A| > R$ *implies that any root* z *of the equation* $p(z) = A$ *satisfies the inequality* $|z - (\bar{\alpha} + A^{1/n})| < \epsilon$, *where* $A^{1/n}$ *is a suitably chosen nth root of* A.

COROLLARY. *Let the polynomial*

$$p(z) \equiv (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

be given. When z *becomes infinite, a root* α *of the equation in* α

$$p(z) = (z - \alpha)^n \tag{1}$$

approaches the center of gravity $\bar{\alpha}$ *of the roots of* $p(z)$. *That is to say, let* $\epsilon > 0$ *be given; there exists* R_1 *depending on* ϵ *such that* $|z| > R_1$ *implies a root* α *of equation* (1) *in the circle* $|\alpha - \bar{\alpha}| < \epsilon$.

So far as the writer is aware, this specific theorem and corollary have never been rigorously formulated and proved. But certain consequences of the theorem have previously been expressed in a looser form, a form that is suggestive but fails to meet modern standards of rigor. Thus Lucas states* without proof that when A becomes infinite the roots of $p(z) = A$ approach the vertices of a regular polygon of n sides. The present theorem implies that for $|A|$ sufficiently large each root of $p(z) = A$ is within a distance ϵ of a vertex of a regular polygon

* Journal de l'École polytechnique, vol. 28 (1879), pp. 1–33.

of n sides inscribed in the circle whose center is $\bar{\alpha}$ and radius $|A|^{1/n}$. Lucas likewise states* without proof that when $|A|$ becomes infinite the locus $|p(z)|=|A|$ approaches a large circle whose center is the point $\bar{\alpha}$. The present theorem implies that for $|A|$ sufficiently large each point of the locus $|p(z)|=|A|$ lies within a distance ϵ of the circle $|z-\bar{\alpha}|=|A|^{1/n}$.

Similarly Lucas studies the curve† $\angle[p(z)]=\phi$, a constant, and proves that the curve has for asymptotes the n half-lines radiating from $\bar{\alpha}$ making with the horizontal respective angles $(\phi+2\pi k)/n$, $k=0, 1, 2, \cdots, n-1$. This result follows from the present theorem by setting $A=\rho e^{i\phi}$, ϕ constant, and allowing ρ to become infinite.

The modulus $|p(z)|$ becomes infinite when z becomes infinite and only when z becomes infinite. Consequently if R is given, the condition $|z|>R_1$ where R_1 is suitably chosen, implies the condition $|p(z)|>R$; conversely if R_1 is given, the condition $|p(z)|>R$ where R is suitably chosen, implies the condition $|z|>R_1$. The corollary is now a direct consequence of the theorem. Let ϵ be given, let R be the number whose existence is asserted by the theorem, and let $|z|>R_1$ imply $|p(z)|>R$. For an arbitrary value of z satisfying the inequality $|z|>R_1$ we set $A=p(z)$; the theorem states that the inequality

$$|z-(\bar{\alpha}+A^{1/n})|<\epsilon$$

is satisfied. Consequently the point $\alpha=z-A^{1/n}$ is a root of equation (1) and satisfies the inequality $|\alpha-\bar{\alpha}|<\epsilon$, as we were to prove. The reader may also notice the reciprocal, that the theorem can be readily established by means of the corollary.

In the proof of the theorem it is convenient to have two lemmas for reference:

LEMMA I. *For sufficiently small values of* ζ, *we have*

$$|\log(1-\zeta)+\zeta| \leqq q|\zeta|^2,$$

where a suitable branch of log $(1-\zeta)$ *is chosen, and where* q *is a constant independent of* ζ.

The branch of the logarithm chosen is defined by the equation

$$\begin{aligned} &\log(1-\zeta) = -\zeta-\tfrac{1}{2}\zeta^2-\tfrac{1}{3}\zeta^3-\cdots, \qquad |\zeta|<1,\\ &\log(1-\zeta)+\zeta = -\zeta^2[\tfrac{1}{2}+\tfrac{1}{3}\zeta+\tfrac{1}{4}\zeta^2+\cdots], \qquad |\zeta|<1. \end{aligned} \tag{2}$$

When we have $|\zeta|<r<1$, the square bracket in this right-hand member has a meaning and possesses a bound independent of ζ, so Lemma I follows.

LEMMA II. *For sufficiently small values of* β *we have*

$$|e^\beta-1| \leqq M|\beta|,$$

* Bull. de la soc. math. de France, vol. 17 (1888), pp. 17–69; compare also Walsh, this MONTHLY, vol. 42 (1935), pp. 1–17.

† The symbol $\angle[\omega]$ is used to denote the angle (argument, amplitude) of the complex quantity ω.

where M is a constant independent of β.

The proof is immediate, by the equations

$$e^{\beta} = 1 + \frac{\beta}{1!} + \frac{\beta^2}{2!} + \cdots ,$$

$$e^{\beta} - 1 = \beta\left[1 + \frac{\beta}{2!} + \frac{\beta^2}{3!} + \cdots \right];$$

the series in the square brackets converges for all values of β, uniformly for all β in modulus less than a given bound, so the sum of the series is bounded whenever $|\beta|$ is bounded.

In the proof of the theorem we assume that the point $\bar{\alpha} = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n$ coincides with the origin; this assumption involves no loss of generality, for such a substitution as $z' = z - \bar{\alpha}$ leaves the content of the theorem essentially unchanged. The equation $p(z) = A$ can be written formally as

$$\log \frac{A^{1/n}}{z} = \frac{1}{n}\left[\log\left(1 - \frac{\alpha_1}{z}\right) + \log\left(1 - \frac{\alpha_2}{z}\right) + \cdots + \log\left(1 - \frac{\alpha_n}{z}\right)\right]. \tag{3}$$

If the numbers $|\alpha_k/z|$ are less than unity, we can use in the right-hand member of (3) the branch of the logarithm defined by (2). Then the right-hand member of (3) has a meaning, and we can consider $A^{1/n}$ to be defined by (3) because we can take the exponential function of both members and solve uniquely for $A^{1/n}$.

From the choice $\bar{\alpha} = 0$, equation (3) can be written

$$\begin{aligned} \log \frac{A^{1/n}}{z} = \frac{1}{n}\bigg\{&\left[\log\left(1 - \frac{\alpha_1}{z}\right) + \frac{\alpha_1}{z}\right] + \left[\log\left(1 - \frac{\alpha_2}{z}\right) + \frac{\alpha_2}{z}\right] \\ &+ \cdots + \left[\log\left(1 - \frac{\alpha_n}{z}\right) + \frac{\alpha_n}{z}\right]\bigg\}. \end{aligned} \tag{4}$$

It follows from Lemma I that for $|z|$ sufficiently large and in any case greater than every $|\alpha_k|$, we have

$$\left|\log \frac{A^{1/n}}{z}\right| \leqq \frac{Q}{|z|^2},$$

where Q independent of z is suitably chosen, and hence it follows from Lemma II that we have

$$\left|\frac{A^{1/n}}{z} - 1\right| \leqq M\left|\log \frac{A^{1/n}}{z}\right| \leqq \frac{MQ}{|z|^2},$$

$$|z - A^{1/n}| \leqq \frac{MQ}{|z|}. \tag{5}$$

Let ϵ be given. There exists R_1 such that $|z| > R_1$ implies both inequality

(5) and the inequality $MQ/|z| < \epsilon$. There exists then a number R such that $|p(z)| > R$ implies $|z| > R_1$, which implies ($\bar{\alpha} = 0$)

$$|z - (\bar{\alpha} + A^{1/n})| < \epsilon,$$

and the theorem is completely proved.

It is worth remarking that the definition of $A^{1/n}$ by means of (3) requires only the inequalities $|\alpha_k/z| < 1$, so whenever z lies exterior to a circle C whose center is $\bar{\alpha}$ and which contains all the points α_k, the value of $A^{1/n}$ can be defined, and depends analytically on z as z remains exterior to C and becomes infinite.

NOTE ON AN INVERSION FORMULA

By E. T. BELL, California Institute of Technology

1. Dedekind's inversion formula in the theory of numbers may be generalized as follows. We first recall some definitions.

If $f(n)$ is single-valued and finite for integer values >0 of n, $f(n)$ is called a numerical function of n. The values of a numerical function may be any real or complex numbers. If $f(1) \neq 0$, the numerical function $f(n)$ is said to be regular. The unit numerical function $\eta(n)$ is defined by $\eta(1) = 1$, $\eta(m) = 0$ for all integers $m > 1$.

I showed elsewhere* that if, and only if, $f(n)$ is regular, there exists a numerical function $f'(n)$ such that

$$\sum f(d)f'(\delta) = \eta(n)$$

for all integers $n > 0$, where the sum extends to all pairs (d, δ) of integers d, $\delta > 0$ such that $d\delta = n$. The $f'(n)$ thus uniquely defined is called the reciprocal of $f(n)$. From this it follows at once that if $f(n)$, $g(n)$, $h(n)$ are numerical functions such that

$$\sum f(d)g(\delta) = h(n) \tag{1}$$

for all integers $n > 0$, then, provided $f(n)$ is regular,

$$g(n) = \sum f'(d)h(\delta), \tag{2}$$

where $f'(n)$ is the reciprocal of $f(n)$.

The special case of (1), (2) in which $f(n) \equiv u(n)$, where $u(n) = 1$ for all integers $n > 0$, is Dedekind's inversion; the reciprocal of $u(n)$ is Möbius' $\mu(n)$, $= 0$ if n is divisible by a square >1, $\mu(n) = 1$ or -1 if n is not divisible by a square >1, according as the number of prime divisors of n is even or odd. As (1), (2) can not be deduced from its special case in which $f(n) \equiv u(n)$, the inversion appears to be a true generalization of Dedekind's.

2. The following, although apparently a generalization of (1), (2), is in fact

* Tôhoku Mathematical Journal, vol. 17 (1920), p. 221. A simplified proof will appear in the same journal. The proof gives the explicit form of $f'(n)$.

a special case, as will be seen immediately. This may be worth noting, as a further specialization* has been called a generalization of Dedekind's inversion.

Let $\Phi(x)$, $\Phi^{-1}(x)$ be a pair of inverse functions, so that

$$\Phi[\Phi^{-1}(x)] = \Phi^{-1}[\Phi(x)] = x,$$

corresponding determinations of Φ, Φ^{-1} being taken (if either is many-valued). Let the range of values of the numerical functions $g(n)$, $h(n)$ be included in, or coincide with, that of x. Write

$$\Phi^{-1}[g(n)] \equiv g_1(n), \qquad \Phi^{-1}[h(n)] \equiv h_1(n),$$

for all integers $n>0$, and let $f(n)$, $f'(n)$ be as in (1), (2). Since $g_1(n)$, $h_1(n)$ are numerical functions, it follows from (1), (2) that if

$$\sum f(d)g_1(\delta) = h_1(n) \tag{3}$$

for all integers $n>0$, then

$$g_1(n) = \sum f'(d)h_1(\delta). \tag{4}$$

From (3) we have

$$\sum f(d)\Phi^{-1}[g(\delta)] = \Phi^{-1}[h(n)];$$

and from (4)

$$\Phi^{-1}[g(n)] = \sum f'(d)\Phi^{-1}[h(\delta)].$$

These are equivalent respectively to

$$\Phi\{\sum f(d)\Phi^{-1}[g(\delta)]\} = h(n), \tag{5}$$

$$g(n) = \Phi\{\sum f'(d)\Phi^{-1}[h(\delta)]\}. \tag{6}$$

Either of (5), (6) thus implies the other. Since (1), (2) imply (5), (6), the generalization in (5), (6) is only apparent. The special case in which $f(n)\equiv u(n)$, as in deducing Dedekind's inversion from (1), (2), is therefore implied by his inversion and is not a generalization of it.

NOTE ON A PAPER BY J. F. THOMSON

By H. E. BUCHANAN, Tulane University

Introduction. Under the caption of *A Type of Oscillation within the Helium Atom*,† Professor J. F. Thomson gives a very interesting discussion of such an atom when both the electrical and gravitational forces act. After Professor

* Due to E. M. Lémeray, Nouvelles Annales de Mathématiques, 4th series, vol. 1 (1901), pp. 163–167; noted in *Dickson's History of the Theory of Numbers*, vol. 1, p. 447. Lémeray's inversion is (5), (6) with $f(n)\equiv u(n)$.

† This MONTHLY, vol. 42 (1935), pp. 487–496.

Thomson's paper was in print, the author* and W. L. Duren, Jr. obtained some new results on the characteristic exponents in certain types of problems in mechanics. In this paper it was shown that Professor Thomson ought to have found that four of the characteristic exponents are 0, 0, and $\pm i\omega$, ω being the angular velocity. These four exponents result from the character of the four integrals of the system of differential equations. The numbers in the next paragraphs refer to Thompson's paper.

A Necessary Restriction. After the two values of ω have been found from equations (3) it is necessary to satisfy equations (4) and (5). Equations(4) are satisfied by $\eta_1=\eta_2=\eta_3=0$. If there is to be an isosceles triangle solution, equations (5) must be satisfied by $\xi_1=0$, $\xi_2=a/2$ and $\xi_3=-a/2$. The first equation of (5) is satisfied identically. The second and third impose the following conditions:

$$\frac{k^2\epsilon_1\epsilon_2+K^2m_1m_2}{2}a+\frac{-k^2\epsilon_3\epsilon_2+K^2m_3m_2}{a^2}=0$$

$$\frac{k^2\epsilon_3\epsilon_1+K^2m_3m_1}{2}a-\frac{k^2\epsilon_3\epsilon_2-K^2m_3m_2}{a^2}=0.$$

Solving for a, we find

$$a=\sqrt[3]{\frac{2(k^2\epsilon_2\epsilon_3-K^2m_2m_3)}{k^2\epsilon_1\epsilon_2+K^2m_1m_2}},$$

and

$$a=\sqrt[3]{\frac{2(k^2\epsilon_2\epsilon_3-K^2m_2m_3)}{k^2\epsilon_3\epsilon_1+K^2m_3m_1}}.$$

These two values of a are inconsistent unless $\epsilon_2=\epsilon_3$ and $m_2=m_3$. *Hence there does not exist an isosceles triangle solution, in which the triangle is symmetrical to the ξ-axis, unless the internal forces of the atom are also symmetrical and there is only one such triangular solution.*

When we let $2\epsilon=\epsilon_1=2\epsilon_2=2\epsilon_3$ and $m_2=m_3=m$, the two values of a become

$$a=\sqrt[3]{\frac{2(k^2\epsilon^2-K^2m^2)}{k^2\epsilon^2+K^2m_1m}}$$

and when this value of a is substituted in equation (7), ω reduces to zero. *Hence there is only one speed of rotation.* The quantity s which is introduced just before equation (11) reduces to a^3/m.

From this point on we are obliged to leave Professor Thomson's paper, since he introduces a slight approximation in equation (11) which we do not wish to make.

* The Duke Mathematical Journal, vol. 1 (1935), pp. 436–441.

The Characteristic Equation. The usual method of substituting $x_i = K_i e^{\lambda t}$, $y_i = L_i e^{\lambda t}$, $z_i = M_i e^{\lambda t}$ gives a characteristic equation of the twelfth degree in λ, the left member being a sixth order determinant:

$$\begin{vmatrix} -\lambda^2 + A_{11} & 0 & 2\omega\lambda & 0 & C_{11} & C_{13} \\ A_{21} & -\lambda^2 + A_{23} & 0 & 2\omega\lambda & C_{21} & C_{23} \\ -2\omega\lambda & 0 & -\lambda^2 & 0 & 0 & 0 \\ 0 & -2\omega\lambda & B_{41} & -\lambda^2 + B_{43} & 0 & 0 \\ A_{51} & A_{53} & 0 & 0 & C_{51} & 0 \\ A_{61} & A_{63} & 0 & 0 & C_{61} & C_{63} \end{vmatrix} = 0,$$

where the constants are defined in equations (10). It is obvious that $\lambda = 0$ is one root. After removing the factor λ, we multiply the third column by $2\omega/\lambda$ and subtract it from the first. Expansion by the minors of the third row shows another factor λ and reduces to a fifth order determinant. This fifth order determinant may be greatly simplified by letting

$$\omega^2 = \left(k^2\epsilon^2 + \frac{m_1 m}{2} K^2\right) u^2, \quad \lambda^2 = \left(k^2\epsilon^2 + \frac{m_1 m}{2} K^2\right) x^2, \quad u^2 = \frac{2(2m + m_1)}{m m_1},$$

dividing the columns by $k^2\epsilon^2 + m_1 m K^2/2$ and remembering that $a^3/m = s$. We have now

$$\begin{vmatrix} -x^2 - \left(1 + \frac{3a^2}{4}\right)u^2 & 0 & 0 & \frac{-3a(4-a^2)}{2m} & \frac{-3a(4-a^2)}{m_1} \\ -\left(\frac{8-3a^2}{2m}\right) + \frac{m_1}{m^2} & -x^2 + u^2 + \frac{6}{m} - \frac{3a^2}{2m} & 2u & \frac{-3a(4-a^2)}{2m} & \frac{3a(4-a^2)}{2m} \\ -\frac{m_1 u^3}{m} & -2ux^2 & -x^2 + u^2 & 0 & 0 \\ -\frac{3a}{2m} & \frac{-3a}{m_1} & 0 & -x^2 - \frac{4-3a^2}{4} u^2 & 0 \\ -\frac{3a}{2m} & \frac{3a}{2m} & 0 & \frac{4-3a^2}{2m} - \frac{2m_1}{m^2} & -x^2 - \frac{6}{m} + \frac{3a^2}{2m} \end{vmatrix} = 0.$$

It is easy to verify that $x = \pm iu$ satisfies this equation and consequently $\lambda = \pm i\omega$ are roots of the original equation. The remaining roots may be found as Professor Thomson showed by expanding as a power series in $1/m_1$.

REMARKS ON "UNSOLVABLE" PROBLEMS*

By A. J. KEMPNER, University of Colorado

This paper represents an effort to make clear to myself the nature of the mathematical and psychological difficulties involved in "unsolvability" problems. Two points have particularly attracted my attention:

(a) It seems to me that it is not frequently and emphatically enough pointed out that there are essentially different ways in which a problem may be "un-

* Presented to The Amer. Math. Society, Ann Arbor, September 1935.

solvable," and that this makes it unnecessarily difficult for those not specializing in "Grundlagenforschung," to follow the literature.

(b) Problems suggested by Brouwer and others suffer from the psychological handicap that one sees no mathematical reason why they should possibly be unsolvable.

I attempt to outline the domain which, in discussions on unsolvability, even the strict classicist must concede to the intuitionist as debatable.

The number $2^{\sqrt{2}}$: The German mathematician G. Hessenberg, in 1906, was one of the first to select some definite unsolved problem and to make remarks on the consequences of its assumed unsolvability on the structure of abstract mathematics. Hessenberg stated* that the proved unsolvability of this problem would confront mathematics with a situation more critical than any heretofore encountered.

He naturally chose a problem for which, at that time, no possible method of attack was known: the question whether $2^{\sqrt{2}}$ is an algebraic or a transcendental number. We shall refer to this as problem (I). Of course, this problem has in the mean time been solved, and $2^{\sqrt{2}}$, and a large class of such numbers, are now known to be transcendental.† (The Fermat problem $x^n+y^n=z^n$ has been only occasionally and half heartedly mentioned in this connection, obviously because there is in the minds of most mathematicians little doubt that the theory of algebraic numbers furnishes machinery which will, sooner or later, lead to a definite decision.)

The decimal expansion of π. A problem which has been frequently used, is Brouwer's‡ question: Are we sure that we can ever decide whether π contains a sequence of digits $012 \cdots 9$? This we refer to as Problem (II). It is unfortunate that in both (I) and (II) and, so far as I am aware, in all other problems which have been proposed as paradigmata for unsolvability, there seems no positive reason for the assumption that they are not solvable. They appear to be difficult problems, and we are—or were—too ignorant to know how to attack them; but there is nothing apparent in their nature to lead one to suspect that they will not finally yield.§

For (I), it may be guessed that Hessenberg had in mind the following: It

---

* G. Hessenberg: *Grundbegriffe der Mengenlehre, Neue Abbandlungen der Friesschen Schule,* 4. Heft, 1906. Also printed separately.

The question, $2^{\sqrt{2}}$ algebraic or transcendental, is the seventh problem of Hilbert's famous lecture: *Mathematische Probleme,* Vortrag Internationaler Mathemat. Congress, Paris, 1900—Göttinger Nachrichten. English: Bull. Am. Math. Soc., vol. 2 (1902), pp. 437–479.

† A. Gelfond, Bull. Acad. Sci. U.R.S.S., vol. 7 (1934). See also T. Schneider, Crelle's Journal, vol. 172 (1935), p. 65.

‡ L. E. J. Brouwer, *Über die Bedeutung des Principiium tertii exclusi in der Mathematik, besonders in der Funktionentheorie.* Crelle's Journal, vol. 154 (1925), p. 1.

§ Unsolvability (under Aristotelian logic) in the sense that both a theorem and its negative can be proved, occurs in an obvious fashion when our axiom system contains a logical contradiction. Conversely, unsolvability in this sense is reducible to inner contradictions in the axiom-system. This eventuality we do not consider.

may be inherently impossible to decide whether $2^{\sqrt{2}}$ is algebraic or transcendental, in the sense that neither one of these results can be derived as a consequence of our set of axioms and a certain (the classical) set of logical rules. There are simply not enough restrictions contained in our set of axioms to enforce the decision, and it is not at all primarily a question of whether it can be done in a finite number of steps, or requires an infinite number.

In cases of this type (the systematic recognition of which constitutes of course a fundamental problem) we shall have, in principle, nothing worse than the same difficulty which confronted mathematicians when it was shown that the Euclidean parallel postulate did not follow from the remaining set of axioms plus the laws of logic, and the remedy would again be the same: just as two new types of geometry were introduced alongside the Euclidean although they are incompatible with each other and with the Euclidean geometry, by admitting either the Euclidean or the Bolyai-Lobatschewski or the Riemann postulate, so, if we had been able to prove that *neither* of the statements "$2^{\sqrt{2}}$ is algebraic" or "$2^{\sqrt{2}}$ is transcendental," logically follows from the axioms of algebra and from the system of logic admitted, we should introduce each of these statements as a new axiom, and build up two mutually exclusive algebras. (The only surprising feature would be to find an axiom on so high a level; but it would conceivably be possible to reduce it to a simpler statement by applying the laws of logic to the enlarged system. However, the Zermelo axiom, if accepted, has familiarized us with the possibility of axioms far removed from intuitive simplicity.) The admission that neither of the two alternatives is a logical consequence of the set of axioms is equivalent to admitting that the set of axioms is not categorical, and has to be supplemented.

Such a situation is conceivable either under our system of classical logic, or, more easily, under the newer logic which forbids the use of the principle of the excluded third in infinite systems, or under other systems of logic. Under the classical logic, $2^{\sqrt{2}}$ is either algebraic or transcendental, but it is assumed that we cannot decide which of these two cases happens; under the new logic, the problem itself may become meaningless, and therefore have no solution. For this particular problem, the assumption that it is meaningless and may therefore be ignored in the systematic building up of mathematics, would seem possible, although painful. In other problems, it seems impossible to admit such an assumption, without wrecking our whole mathematical system.

For (II), we have a different situation. The familiar argument of Brouwer is that, however far we may have verified that the sequence $012 \cdots 9$ has not occurred in the decimal representation of π, we can never be certain that it will not some time occur and that an infinite number of steps would have to be carried out to arrive at a decision.

In this argument, the important assumption is silently made that this step by step examination of the digits is the only possible means of deciding the question; in other words, it is assumed that it is not possible by some "finitary" calculus to gain sufficient insight into the structure of the decimal expansion of π

to guarantee the existence of such a sequence, or, possibly, of an infinite number of them.

This is quite different from what we found in (I). It is now not possible to admit as a new axiom either that the sequence occurs or that it does not occur. If we assume that it does not occur, we may at any time discover that it does occur; if we assume that it does occur, we may never be able to verify this assumption. This case reminds one more of the situation we face in the theory of distribution of primes, in connection with the Riemann hypothesis concerning the complex zeros of the zeta-function, where we may derive important results under the assumption of its validity, although the hypothesis has so far withstood all attempts to prove it. Now, the results derived under its assumption may at any time have to be thrown out, and one sympathises with Landau's dictum (quoted from memory): "Ich mache es nicht wie manche, die an einen unübersteigbaren Zaun kommen, und auf der anderen Seite weitergehen." Although the special problem $2^{\sqrt{2}}$ is not well chosen (there was no particular reason for selecting this rather than any of hundreds of problems; there was nothing about it except the lack of success in finding methods to attack it, which prompted the selection), case (I) actually can occur, as we know from the parallel postulates, and it has been frequently used in building postulate systems.

The Problem (II) labours under the psychological handicap referred to. While it may at first strike one as reasonable to expect that this question could never be settled except by actually carrying out the expansion farther and farther and hoping to eventually reach the sequence $01 \cdots 9$ (which would in turn simply suggest modifying the question to: "Is there a finite bound for the number of such sequences?," and our misery starts all over again!), yet, it is at once clear that a sequence $01 \cdots 9$ is only a simple digit to base 10^{10}, and the question would certainly be settled if we knew: "In the expansion of π to any base g, all digits $0, 1, \cdots, g-1$ actually occur." Nobody will suspect this to be a simple problem; possibly nobody would at present know how to attack it; but there is also nothing about it to lead us to suspect any barrier but our own ignorance. One detects no indication of any particular features which should make the problem unsolvable.

Goldbach's Problem (III): It is of interest to have a problem of type (II), where there is some underlying reason for understanding why it may conceivably be unsolvable. After constructing some very artificial problems based on complicated systems of numeration, it seems to me at present that Goldbach's conjecture (every even integer = sum of at most two primes) offers a fairly satisfactory example.

If we write $2n = p_i + p_j$, and call $N(2n)$ the number of solutions, that is, the number of ways Goldbach's problem can be satisfied for the number $2n$, we find that, as far as has been computed, $N(2n)$ behaves irregularly, as do most number-theoretic functions, but increases on the whole with n. Felix Klein, in his lectures on the theory of numbers, used to speak of an "embarras de richesse," to indicate this situation.

Hardy and Littlewood,* by their remarkably powerful function-theoretic tools, were the first to make important advances toward the solution of this old problem. However, their methods did not even show whether a limited number, however large (say μ) of primes is sufficient to represent all even numbers.

In the meantime, extremely important results concerning Goldbach's problem have been derived. The present status is the following: *every "sufficiently large" number is the sum of at most* 71 *primes.*†

In many problems where we encounter an embarras de richesse, considerations of the following type may be applied.

It is thinkable, although not likely, that Goldbach's problem may be unsolvable for the following reason: Since the number of odd primes $\pi(n)$ is given by $n/\log n$ with a percentage error which approaches 0 as n becomes infinite, there are (for large n) more than $[n/\log n]^2$ sums of two primes $<2n$, and hence the average number of ways, $N(2n)$, in which a given number $2n$ can be represented as sum of two primes, becomes infinite with $2n$. However, we have no knowledge concerning $N(2n)$ to determine the amount of duplication or number of ways which may be absorbed by some numbers, possibly leaving nothing for others. It is obviously impossible to derive from such averages any information concerning the actual representation of all even numbers as a sum of two primes. (The *average* density of the number of ways of representing a number as the sum of two odd primes still grows infinite with n, even if we count both even and odd numbers, although not a single odd number can be so represented.)

However, it is quite conceivable that, after the elimination of the obviously impossible odd numbers, the possibility of the representation of the even numbers as a sum of two primes, if true, is a merely statistical fact in the sense that there is no general principle from which it follows for all even numbers. It may happen that the set of even numbers breaks up into classes, each one of which has to be treated by a separate method. This is a situation with which we are thoroughly familiar in mathematical proofs, provided we have only a finite number of classes. However, we may conceivably have an infinite number of classes requiring individual treatment. Even, as an extreme case, it might be necessary to prove it for each even number separately. (For any given number, the problem is verified or disproved by a finite number of steps). The problem would then obviously be unsolvable, in the following sense:

* Hardy and Littlewood: (Some problems of partitio numerorum). *Goldbach's problem*, Proceedings London Math. Proc. (2), vol. 22 (1923), pp. 46–56. As is known Hardy and Littlewood have proved that under certain important assumptions every sufficiently large *odd* number is the sum of at most *three* odd primes. An attack on Goldbach's problem from another direction is due to V. Brun, *Le Crible d'Eratosthène et le théorème de Goldbach*, Christ. Vidensk. Selsk. Skr., 1920, m. 3, who proved that every sufficiently large number = sum of two numbers each of which contains at most 9 prime factors.

We also refer to two addresses by A. Dresden, Bull. Am. Math. Society, vol. 30 (1924), pp. 31–40, and vol. 34 (1928), pp. 438–52, and one by E. R. Hedrick, Science, vol. 77 (1933), pp. 335–343.

† H. Heilbronn, E. Landau, P. Sherk, Čas. mat. fys., vol. 65 (1936).

If the theorem is true, it can never be proved. If the theorem is not true, we have several alternatives:

(a) this fact may never be established, since it may be covered by one of the cases which will never happen to be tested;

(b) it may happen, sooner or later, that the theorem does not hold for some numbers tested. There is a greatest number for which this is known to occur; the problem may then be modified to: *From some point on*, every even number = sum of two primes; and this takes the place of the original problem, with all the questions of unsolvability carried over. A quite similar discussion applies when we have an infinite number of classes (each containing a finite or an infinite number of even numbers), each requiring individual methods.

Under all circumstances, assuming the structure of the problem as indicated, we are confronted with an "unsolvable" problem. If it is preferred to have a problem asking for a definite number, we may assume that an existence proof may be derived for μ (above), and that the existence proof actually gives us a definite number, say 50, so that it is known that every even number is the sum of at most 50 primes. Then, if Goldbach's problem should be unsolvable, there must be one largest number k between 2 and 50 for which it cannot be proved that every even integer is the sum of k or fewer primes.

This example avoids, in a fairly natural manner, the psychological objections mentioned above.

We thus have so far two types of "unsolvability." In the first type (I) the consequent axiomatic treatment is completely indicated, and no philosophical questions are raised (except the fundamental one of recognizing when we are dealing with a situation for which the set of axioms is not categorical). In the second type, (II) and (III), nothing at all is really settled: the one alternative can never be proved; but it may at any time be decided by an example which shows that it does not hold. On the other hand, we never know whether such an example exists or not, until we have actually found one.

There is a marked lack of symmetry concerning the two alternatives in this last case. To enforce symmetry, we need an example (IV) in which an infinite number of steps is required to decide either alternative. Consider for this purpose $e = 2.71828 \cdots$. We ask whether a certain digit α, say 7, occurs a finite or an infinite number of times (infinite = larger than an arbitrarily assigned number). For the sake of argument we again have to assume that it is known that the only way of settling the question is to check digit by digit in the expansion of e; in particular, that the question cannot be decided by a finite number of steps—permitting the ordinary telescoping processes of analysis. (To allow an infinite number of individual steps, would admit the digit by digit checking.)

The average density would allow an infinite number of occurrences of each digit. If some digits should occur only a finite number of times, one may suspect some specific underlying reason which can be isolated by classical methods in a finite number of steps. However, if each digit occurs an infinite number of times, as is at first to be expected from probability considerations, it may conceivably

happen that it is really a statistical phenomenon, in the sense mentioned above, although any given digit is uniquely determined. (More likely of course is that it will be possible to prove a theorem such as: "Consider the first n digits; then the ratio of occurrences of each digit 0, 1, 2, $\cdots$ will be $1/10+\epsilon(n)$, where $\lim_{n\to\infty}\epsilon(n)=0$.")

We may say, the question is meaningless, since it can never be decided. This leads us far along the road to defeat, since it implies the elimination of large portions of our classical mathematics, as Brouwer and his school explain. Or we may attempt to admit either alternative as a new axiom. We should have at least two values of e, say e_1 with an infinite number of digits α, e_2 with only a finite number. Excluding $\alpha=0$, 9, to avoid situations involving $.999\cdots=1.000\cdots$, we shall have $e_2-e_1\neq 0$.

If we maintain from classical mathematics those portions which guarantee the existence and uniqueness of $e=1+1/1!+1/2!+\cdots$ (Archimedean axiom, existence of limits, etc.), then although neither one of our two assumptions can ever be proved or disproved, yet, one has to be accepted as true, one rejected as false, without our ever being able to decide which is true and which is false.

The situation differs now from the case of Euclidean and non-Euclidean geometries exactly in this point that our arithmetic axiom systems are built to guarantee uniqueness; a weakening of the axioms to secure a situation corresponding to the geometrical systems, admitting the possibility of more than one value of e, would hopelessly emasculate our arithmetic systems.

The classicist in mathematics will look for the solution of the dilemma in the direction of the attempts of Hilbert and his school to prove that such situations never arise, and that such a theorem can be proved by "finitary methods."

NECESSARY CONDITIONS FOR POLYNOMIAL SOLUTIONS OF CERTAIN RICCATI EQUATIONS*

By E. D. RAINVILLE, U.S. Bureau of Reclamation, Denver, Colorado

1. The differential equation

$$\frac{du}{dx} = B_0 + B_1u + B_2u^2, \tag{1.1}$$

in which the B's are functions of x, is known as the generalized Riccati equation.† By the transformation $u=y/B_2$ this becomes

$$\frac{dy}{dx} = A_0 + A_1y + y^2, \tag{1.2}$$

where $A_0=B_0B_2$ and $A_1=B_1+B_2'/B_2$. This latter is the form of the equation which we shall use in most of what follows. It is assumed throughout that A_0

* Presented, under the title *Notes on Riccati's Differential Equation*, to the Rocky Mountain Section of the Association, April, 14, 1933.

† See Forsyth: *A Treatise on Differential Equations*, 6th ed., pp. 191–194.

is not identically zero (a case of the Bernoulli equation), since the complete primitive could then be readily obtained.

Equation (1.2) has many interesting properties. First, it is the most general equation of the form $y'=f(x, y)$, with f rational in y, the solution of which has no movable singularities other than poles.* Second, any homogeneous and linear, ordinary differential equation of the second order can be transformed into a Riccati equation by means of the transformation $w=e^{\int y dx}$.

In solving either (1.1) or (1.2), it is most desirable to find a comparatively simple particular solution, usually referred to as an "inspection" solution. There are general theorems giving the complete integral of a Riccati equation whenever one, two or three particular solutions are known.† If A_0 and A_1 in (1.2) are polynomials is x, it is natural to search for one or more polynomial solutions.

In these notes we show that in general only one or two perfectly well defined polynomials need be tested as possible solutions. No polynomials other than these can be solutions.

2. *Definition.* By the symbol $[\sqrt{P(x)}]$, where $P(x)$ is a polynomial of even degree, we shall mean the polynomial part of the expansion of $\sqrt{P(x)}$ in a series of descending integral powers of x. For example:

$$[\sqrt{x^4 - 2x^3 + x - 6}] = x^2 - x - \tfrac{1}{2},$$
$$[\sqrt{x^6 + 8x^4 - x^3 + 7x - 2}] = x^3 + 4x - \tfrac{1}{2}.$$

With this notation we may more easily state

THEOREM I. *If in*

$$\frac{dy}{dx} = A_0 + y^2 \tag{2.1}$$

$A_0(x)$ is a polynomial of even degree, then no polynomial other than

$$y = \pm [\sqrt{-A_0}]$$

can be a solution of (2.1). *If the degree of A_0 is odd, there is no polynomial solution of* (2.1).

Proof. Let the degree of A_0 be even and write

$$S = [\sqrt{-A_0}];$$

further, define Q by the relation

$$-A_0 = S^2 + Q.$$

It is evident that the degree of Q is less than that of S. We may now write (2.1) in the form

* See Picard: *Traité d'Analyse*, Second Edition, vol. 2, pp. 373–374.

† The results are due to Euler and were later elaborated by Weyr and Picard. See Forsyth, loc. cit.

$$\frac{dy}{dx} = (y - S)(y + S) - Q. \tag{2.2}$$

Let y be a polynomial in x. Unless $y=S$ or $y=-S$, the degree of the right-hand member of (2.2) will be at least as great as the larger of the degrees of y and S. The degrees of y' and Q are each less than the larger of the degrees of y and S. It therefore follows that the equation cannot be satisfied by any polynomial other than $\pm S$. Next, let the degree of A_0 be odd. But, since the degree of y^2 is even and is greater than that of y' and since, in order that the equation be satisfied, the terms in A_0+y^2 of degree greater than that of y' must be identically zero, therefore the degrees of A_0 and y^2 must be the same which is impossible. Thus there is no polynomial solution of (2.1) when A_0 is a polynomial of odd degree. This completes the proof of the theorem.

Example 1.

$$y' = -x^2 + y^2$$

has no polynomial solution.

Example 2.

$$y' = -x^2 + 1 + y^2$$

has only the one polynomial solution $y=x$.

Now, by the transformation $y=u-A_1/2$, equation (1.2) is transformed into

$$u' = B_0 + u^2,$$

where

$$B_0 = -\tfrac{1}{4}\left\{A_1^2 - 4A_0 - 2\frac{dA_1}{dx}\right\}.$$

Consequently Theorem I is applicable and we may state

THEOREM II. *If in* (1.2) A_0 *and* A_1 *are polynomials in* x *and if the degree of*

$$\Delta = A_1^2 - 4A_0 - 2A_1'$$

is even, then no polynomial other than

$$y = -\tfrac{1}{2}(A_1 \pm T),$$

where

$$T = [\sqrt{\Delta}],$$

can be a solution of the equation. If the degree of Δ *is odd, there is no polynomial solution of* (1.2).

Example 3. Consider

$$y' = 2x^3 - 3x^2 - 4x + 6 + (x^3 - 2x + 4)y + y^2.$$

The trial polynomials are $y_1=-x^3+2x-2$ and $y_2=-2$. Of these y_1 is the only solution.

3. It is natural to enquire into the conditions under which both trial polynomials would satisfy the equation. By way of answering this question we state and prove

THEOREM III. *A necessary and sufficient condition that both trial polynomials of Theorem* II *be solutions of* (1.2) *is that* Δ *reduce to a constant.*

To prove necessity we assume that $y_1=-(A_1+T)/2$ and $y_2=-(A_1-T)/2$ both satisfy equation (1.2); and by substituting these into the equation, we get

$$T' = (T^2 - \Delta)/2 \tag{3.1}$$

and

$$- T' = (T^2 - \Delta)/2. \tag{3.2}$$

From this it follows at once that necessarily $T'=0$ and hence $T=k$. Since T is constant, so also is Δ, and the first part of the theorem follows.

Conversely, let Δ be constant. Then, retracing our steps, we see that equations (3.1) and (3.2) reduce to the same equation; hence, if y_1 satisfies (1.2), then so does y_2 and the sufficiency of the condition is proved.

We shall proceed in this case to obtain the general solution. Let the form of y be

$$y = - \tfrac{1}{2}(A_1 + k) + v^{-1}, \tag{3.3}$$

where v is a function to be determined so that (3.3) satisfies (1.2) with $T=k$. Substituting this value into the equation and reducing, we obtain $v'-kv+1=0$ as the condition which v must satisfy. Solving this linear differential equation for v we get

$$v = ce^{kx} + k^{-1}$$

where c is an arbitrary constant. Then the general solution of our equation takes on the form

$$y = - \tfrac{1}{2}(A_1 + k) + (ce^{kx} + k^{-1})^{-1}.$$

It is of interest to point out that for $c=0$ and $c=\infty$, the above reduces to the two polynomial solutions. If it should happen that $\Delta=0$, then the general solution is of the form

$$y = - \tfrac{1}{2}A_1 + 1/(c_1 - x).$$

4. Many forms, other than polynomials, for A_0 and A_1 suggest possible "inspection" solutions. Often the number of such solutions of given type which need be tried can be greatly restricted by conditions similar to those in these notes.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

FORMULAS FOR REDUCING A QUADRATIC FORM TO A SUM OF SQUARES

By TOMLINSON FORT, Lehigh University

In the following note a theorem of Sylvester on determinants and the elementary device of completing the square leads to simple explicit formulas for transforming a definite quadratic form to a sum of squares. Formulas for the coefficients in the resulting normal form are compact in expression and suitable for numerical computation. We also conclude from them a necessary and sufficient condition that a form be definite. Results are not new, but the elementary and direct method of the proofs and the importance of the subject seem to warrant publication.

1. Let

$$
{}_iD_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,i-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2\,i-1} & a_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,i-1} & a_{i-1\,k} \\ a_{j1} & a_{j2} & \cdots & a_{j\,i-1} & a_{jk} \end{vmatrix}
$$

where $i > 1$, $k \geqq i$ and $j \geqq i$ and let ${}_1D_{ij} = a_{ij}$.

The theorem of Sylvester* in question is as follows:

When $i \leqq n$

$$
[{}_nD_{nn}][{}_{i-1}D_{i-1\,i-1}^{n-i}] = \begin{vmatrix} {}_iD_{ii} & {}_iD_{i\,i+1} & \cdots & {}_iD_{i\,n} \\ {}_iD_{i+1\,i} & {}_iD_{i+1\,i+1} & \cdots & {}_iD_{i+1\,n} \\ \cdot & \cdot & \cdot & \cdot \\ {}_iD_{n\,i} & {}_iD_{n\,i+1} & \cdots & {}_iD_{n\,n} \end{vmatrix}.
$$

For a ready understanding of this theorem a proof is given for $n = 4$.

Consider a three-rowed determinant, $|a_{ij}|$. It is immediate that

$$
(1) \qquad a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}\,a_{22} - a_{12}\,a_{21} & a_{11}\,a_{23} - a_{13}\,a_{21} \\ a_{11}\,a_{32} - a_{12}\,a_{31} & a_{11}\,a_{33} - a_{13}\,a_{31} \end{vmatrix}
$$

$$
= \begin{vmatrix} {}_2D_{22} & {}_2D_{23} \\ {}_2D_{32} & {}_2D_{33} \end{vmatrix}.
$$

* See Kowalewski: *Determinantentheorie*, ed. 1909, pp. 83.

Now consider a four-rowed determinant, $|a_{ij}|$, where $a_{11} \neq 0$. We readily carry out the reduction

$$(2)\quad {}_4D_{44} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \frac{1}{a_{11}^3} \begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & {}_2D_{22} & {}_2D_{23} & {}_2D_{24} \\ a_{31} & {}_2D_{32} & {}_2D_{33} & {}_2D_{34} \\ a_{41} & {}_2D_{42} & {}_2D_{43} & {}_4D_{44} \end{vmatrix} = \frac{1}{a_{11}^2} \begin{vmatrix} {}_2D_{22} & {}_2D_{23} & {}_2D_{24} \\ {}_2D_{32} & {}_2D_{33} & {}_2D_{34} \\ {}_2D_{42} & {}_2D_{43} & {}_2D_{44} \end{vmatrix}$$

where

$${}_2D_{ij} = \begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}.$$

If we now apply (1) to the three-rowed determinant in the right-hand member of (2) and denote

$$\begin{vmatrix} a_{11} & a_{12} & a_{1j} \\ a_{21} & a_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix} \text{ by } {}_3D_{ij}, \text{ we have}$$

$$(3)\qquad {}_4D_{44} = \frac{1}{{}_2D_{22}} \begin{vmatrix} {}_3D_{33} & {}_3D_{34} \\ {}_3D_{43} & {}_3D_{44} \end{vmatrix}, \quad {}_2D_{22} \neq 0.$$

We can, however, write (2) and (3) in a form cleared of fractions without any assumptions relative to the non-vanishing of any element of the determinants involved. Observe that if $a_{11}=0$ the determinant in the right-hand member of (2) is zero having two columns proportional. A like situation exists in (3).

Formulas (2) and (3), when cleared of fractions, are what we wished to prove.

2. Consider the real non-singular definite form

$$(4)\qquad f(x_1, \cdots, x_n) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}x_ix_j,$$

where $a_{ij} = a_{ji}$.

THEOREM: *The transformation*

$$(5)\qquad \begin{aligned} X_1 &= {}_1D_{11}x_1 + {}_1D_{12}x_2 + \cdots + {}_1D_{1n}x_n \\ X_2 &= {}_2D_{22}x_2 + {}_2D_{23}x_3 + \cdots + {}_2D_{2n}x_n \\ X_3 &= {}_3D_{33}x_3 + {}_3D_{34}x_4 + \cdots + {}_3D_{3n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ X_{n-1} &= {}_{n-1}D_{n-1\ n-1}\, x_{n-1} + {}_{n-1}D_{n-1\ n}\, x_n \\ X_n &= x_n \end{aligned}$$

carries (1) *into*

$$
\begin{aligned}
(6)\qquad & \frac{1}{{}_1D_{11}} X_1^2 + \frac{1}{{}_1D_{11}\ {}_2D_{22}} X_2^2 + \frac{1}{{}_2D_{22}\ {}_3D_{33}} X_3^2 + \cdots \\
& + \frac{1}{{}_{n-2}D_{n-2\ n-2}\ {}_{n-1}D_{n-1\ n-1}} X_{n-1}^2 + \frac{{}_nD_{nn}}{{}_{n-1}D_{n-1\ n-1}} X_n^2 .
\end{aligned}
$$

The proof is carried through for $n=4$. This permits certain brevity in writing and it will be apparent that the method is quite general. However, a person who wishes to carry through a complete proof in detail will have no difficulty in doing so by mathematical induction.

We write

$$
\begin{aligned}
(7)\qquad f(x, y, z, t) = {} & a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + 2a_{14}xt \\
& + a_{22}y^2 + 2a_{23}yz + 2a_{24}yt \\
& + a_{33}z^2 + 2a_{34}zt \\
& + a_{44}t^2,
\end{aligned}
$$

and proceed formally.

Note that ${}_kD_{ij} = {}_kD_{ji}$ for all i, j, k. By completing the square

$$
\begin{aligned}
(8)\qquad f(x, y, z, t) = {} & \frac{1}{a_{11}} (a_{11}x + a_{12}y + a_{13}z + a_{14}t)^2 \\
& + \frac{1}{a_{11}} ({}_2D_{22}y^2 + 2{}_2D_{23}yz + 2{}_2D_{24}zt \\
& + {}_2D_{33}z^2 + 2{}_2D_{34}zt \\
& + {}_2D_{44}t^2).
\end{aligned}
$$

Replace the squared parenthesis by X^2 and complete the square in exactly the same way as previously on the form

$$
\begin{aligned}
& {}_2D_{22}y^2 + 2{}_2D_{23}yz + 2{}_2D_{24}yt \\
& + {}_2D_{33}z^2 + 2{}_2D_{34}zt \\
& + {}_2D_{44}t^2.
\end{aligned}
$$

We get

$$
\begin{aligned}
(9)\qquad f(x, y, z, t) = {} & \frac{1}{a_{11}} X^2 + \frac{1}{a_{11}\ {}_2D_{22}} ({}_2D_{22}y + {}_2D_{23}z + {}_2D_{24}t)^2 \\
& + \frac{1}{a_{11}} \frac{{}_2D_{22}\ {}_2D_{33} - {}_2D_{23}^2}{{}_2D_{22}} z^2 + \frac{2}{a_{11}} \frac{{}_2D_{22}\ {}_2D_{34} - {}_2D_{23}\ {}_2D_{24}}{{}_2D_{22}} zt \\
& + \frac{1}{a_{11}} \frac{{}_2D_{22}\ {}_2D_{44} - D_{24}^2}{{}_2D_{22}} t^2.
\end{aligned}
$$

Replace the expression in parentheses by Y and apply Sylvester's theorem. We have

$$f(x, y, z, t) = \frac{1}{a_{11}} X^2 + \frac{1}{a_{11}\,{}_2D_{22}} Y^2 + \frac{1}{{}_2D_{22}} ({}_3D_{33}z^2 + 2{}_3D_{34}zt + {}_3D_{44}t^2). \tag{10}$$

We can in turn write this

$$f(x, y, z, t) = \frac{1}{a_{11}} X^2 + \frac{1}{a_{11}\,{}_2D_{22}} Y^2 + \frac{1}{{}_2D_{22}\,{}_3D_{33}} ({}_3D_{33}z + {}_3D_{34}t)^2 + \frac{1}{{}_2D_{22}} \frac{{}_3D_{33}\,{}_3D_{44} - {}_3D_{34}^2}{{}_3D_{33}} t^2. \tag{11}$$

Replace the squared parenthesis by Z^2, t by T and apply Sylvester's theorem to the coefficient of T^2 and we have, after replacing a_{11} by ${}_1D_{11}$,

$$f(x, y, z, t) = \frac{1}{{}_1D_{11}} X^2 + \frac{1}{{}_1D_{11}\,{}_2D_{22}} Y^2 + \frac{1}{{}_2D_{22}\,{}_3D_{33}} Z^2 + \frac{{}_4D_{44}}{{}_3D_{33}} T^2,$$

which is the desired formula when $n=4$.

Since the form is non-singular definite it can not happen that any member of the sequence ${}_1D_{11}, \cdots, {}_4D_{44}$ is zero. If, for example, ${}_1D_{11} \neq 0$ but ${}_2D_{22}=0$ then if ${}_2D_{23}=0$ and ${}_2D_{24}=0$ the number of variables is reduced to three, namely X, z, t, and the form is singular. If, on the other hand, for example, ${}_1D_{11} \neq 0$ and ${}_2D_{22}=0$, but ${}_2D_{23} \neq 0$, let x, y, z and t be given such values that $X=0$, $t=0$, and y is numerically so much larger than z that $(1/a_{11}){}_2D_{23}\,yz$ dominates in sign what terms remain, then the form will be of the same sign as $(1/a_{11}){}_2D_{23}$ if y and z are of the same sign or of opposite sign to $(1/a_{11}){}_2D_{23}$ if y and z are of opposite sign and the form is not definite. Such choices of x, y, z and t are always possible since we can solve the equations

$$\begin{aligned} X = a_{11}x + a_{12}y + a_{13}z + a_{14}t &= 0 \\ t &= 0 \\ y &= b \\ z &= c \end{aligned}$$

for arbitrary values of b and c. We, consequently, are fully justified in all steps of our procedure and the proof of the theorem for $n=4$ is complete.

From formula (6) we immediately deduce the following theorem:

A sufficient condition that (4) *be non-singular positive definite is that the terms of the sequence* $1, {}_1D_{11}, {}_2D_{22}, \cdots, {}_nD_{nn}$ *be all of the same sign. A sufficient condition that* (1) *be non-singular negative definite is that the terms of this sequence alternate in sign.*

In fact we readily prove the following necessary and sufficient condition:

Given that (4) *is of rank m, in order for it to be positive definite it is necessary and sufficient that it be possible so to arrange its terms that the terms of the sequence*

$$1, {}_1D_{11}, {}_2D_{22}, \cdots, {}_mD_{m\,m} \tag{12}$$

be all of the same sign with ${}_iD_{ii} = 0$ *when* $i > m$; *for it to be negative definite, that the terms of* (12) *alternate in sign.*

Such an arrangement can be carried out as follows. Given (4), form the sequence ${}_1D_{11}, {}_2D_{22}, \cdots$. If ${}_iD_{ii}=0$ but previous members of the sequence are different from zero, then either $i=m+1$ or some ${}_iD_{jj}\neq 0$ where $j>i$. If this were not the case but some ${}_iD_{kp}\neq 0$, $k+p>2i$, the form is readily shown to be non-definite by reasoning already employed in a similar situation. Rearrange the order in the discriminant of (4) so that x_{jj} replaces x_{ii}. This does not affect the sequence (12) to this point. We proceed forward as formerly noting that from this point on the successive D's of (12) must be calculated from the rearranged discriminant.

On Integral Solutions of $A^2+B^2=C^2$

By William Sell, Ottawa, Ill.

Given

$$A^2 + B^2 = C^2 \tag{1}$$

or its equivalent

$$A^2 + [A + (B - A)]^2 = [A + (C - A)]^2,$$

consider

$$x^2 + [x + (B - A)]^2 = [x + (C - A)]^2. \tag{2}$$

The two roots of (2) are

$$A \text{ and } (-A - 2B + 2C).$$

We write

$$\begin{aligned} A' &= -A - 2B + 2C, \\ B' &= A' + (B - A), \\ C' &= A' + (C - A). \end{aligned}$$

Eliminating A' in the right members, we obtain

$$T_1: \begin{cases} A' = -A - 2B + 2C, \\ B' = -2A - B + 2C, \\ C' = -2A - 2B + 3C. \end{cases}$$

Evidently A', B', C' satisfy the relation

$$A'^2 + B'^2 = C'^2. \tag{3}$$

Each of the transformations

$$\begin{aligned} &T_2\colon \quad A' = B, \quad\ \ B' = A, \quad C' = C, \\ &T_3\colon \quad A' = -A, \ B' = B, \quad C' = C, \\ &T_4\colon \quad A' = A, \quad\ \ B' = B, \quad C' = -C, \end{aligned}$$

gives A', B', C' satisfying (3).

If A, B, C are integers, so too are the A', B', C' obtained from $T_i, i=1, 2, 3, 4$.

If A, B, C are relatively prime, so too are A', B', C'. Suppose, otherwise, that A', B', C' have the common factor k. Since each T_i is of order two, A, B, C are obtained from A', B', C' by the same transformation, and hence A, B, C have the common factor k, contrary to assumption.

Every set A', B', C' of relatively prime integers satisfying (3) can be obtained from the set 0, 1, 1 by means of T_1, T_2, T_3, T_4 and their products.

Let A', B', C' be such a set. By means of T_2, T_3, T_4 we obtain a set A, B, C such that $0<A<B<C$. Transform A, B, C by T_1 to obtain A_2, B_2, C_2, where $A_2=-A-2B+2C$.

Let $B=kA$, $k>1$. Then

$$C = A\sqrt{k^2+1}, \qquad A_2 = A(-1-2k+\sqrt{k^2+1}).$$

Since $k>1$,

$$\begin{aligned} 4 + 4k^2 &< 1 + 4k + 4k^2 \\ 2\sqrt{1+k^2} &< 1 + 2k; \end{aligned}$$

and since $A>0$, $A_2<0$. Moreover $B<C$, and hence

$$\begin{gathered} A + 2B - 2C < A, \\ -A_2 < A \quad \text{or} \quad |A_2| < A. \end{gathered}$$

A_2, B_2, C_2 may be transformed by T_2, T_3, T_4 into A_3, B_3, C_3, where $0<A_3<B_3<C_3$ and $A_3<A$. The process can be repeated on A_3, B_3, C_3 and ultimately will terminate in A_i, B_i, C_i where $|A_i|=1$, i.e. the set -1, 0, 1, from which can be obtained 0, 1, 1. By transforming -1, 0, 1 by the same transformations in reverse order, any set A', B', C' of relatively prime integers may be obtained.

While each T_i, $i=1, 2, 3, 4$, is of order two, the group generated by them is infinite. This follows from the preceding paragraph, for an unlimited number of

sets A', B', C', can be obtained from any one set A, B, C by means of T_1, T_2, T_3, T_4 and their products.

Note by the Editor. This very interesting result can be interpreted geometrically so that A, B, C are homogeneous coordinates in the plane. Then the integral solutions of $A^2+B^2=C^2$ correspond to points with integral (or rational) coordinates on the conic $A^2+B^2=C^2$; which according to the results of the present paper can all be obtained from any one of them by the operations of an infinite group of linear transformations. It is interesting to note that the transformations T_1, T_2, T_3, T_4 are all of the same type, each being a plane involution carrying the conic $A^2+B^2=C^2$ into itself. But with this interpretation they are no longer independent since it is readily seen that $T_4=(T_2T_3)^2$ so that the involutions T_1, T_2, T_3 alone constitute a fundamental set. Moreover it is clear that if Q be any linear transformation with rational coefficients that leaves the conic $A^2+B^2=C^2$ invariant then the set of three involutions

$$T_i^1 = Q^{-1}T_i Q \qquad i = 1, 2, 3$$

will also be a fundamental set of transformations which will generate the whole set of rational points on the conic from any one of them.

The involutions T_i are not a symmetric set. The center of T_1 is on the axis of T_2 and the center of T_2 is on the axis of T_1 and since these incidence relations are projective the same is true for any set T_i^1. It would be interesting to find a symmetric set of three involutions, generating all the rational points of a conic or else to prove its non-existence.

On Triangle Constructions

By A. J. KEMPNER, University of Colorado

One frequently encounters triangle constructions of a type different from the classical: Given of a triangle three data (lengths, angles, areas) sufficient to determine the triangle, to construct it.

Such problems are, in recent issues of the MONTHLY:

Problem 3668, March, 1934: Construct a triangle ABC, given the circumcenter, the foot of the altitude from A, and the point common to BC produced and the bisector of the exterior angle at A.

Problem E144, March, 1935: Show that a triangle must be equilateral if any pair of the following centers coincide: Incenter, Circumcenter, and Centroid.

In these, and in many other problems, points of the triangle are among the given data.

In the classical problems, three data are sufficient to determine the triangle (uniquely, or with a limited number of solutions), unless some relation exists between the three data. If points are admitted, the situation is not obviously clear, as the simplest cases show.

Assume given one side, a, and the centroid. If this is to mean that the position of the centroid is given relative to a, it is clear that these two data com-

pletely determine the triangle, so that the two are equivalent to three data in classical constructions. If it means that the position of the centroid in the plane and the length of a are given, the centroid has added nothing to our knowledge of size and shape of the triangle, so that the two are equivalent to *one* datum in classical constructions. On the other hand, the location of the centroid in the plane to a certain extent fixes the *position* of the triangle.

The problems may in certain cases appear equivalent to classical ones; for example, the three vertices of a triangle determine the triangle; or, compare problem 3668. But this similarity is only superficial, since the three vertices given as points in the plane, determine not only shape and size of the triangle, but also its position in the plane.

These obvious remarks lead to the following formulation:

When the given data consist only of lengths, angles, areas, but not of points, three (independent) data essentially determine, (i.e. except for a possible finite number of solutions) size and shape of the triangle; but no information is furnished concerning the position of the triangle.

As soon as points are contained among the given data, the problem is to determine shape, size and position in the plane, of the triangle. For this, six data are required (for instance, the six coordinates of the vertices). In counting the number of given data, lengths, angles, areas still will be assigned weights 1, but each given point must be assigned weight 2.

This enlarges in an obvious fashion the scope of elementary construction problems. Instead of giving one point, we may give one coordinate each of two points, such as the x-coordinate of A and the y-coordinate of the centroid. Or, instead of using a coordinate, we may specify that the point is to lie on an assigned curve in the plane; say, the vertex A on the line $x+y-1=0$, the centroid on the circle $x^2+y^2-4=0$, etc. Of course, this amounts to assigning a geometrical locus for a point. (Of this type is problem 3749, October, 1935: Construct, similar to a given triangle, a triangle whose vertices lie each on a corresponding curve in a set of three given curves in a plane. I confess that I am puzzled to know how to attack the problem in this generality by geometrical methods.) This immediately suggests the admission of dual elements of the following character: a certain line such as a side, or an altitude, has to pass through an assigned point. It is clear that such points will have to be assigned a weight 1.

We shall therefore have:

To determine a triangle in shape, size, and position, six data of weight one or their equivalent in total weight six are required. Here, the following are of weight one: any length; angle; area; any single coordinate or locus of a point; a point through which an assigned line is to pass. Of weight two is a given point of the figure.

If the triangle admits of no solution, the given data contradict each other; if it permits an infinite number of solutions, the data are not independent.

The position of the triangle, or, what amounts to the same, the location of the coordinate system in the plane, correctly absorbs three of the given data

as they are weighted (say, two for the origin of the system, one for the angle determining the direction of the x-axis).

Each degree of freedom in the solution absorbs one of the six data. For example, given one side (1) and the centroid (2), there are ∞^3 solutions.

In problem 3749, the six data are accounted for as follows: the three vertices to lie each on a given curve (3); given two angles (2); and one unit of weight is absorbed by the fact that there is generally a solution with any assigned point on one of the curves as a vertex of the triangle (1 degree of freedom, accounting for a weight 1).

For problem E144: 2 weights each for Incenter and Circumcenter; and, when the common point representing them is given in the plane, two degrees of freedom remain to account for the size of the equilateral triangle and for its orientation through rotation about this point.

If in a problem a side is given in length and in position, it absorbs four of the six data.

The extension of the classical problem indicated by this outline offers a large field for the selection of new problems and their systematic examination concerning solvability (lack of contradiction in the given data), constructability by ruler and compass constructions, etc., and determination of relations between the given data.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

The Revolution in Physics. By Ernst Zimmer. New York, Harcourt, Brace and Company, 1936. xiv+240 pages. $3.75.

This book presents in non-technical terms the historical development of the quantum theory. It begins with a brief summary of classical physics, roughly that prior to 1900 but including the theory of relativity. The author then discusses the beginnings of the quantum hypothesis by Planck and Einstein, the Bohr theory of the atom, the invention of matrix and wave mechanics and the probability interpretation of the wave equation with its related uncertainty principle. Then follows a briefer account of more recent discoveries, such as the London-Heitler theory of valence.

The concluding chapter on science and philosophy is an interesting and impartial exposition of the positivist and metaphysical points of view, together with a possible compromise attitude which probably represents that of most working scientists.

The author is to be commended on his selection of material, his comprehensive grasp of the subject, and the clarity of his exposition. The book proba-

bly contains as much detail of the quantum theory as can be appreciated by a reader with only rudimentary mathematical training. Unlike many recent books on physics and astronomy intended for popular consumption, there is no distortion of emphasis in order to shock the reader with paradoxical statements. To the technical reader, the book will prove useful as a supplement to the mathematical expositions which often contain all the relevant equations, but are lacking in the historical and experimental background.

A few notes have been added by H. Stafford Hatfield, who translated the book from German into English.

PHILIP FRANKLIN

Tables of the Higher Mathematical Functions. Computed and compiled under the direction of Harold T. Davis. Vol. II. Bloomington, Indiana, The Principia Press, Inc., 1935. xiii+391 pages. $6.50.

Volume I of this work was chiefly concerned with tables of the Gamma and Psi Functions, together with historical, explanatory, and biographical matter relating to these functions as well as to computation and interpolation in general. The second volume contains a supplementary bibliography, and a short list of errata for volume I.

The tables of the second volume fall into three main categories, namely polygamma functions, Bernoulli and Euler numbers, and polynomial approximation. The polygamma functions are the derivatives of the function psi x, or d log gamma x/dx. The first four derivatives of this function are given from -10 to 100. The tables give a minimum of ten places, and include the common logarithms for the first part of the range, and central differences for the latter part of the range.

The second and third sections deal respectively with the Bernoulli and Euler polynomials and numbers. The principal table in each section gives the first 250 numbers to nine places, and their logarithms to ten places. Certain sums of powers related to these numbers are also tabulated.

The fourth section is concerned with Gram polynomials. Tables for computing approximating polynomials of the first seven orders by the method of Gram, or finite sequences of orthogonal polynomials are given. The last section gives corresponding tables for polynomial approximation by the method of moments, for polynomials of the first three orders.

In each section the tables of functions are preceded by a discussion of the formulas used in their computation and application. Many of the methods of computation were especially devised by the director for this work and show considerable originality and ingenuity.

The tables themselves will undoubtedly prove a useful source of data for statisticians and biometricians. Mathematicians not actually engaged in computing will find much of interest in the explanatory pages.

PHILIP FRANKLIN

Back to Newton. By Georges Bothezat. New York, G. E. Stechert and Co., 1936. vii+152 pages. $2.50.

This is an attack on Einstein's theory of relativity. The author's chief weapons are a few dogmatic statements about the existence of absolute space and time, which meet none of the original objections of Einstein. The value of the author's argument may be judged from the fact that on p. 55 he "proves" that the points of a line segment form an enumerable set, and on p. 66 he "proves" a formulation of Euclid's postulate on parallels.

PHILIP FRANKLIN

Erreurs des Mathématiciens des Origines à nos Jours. By Maurice Lecat. Brussels and Louvain, Ancienne Librairie Castaigne, 1935. xii+136 leaves (printed on one side only)+pages 137–167.

Since Monsieur Lecat's *Leçons sur la Théorie des Déterminants à n Dimensions* appeared at Paris in 1910, he has published a number of volumes and pamphlets besides numerous articles in periodicals. His bibliographies of Relativity, of n-dimensional Determinants, of the Calculus of Variations, and of Fourier Series, are well known.

In the work before us, 476 errors are noted in writings of 336 authors from the fourth century B.C. to 1916 A.D., with corrections made during the years 1509–1934. Many of the authors referred to are quite unimportant in the history of mathematics; for example, L. Aubry, Saint Augustin, E. W. Davies, C. Gill, J. Gough, J. Hill, V. Spunar, S. P. V., and G. Stecz.

The general nature of the errors listed is set forth on pages 139–143. They are classified under the headings: 1, Algebra; 2, Analysis; 3, Theory of Numbers; 4, Astronomy; 5, Probabilities; 6, Calculus of Variations; 7, Geometry; 8, Mechanics; and 9, Optics. There are at least four times as many references under the heading no. 3 as under any other, and many of these are to highly trivial things such as incorrect listing of perfect numbers in unimportant works, and a long list of insignificant statements about Fermat's Last Theorem. "Errors" are listed even when authors themselves corrected the error before it was noticed by anyone else, so far as Lecat shows; e.g., Gauss (page 41), Fermat (page 34). References are frequently given to errors found by editors of collected works. Many errors are simply selected from the thousands to be found listed in Dickson's great work on the *History of the Theory of Numbers*. There does not seem to be any plan in making the selection of errors published. For example, only two of the various misstatements made by Mersenne with reference to his numbers are mentioned (page 78).

The material is displayed, mainly, according to authors arranged alphabetically (pages 1–115) with "Compléments" (pages 116–122). An attempt is made to give the exact dates and places of birth and death of each author, and also a few words as to his academic activities. Information is then found under the four heads (i) "Matière de l'erreur"; (ii) "Références bibliographiques";

(iii) "Découverte et nature de l'erreur"; (iv) "Références bibliographiques." An attempt is made to give exact references.

While page 39 is devoted entirely to Galois, and not in the form of the earlier items, the references are not to any error of Galois, but to errors of judgment or negligence of Cauchy, Fourier, and Poisson connected with manuscripts of Galois. Pages 123–136 are given over to a summary account of Chasles's gullibility where pseudo-historical manuscripts were in question. "Corrigenda" are listed on pages 137–138. On pages 144–147 there is a list of periodicals containing errors, and on pages 148–151, a list of the periodicals indicating the errors. An index-chronology of errors is given on pages 152–153, and a similar index (pages 154–155) for the chronology of corrections of errors. Then two more lists (pages 156–163) of authors of errors, and of discoverers of errors.

It is not evident that this latest work of M. Lecat can be of any general interest or service to mathematicians. Indeed it seems a futile kind of work. In the course of a few years every mathematician in the ordinary round of his research encounters far more errors than are listed in this volume. The published record of such a miscellany can be of real service for reference only if part of an important scheme, such as Dickson's monumental work, or Cunningham's collection of errors in factor tables.

R. C. ARCHIBALD

A Text Book of Algebra for Colleges and Engineering Schools. By W. H. H. Cowles and J. E. Thompson. New York, D. Van Nostrand Company, 1935, xi+402 pages. $2.25.

College teachers over the country are asking themselves—"What is to be done about algebra?" The entering students seem to know a little less each year concerning algebraic manipulation, and the demand for unified courses, for an early introduction to calculus, and in the case of business students, for topics in mathematics of finance and statistics, is increasing steadily—at least, for such as continue mathematics in college. Here is one proposed answer (perhaps correct, but not subscribed to by the reviewer). Complete a good course in trigonometry. Follow this by a 400-page course in algebra (style of 1870). Take an analogous course in the rudiments of analytical geometry. Then, if you survive, meet the derivative of a polynomial.

The authors state in their preface, "Although somewhat more thorough than the usual texts in those particular subjects which it covers, an effort has been made to relieve the teacher of the burden of lecture work by making the explanatory material quite full when necessary." A complete perusal of this rather bulky volume reveals how consistently the authors have endeavored to smooth the way, not by omitting mention of inevitable difficulties, but by pointing out the various cases that must arise, and handling each with explanation and numerous suitable illustrative exercises. Chapters on the slide rule, imaginary and complex numbers, binomial, exponential and logarithmic series, cubic, quartic and higher equations are included, while determinants, partial fractions, con-

tinued fractions, Horner's method, etc., are omitted. The book assumes familiarity with trigonometry—even where this is not needed. In many details such as the treatment of significant figures, the extensive explanation is to be commended for clarity, accuracy and psychological motivation.

Despite these qualities which will appeal to every slow-plodding conscientious student, and hence to the teachers responsible for their progress, the reviewer is forced to conclude that the book reflects but little credit upon the mathematical teaching profession.

Save for the discussion of the slide rule, significant figures, the mortality table and Graeffe's method, the book offers few items that have not been traditional (if not outmoded) for generations. This might be accepted in view of the announced purpose of this text. But the state of mind of the critical reader who delights in the logical beauty of mathematics can best be imagined when he meets repeatedly such statements as the following, (quoted verbatim, including the use of italics).

P. 2. "All the numbers of arithmetic and algebra may be divided into *positives* and *negatives*, which are opposite in "sense" or kind. These are designated respectively by plus $(+)$ and minus $(-)$ signs, thus $+1$, $+2$, $+3$, $+a$, $+n$, and so on; and -1, -2, -3, $-a$, $-n$. *The symbol* 0, *for zero, represents a number which is neither positive nor negative.*"

P. 3. "*If the product of two numbers is zero, one of them must be zero.* For if $a\times b=0$, either $a=0$, or $b=0$."

P. 3. "For if $a=a$, then $a-a=0$."

P. 6. "*Subtraction* is the converse of addition."

P. 9. "*Division* is the inverse of multiplication."

P. 21. "Therefore $a^0=1$."—and so on and on.

A. A. BENNETT

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

THE MATHEMATICS LABORATORY AT WESTMINSTER COLLEGE

By A. C. BURR, Westminster College, New Wilmington, Pa.

The Mathematics Laboratory at Westminster College was established last year and is now in its second year of operation. The purpose of this laboratory is to provide the student training in the solution of such problems as arise through the application of mathematics in other fields.

The laboratory itself is a large room set aside expressly for this purpose. Its furniture consists of one large tilting drawing table, two large flat top tables, sufficient chairs and stools, a set of three commodious storage cases and a blackboard running the entire length of one side. Twelve to fifteen students can be

accommodated at one time. The laboratory may be entered from the main hall of the building and is directly connected with a large class room assigned to the mathematics department.

The equipment available for student use consists of two electrically operated Munroe calculating machines, one with automatic multiplication and division, the other an older model, two small Burroughs adding machines, a set of drawing instruments, a large Browne and Sharp protractor, a good K. & E. planimeter, a Bausch and Lomb tangentmeter, a selection of French and ship curves, triangles, etc.

A selection of books, including Scarborough, "Numerical Mathematical Analysis"; Lipka, "Graphical and Mechanical Computation"; Whitaker and Robinson, "Calculus of Observation"; Running, "Empirical Formulas" and "Graphical Mathematics"; Lindow, "Numerische Infinitesimalrechnung"; Willers, "Methoden der praktischen Analysis"; Swett, "Alignment Charts"; and books on statistical methods are available for student use. Standard sets of tables, Barlow, Pierce, etc. are included. Several forms of log tables, ordinary, La Croix-Ragot, simplified, etc., are at hand.

The laboratory is used by two groups of students. The first group consists of those taking the non-major course in mathematics of statistics. This is a semester course taken principally by those students majoring in either business administration or secretarial science. It is preceded by at least one semester of college mathematics and usually followed by advanced work in business statistics. The purpose of the laboratory work is to train the students in the use of various statistical techniques and to solve problems too complicated for out-of-class assignment.

The second group consists of those junior and senior majors in mathematics taking the course in mathematics laboratory. This course is designed to give the student insight into and training in the mathematical techniques available for analyzing data. Equal emphasis is placed upon graphical and numerical methods. Topics dealt with in any one semester vary from term to term but are selected from the following list: slide rule, plotting of data, smoothing of data, empirical equations, interpolation, graphical and numerical differentiation and integration of functions represented only by tabulated data, graphical and numerical solution of differential equations, alignment and other charts.

In the laboratory work particular attention is paid to comparative studies of the precision, accuracy, speed and convenience of various methods available for the solution of a given problem. Special study is now being given to the problems associated with the differentiation of a function defined by tabulated data and to the problems associated with the use of logarithmic tables.

Frequently the data used in problems has been obtained by either members or non-members of the group in other courses such as physical chemistry or advanced physics. This cooperation with advanced undergraduate work in other departments creates a very desirable academic situation and adds life and interest to the problems used.

Dr. Harold Black, head of the department of mathematics, conducts one course and Dean Burr the other. It is felt that the laboratory adds greatly to the effective work of the mathematics department and contributes an essential type of training and point of view frequently absent from the usual undergraduate curriculum in mathematics.

Undergraduate Publications

Among the numerous bulletins issued by undergraduate clubs, two have devoted themselves to the publication of papers on mathematical subjects, written by undergraduates. The first of these to be established is "X," sponsored by the Mathematics Club of Washington Square College. For six years this bulletin has appeared regularly, a typical fifty-four page issue containing half a dozen expository articles, the subjects ranging from determinants to finite geometry and nomography. The papers published are, for the most part, papers that have been read before the club. The publication is offered for sale to the students and has found considerable support.

This year appeared for the first time the "*Bulletin of the Yale Undergraduate Club*." This trim little magazine opens its columns with the following paragraph worthy of the consideration of our clubs: "The purpose of this magazine may be best expressed by a sincere use of the time-worn phrase 'to stimulate interest and original work in mathematics.' It is not to be expected that undergraduates will make significant contributions to the science of mathematics; but if some students will be induced to do independent thinking—thinking based on their own ideas or on an extension of subjects covered in class-work—this magazine will serve a purpose."

The progress of both these magazines will be watched with interest and may induce some of our clubs which are already issuing commendable monthly or annual bulletins to expand their papers.

Club Reports

Since the number of our contributing clubs has increased so very satisfactorily, the presentation of programs must assume a more abbreviated form. We hope, as far as possible, to present all subjects discussed and to give all activities which have helped to make the clubs interesting to their members. Unless otherwise specified, all programs were presented during the academic year 1935–1936 and all officers were for that year.

Mathematics Club of Bowdoin College

President, J. F. Presnell, Jr.; Vice President, C. J. Hawkes; Secretary-Treasurer, E. E. Morse. Student and faculty members of the club discussed Mathematics in anti-aircraft artillery; Apollonius and his work on conics; Space, time and space-time; The mathematical game "Nim"; Probability. The club had as guest speaker, Dr. Arthur Haas, professor of theoretical physics at the University of Vienna. Dr. Haas discussed several cosmological problems.

Kappa Mu Epsilon, Iowa State Teachers College

Club meetings were held during the summer sessions for the first time. A feature of two initia-

tions was the discussion by the initiates of assigned problems. A homecoming breakfast brought back ten former members. Some of the topics discussed at meeting were: John Napier and his invention of logarithms; Four region planes; Integration of the probability function; Fredhom's equation. Special attention was given to the solution of problems appearing in the MONTHLY

Van Vleck Club, Wesleyan University, Middletown, Connecticut

President, E. Hammerstrom; Vice President, W. A. Wells; Secretary-Treasurer, C. H. Knapp. Subjects discussed included: Demonstration of the Hammond electric organ; Theory of the integraph; The problem of the bank teller, explained by the method of positive integers; Problem of finding buried treasure; Application of DeMoivre's theorem; Two marbles in a cone; Orthogonal properties of functions of a complex variable; Actuarial examination problems; Envelope of reflected rays from the inner surface of a napkin ring; History and development of mathematical machines; Algebraic solution of cubic equations.

The year closed with an outing at Professor B. H. Camp's cottage on the Sound.

Harvard Mathematical Club, Harvard University

President, C. Herpel; Secretary-Treasurer, J. C. Oxtoby; Faculty Adviser, Professor W. C. Graustein.

Thirteen meetings were held, at one of which the club entertained eighty guests when G. E. Roosevelt of New York City discussed Progress in Navigation. Two other meetings had guest speakers. Professor R. E. Langer spoke on "The life of Sir Isaac Newton," and Harvey Brooks of the Yale Mathematics Club discussed "Problems in theoretical meteorology." All other speakers were student or faculty members of the club. Topics discussed were: Some European mathematical centers; Linear symmetric transformations in Hilbert space; Infinite Abelian groups; Recent developments in interpolation; Taylor's theorem and absolutely monotonic functions; History of three body problem; The zeros of analytic functions; The dimensionality of Euclidean n-space; Some topics in sampling; Life and work of Maxime Bôcher.

The winners of the Robert Fletcher Rogers prizes were H. E. Robbins and Z. I. Mosesson.

Denison Mathematics Club, Denison University

Five classes in advanced mathematics each furnished a program for the club; an annual banquet and initiation of new members occupied two more; and the solution of mathematical problems and puzzles filled two meetings. Yet this club found time to listen to discussions of the following topics: Leibniz's theorem on derivatives; Mathematical induction; Inversion; Calculus of variations; Gamma and Beta functions.

Nicholson Mathematics Club, Louisiana State University

The topics discussed were: Mathematics in early days of L. S. U.; The National Mathematics Magazine; The early mathematicians of America; The magnitude of the projection of an angle on a plane; Osgood and his curve; Life and work of E. H. Moore; Proof of theorem concerning possibility of differentiating under integral sign; The Gauss total curvature of a surface from several view points; Connection between the probability curve $ae^{-k^2t^2}$ and the expansion of the binomial $(p+q)^n$ expressed as combinations ${}_nC_0p^n+{}_nC_1p^{n-1}q+\cdots$; A general theory of roulettes given in parametric form.

Pi Mu Epsilon, St. Lawrence University

The meetings of Pi Mu Epsilon are held with the Mathematics Club, though each club holds separate business meetings. During the past year the following papers were presented: Puzzles; New numbers; Punch card technique; Mathematical approach to gambling; History of mathematics; Math prodigies; Impossible geometrical constructions; Mathematics and the universe.

The officers were: President, Doris Berry; Secretary-Treasurer, Ruth Husing.

Mathematics Club, Yakima Junior College

The topics presented at the eight meetings were: The complete solution of the Diophantine equation, $x^2+y^2=z^2$; Trigonometric solution of the cubic equation using hyperbolic functions; Non-differentiable functions; Algebraic invariants and the problem of relativity; Problems involving probability; Magic squares; Problems of celestial mechanics; Permutations and combinations.

Pi Mu Epsilon, University of Nebraska

The subjects discussed at the monthly meetings were: Experimental determination of π; Application of the hyperboloid in industry; Demonstration of the electric calculating machine; Demonstration of the slide rule; Weather forecasting; Application of mathematics to discharge of electricity through gases; Time telling through the ages.

The fraternity sponsors an annual contest. The winners this year were: R. Schucklebier in Calculus and C. Eberline in Analytics. The year closed with an initiation picnic.

Pi Mu Epsilon, New York University

The society held three business and social meetings during the year. Professor Richard Courant spoke at the December meeting on "The rise of European universities." Professor William West of the Chemistry Department spoke at the March meeting on "Some ideas in quantum mechanics." Mr. Max Shiffman spoke at the April meeting on "The theory of surface areas." These talks were followed by regular meetings after which refreshments were served.

The Third Annual Pi Mu Epsilon Interscholastic Mathematics Contest was held on April 25, 1936. There were 115 high schools entered, 92 of which sent full teams of 4 students each. The highest score was made by the team from Boys High School of Brooklyn, N. Y., which was awarded a large cup. The team placing second was from Stuyvesant High School of New York, and team placing third was from Abraham Lincoln High School of Brooklyn. There were four smaller cups awarded to the teams making the highest score in their sections.

The members of Pi Mu Epsilon also assisted in the preparation and demonstration of a mathematical exhibit which was shown after the contest. The purpose of this exhibit was to make clear and attractive the principles of mathematics. This was done by means of charts and apparatus which were designed for that purpose.

The annual dinner was held on May 16, 1936, at the Holley Hotel. The guest of honor was Dr. R. S. Britton who gave a talk on "Chinese mathematics." Sixteen newly elected members were initiated at this time.

Mathematics Club, Hunter College of the City of New York

At the meetings of the Mathematics Club of Hunter College during year 1935–1936 the following topics were discussed: Shots and shells; Monte Carlo roulette; The school-girl problem; Dice playing; The dog biscuit problem; Finite geometry; Analysis situs; π and probability.

Two of the meetings were enlivened by screen projections of photographs of the staff and students when very young, a prize being awarded to the member who was able to identify the greatest number.

The big event of the season was one which was inaugurated last Spring and which has proved so successful that it will probably become a regular custom. On May 16th, some four hundred Mathematics students were present at the Mathematics Department Party held in the Student Hall of the Bronx Buildings and sponsored by the combined Mathematics Clubs and the Staff. The entertainment at this party was presented in the form of an "Amateur Hour" and our young mathematicians may well be proud of their talents in other directions.

The officers for the year were: President, Jessie Schroeder; Vice President, Mildred Wurman; Treasurer, Ruth Koch; Publicity Manager, Rose Shor.

Pi Mu Epsilon, Hunter College of the City of New York

The society had another successful year under the direction of Professor Jewell Hughes Bushey.

There were twenty new members initiated in the Fall and Spring terms bringing the membership up to forty-one, exclusive of faculty members. The talks for the year were devoted to mathematical economics. Following is a list of the topics presented by student members at the eight program meetings: Monopoly competition; Taxation, Edgeworth's taxation phenomenon; Indifference curves, Law of demand subject to budget restrictions, Income taxes; Economics of exhaustible resources; Calculus of variations and mathematical economics; A general theory of depreciation; Application of mathematics to psychology, Factor analysis; and Postulational development of the theory of probability.

In accordance with the usual custom, a prize was awarded to the best speaker of each semester. Rose Shor and Claudia Cohen each received a copy of *Statistical Methods for Research Workers* by R. A. Fisher.

There were three main social events during the year. The first was the fall initiation dinner at which Professor Harold Hotelling of Columbia University was the guest speaker. His topic was "Some little known applications of mathematics." The second was the spring initiation tea which was held at the Panhellenic Building. The third was the usual spring reunion, which probably aroused the most interest because of its novelty and its appeal to our alumnae members. This was held in the new quarters of the College at Two Park Avenue, our near approach to a skyscraper university. The members were conducted on a tour of the floors occupied by the College, and all enjoyed the lovely sight of the lights of Manhattan from that high place.

The officers for the year were: Vice-Director, Esther Harris; Treasurer, Hazel Purdy; Corresponding Secretary, Lillian Borer; Recording Secretary, Rose Shor.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

E 232. *Proposed by V. Thébault, Le Mans, France.*

Prove that no perfect square can be written in the scale of ten with just five digits which are distinct, but congruent modulo two.

E 233. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

If P is a point in the plane of the triangle ABC, and if circles are circumscribed about the triangles PAB, PBC and PCA, prove that the tangents to these circles at P meet the respective sides AB, BC and CA produced, in three collinear points whose common line is perpendicular to PO, where O is the center of the circle circumscribed about triangle ABC.

E 234. *Proposed by Leon Recht, College of the City of New York.*

Ten years from now Tim will be twice as old as Jane was when Mary was

nine times as old as Tim. Eight years ago Mary was half as old as Jane will be when Jane is one year older than Tim will be at the time when Mary will be five times as old as Tim will be two years from now. When Tim was one year old, Mary was three years older than Tim will be when Jane is three times as old as Mary was six years before the time when Jane was half as old as Tim will be when Mary will be ten years older than Mary was when Jane was one-third as old as Tim will be when Mary will be three times as old as she was when Jane was born. How old are they now?

E 235. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Prove that the following system of equations is consistent, and solve it.

$$(ad + be)x + (ae + bf)y + (af + bd) = 0$$
$$(bd + ce)x + (be + cf)y + (bf + cd) = 0$$
$$(cd + ae)x + (ce + af)y + (cf + ad) = 0.$$

E 236. *Proposed by J. M. Feld, New York City.*

On the sides of triangle $A_1B_1C_1$ points A_2, B_2, and C_2 are chosen so that $A_1B_1 = nA_1A_2$, $B_1C_1 = nB_1B_2$, $C_1A_1 = nC_1C_2$. On the sides of triangle $A_2B_2C_2$ points A_3, B_3 and C_3 are chosen so that $A_2B_2 = nA_2A_3$, $B_2C_2 = nB_2B_3$ and $C_2A_2 = nC_2C_3$. This process is repeated indefinitely. Show that the vertices of the sequence of triangles approach the centroid of $A_1B_1C_1$ as a limit.

E 237. *Proposed by C. W. Trigg, Cumnock College, Los Angeles.*

Prove that there exists just one six-place, palindromic square, and determine its value (Palindromic means reading backwards and forwards alike.)

SOLUTIONS

E 192 [1936, 103]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A merchant marked his merchandise in code, replacing each real digit by a preassigned code digit. His new bookkeeper made out the following bill, *every* digit in which is thus coded.

4 Gimcracks at 81¢	\$7.85
2 Baubles at 70¢	.37
6 Gee Gaws at 9¢	.16
	\$8.43

What were the correct figures in this bill?

Solution by C. W. Trigg, Cumnock College, Los Angeles

(a) From the multiplications it is evident that no one of "0", "1", "2", "3", "4", "6" or "9" can represent 0 or 1. Nor can "7" or "8" stand for 0. Therefore "5" represents 0.

(b) From the addition, "7" is less than "8", so that "7" represents 1, and "8" represents 2 or 3.

(c) "4"×"8" "1" = 1 "8" 0, so either "4" or "1" represents 5, and the other is even.

(d) "6"×"9" = "1""6", so "9" represents 6, "6" is even, and "1" is less than 5. Therefore "4" represents 5.

(e) By (b) and (c), 5×"8""1" = 1 "8" 0, whence "8" represents 2 and "1" represents 4.

(f) By (d), "6" is even, so it must represent 8.

(g) It now follows from the addition that "3" represents 9.

(h) Finally the product "2"×1 "0" = 91 establishes "2" as 7 and "0" as representing 3.

The unique decoding of the bill is then

5	Gimcracks	at 24¢ =	\$1.20
7	Baubles	at 13¢ =	.91
8	Gee Gaws	at 6¢ =	.48
			\$2.59

Simon Vatriquant points out that the problem would still have had a unique solution if the total of the bill had not been given at all.

Also solved by J. W. Boyce, W. E. Buker, E. H. Carlson, Mary L. Constable, Fred Discepoli, Wm. Douglas, Russell Gettinger, L. M. Kelly, Leon Recht, D. P. Richardson, J. E. Trevor, M. J. Turner and the proposer.

E 193 [1936, 104]. *Proposed by A. Gloden, l'Athéné de Luxembourg.*

Find a perfect square of six digits, such that if we consider the numbers formed by the successive pairs of its digits, the second of these three two-place numbers exceeds by unity the sum of the other two.

Solution by Simon Vatriquant, Brussels, Belgium

Let $abcdef = x^2$. Since $0 < a$, therefore $316 < x$. This may be written $10{,}000ab + 100cd + ef = x^2$, with the condition that $cd = ab + ef + 1$. The elimination of cd between these last two equations gives

$$101abef = (x + 10)(x - 10).$$

Since 101 is a prime, it must divide one of the two right-hand factors. Hence the only possible values for x are 394, 414, 495, 515, 596, 616, 697, 717, 798, 818, 899 and 919. Actual trial with these values eliminates 414, 717 and 919, so that there remain the following nine values for x^2: 155,236; 245,025; 265,225; 355,216; 379,456; 485,809; 636,804; 669,124; and 808,201.

Editorial note. Several solutions were received for this problem showing six-place squares beginning with one or more cyphers, such as 000,100. There seems to be no more justification for this than to claim that every linear equation is a quadratic equation in which the coefficient of x^2 is zero.

Also solved by W. E. Buker, Mary L. Constable, E. P. Starke, W. R. Talbot, C. W. Trigg and the proposer.

E 194 [1936, 104]. *Proposed by Abe Gelbart, Paterson, New Jersey.*

Of all the lines through vertex A of triangle ABC, there is just one which meets BC in the point D such that the incircles of triangles ABD and ACD are equal. Show that this line AD may be constructed by ruler and compasses.

Solution by H. Tate, McGill University, Montreal, Canada.

Let the angle ADC be θ, so that the angles ADB, BAD and DAC are $\pi-\theta$, $\theta-B$ and $\pi-\theta-C$ respectively. Using the well-known formula, $r=a\sin(B/2)\sin(C/2)\sec(A/2)$ for the radius of the incircle of a triangle ABC, we apply it to each of the triangles ABD and ADC and equate the results, obtaining

$$c\sin\frac{B}{2}\sin\frac{\theta-B}{2}\sec\frac{\pi-\theta}{2}=b\sin\frac{C}{2}\sin\frac{\pi-\theta-C}{2}\sec\frac{\theta}{2}.$$

But $c=2R\sin C=4R\sin C/2\cos C/2$, with a similar equation for b. When these values are substituted in the above equation, there results after cancellation and reduction of functions of complementary angles:

$$\cos\frac{C}{2}\sin\frac{\theta-B}{2}\csc\frac{\theta}{2}=\cos\frac{B}{2}\cos\frac{\theta+C}{2}\sec\frac{\theta}{2}.$$

Expansion of the second factor on each side, and further cancellation and reduction leads to the equation

$$\tan^2(\theta/2)=\tan(B/2)\cot(C/2).$$

This equation can give but one acute value for $\theta/2$.

If now in this equation we replace $\tan(B/2)$ and $\cot(C/2)$ by their values in terms of the sides of triangle ABC, we find that

$$\cos\theta=\frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}=\frac{c-b}{a}.$$

Since the angle θ is now readily constructed by ruler and compasses, the problem is solved.

Also solved by Wm. Douglas, L. M. Kelly, E. P. Starke and C. W. Trigg.

E 195 [1936, 104]. *Proposed by V. Thébault, Le Mans, France.*

Show that there is just one number whose square, expressed in the scale of six, contains each of the six different digits just once.

Solution by J. Rosenbaum, Hartford Federal College.

It is easily seen that the rule for divisibility by 9 in the decimal system holds correspondingly in every system whose base is an integer greater than one. Hence, since the sum of 0, 1, 2, 3, 4 and 5 is divisible by 5, the square of the number in question is divisible by 5, and hence the number itself is divisible by 5. Then the number in question is a three-digit number (in the scale of six), the sum of whose digits is either 5 or 10 or 15. Furthermore, since the square

does not end in two zeros, the number itself does not end in a zero. Also, since the square root of 6, written in the scale of six, is $2.24 \cdots$, the smallest three-digit number whose square has six digits is 225. With these considerations, the number of eligible numbers reduces to twenty-one. Squaring each of these numbers in the scale of six, it appears that, in the scale of six, $523^2=452013$, and no other solution exists.

Also solved by W. E. Buker, Mary L. Constable, Fred Discepoli, Daniel Finkel, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 196 [1936, 104]. *Proposed by J. Rosenbaum, Hartford Federal College.*

The construction by ruler and compasses of the edge of a cube, whose volume equals the sum of the volumes of two cubes of arbitrary edges, is known to be impossible. However, the construction is possible in particular cases, as for instance when the two given cubes have edges proportional to $9+\sqrt{5}$ and $9-\sqrt{5}$. Derive a general formula for determining such constructible cases.

Solution by E. P. Starke, Rutgers University.

Let a and b be the edges of the given cubes, with $b<a$. Let $a^3+b^3=c^3$. It will appear that the necessary and sufficient condition that a cube of edge c can be constructed is that the ratio $a:b$ be expressible in the form

$$\frac{\sqrt{6m^3}+\sqrt{1-2m^3}}{\sqrt{6m^3}-\sqrt{1-2m^3}},$$

where m is rational, or is derived from rational numbers by rational operations and a finite number of square roots, and where $1<2m<\sqrt[3]{4}$.

To show this, put $a/c=m+n$ and $b/c=m-n$. For a given a and b, c is constructible if only m and n are constructible. We have then $(m+n)^3+(m-n)^3=1$, whence $n=[(1-2m^3)/6m]^{1/2}$, where the condition that $0<b<a$ requires $1<2m<\sqrt[3]{4}$. If then m is constructible and satisfies this inequality, n is constructible and c is constructible. Finally $a/b=(m+n)/(m-n)$ $=(\sqrt{6m^3}+\sqrt{1-2m^3})/(\sqrt{6m^3}-\sqrt{1-2m^3})$.

Thus corresponding to $m=2/3$, we have $a/b=(12+\sqrt{33})/(12-\sqrt{33})$; similarly $m=1/\sqrt{2}$ gives

$$\frac{a}{b}=\frac{\sqrt{3}+\sqrt{\sqrt{2}-1}}{\sqrt{3}-\sqrt{\sqrt{2}-1}}$$

and $m=3/4$ gives the case cited in the problem.

Also solved by the proposer.

E 197 [1936, 104]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Construct a parallelogram, given both its altitudes and the acute angle between its diagonals.

Solution by L. M. Kelly, Lawrence, Mass.

Calling the adjacent sides a and b, and their respective altitudes h and k, we have $ah=bk=area$, so that $a/b=k/h$.

We may now construct a parallelogram similar to the desired one, as follows. Choose any arbitrary length for one of the diagonals. Construct the locus of points whose distances from the ends of this diagonal are in the ratio h/k. This is the circle of Apollonius. Also construct the second diagonal bisecting the first and making the given angle with it. The second diagonal must terminate on the Apollonian circle, giving two solutions in the general case, with but one if the second diagonal is tangent to the circle, and none if there is no real intersection. Having a parallelogram similar to the desired one, the construction of the latter follows immediately.

Also solved by Leon Recht, E. P. Starke, H. Tate, C. W. Trigg, Simon Vatriquant and the proposer.

E 198 [1936, 104]. *Proposed by J. E. Trevor, Cornell University.*

A multiplication of a three-place number by a two-place number has the form

$$\begin{array}{r} p\ p\ p \\ \underline{p\ p} \\ p\ p\ p\ p \\ \underline{p\ p\ p\ p\ \ \ } \\ p\ p\ p\ p\ p \end{array}$$

The p's are all prime digits, different from unity. Determine their values and show that the solution is unique.

Solution by W. E. Buker, Leetsdale High School, Pa.

Each of the digits is either 2, 3, 5 or 7. The conditions of the problem require that we find a three-digit and a one-digit number whose product has four digits, with the condition that only the above four digits be used. There are only the following four solutions to this problem, which are readily found from a few moments of trial: $3\times 775=2325$, $5\times 555=2775$, $5\times 755=3775$ and $7\times 325=2275$.

Since no three-digit number occurs for more than one multiplier, the multiplier of the problem must consist of two identical digits. Examination of the above four possibilities yields the unique solution

$$\begin{array}{r} 7\ 7\ 5 \\ \underline{3\ 3} \\ 2\ 3\ 2\ 5 \\ \underline{2\ 3\ 2\ 5\ \ \ } \\ 2\ 5\ 5\ 7\ 5 \end{array}$$

Also solved by R. K. Allen, J. W. Boyce, Mary L. Constable, Fred Discepoli, Wm. Douglas, Daniel Finkel, L. M. Kelly, W. R. Ransom, E. P. Starke, C. W. Trigg, M. J. Turner, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3796. *Proposed by Don Wallace, Charlottesville, Va.*

Solve the following matric-differential equation using only ordinary methods, i.e., without the use of the matric product integral,

$$\frac{dZ}{dt} + ZA + BZ + ZPZ = 0,$$

where A and B are constant matrices and P is a function of t.

3797. *Proposed by Don Wallace, Charlottesville, Va.*

Show that the locus of points collinear with their isogonal and isotomic conjugates is a conic passing through the median and exmedian points and the incenter and excenters. Show that its center is Steiner's point and that it is a rectangular hyperbola. Show also that the tangents at the median and exmedian points pass respectively through the symmedian and exsymmedian points, and that any point, its isogonal and isotomic conjugates are conjugate points with respect to the conic.

See in this Monthly, *On isogonal points*, by J. H. Weaver, 1935, pp. 496–499, and *Isogonal and isotomic conjugates and their projective generalization*, by P. H. Daus, 1936, pp. 160–164.

3798. *Proposed by N. A. Court, University of Oklahoma.*

Construct a sphere belonging to a given coaxial pencil and passing through the inaccessible point of intersection of a given line with a given plane.

Note. A similar problem in plane geometry was discussed in the Educational Times, Reprints, vol. 5 (1904), p. 83, Q. 15401.

3799. *Proposed by E. G. Olds, Carnegie Institute of Technology.*

From an urn containing ten balls, numbered from one to ten, balls are drawn one by one and placed in a row of holes, numbered from one to ten, each ball being placed in the proper hole. What is the probability that there will not be an empty hole between two filled ones at any time of the drawing?

3800. *Proposed by D. C. Duncan, University of California, Berkeley.*

Solve the equation

$$Ax^4 + (x-1)^3 = 0, \qquad A \neq 0,$$

given that it has a double root. Also prove the generalized theorem:

For just one set of real values of A and a_i the equation

$$Ax^{2k} + (x-1)^3 \prod_{i=1}^{k-2} (x-a_i)^2 = 0, \qquad A \neq 0, \qquad k > 2,$$

has $k-2$ double roots.

Editorial Note. Problem 3549 [1932, 239] is similar to the above, and the theorem in this latter problem is used in the proposer's article *A plane elliptic curve of order* $4k+2$, *with singularities all real and distinct, and autopolar by* $4k+4$ *conics*, Bull. Amer. Math. Soc., vol. 39 (1933), p. 810. No solution of problem 3549 has been received.

SOLUTIONS

3709 [1934, 582]. *Proposed by E. B. Escott, Oak Park, Ill.*

Determine the values of A in the trinomial

$$x^{12} + Ax^6y^6 + y^{12}$$

so that it will have two polynomial factors of the sixth degree with rational coefficients.

Solution by E. P. Starke, Rutgers University.

The required values of A are given by $R+R^{-1}$, $2-R^2$, and $-2-R^2$, where R is any rational number. The factors are then $(x^6+Ry^6)(x^6+R^{-1}y^6)$, $(x^6+Rx^3y^3+y^6)(x^6-Rx^3y^3+y^6)$ and $(x^6+Rx^3y^3-y^6)(x^6-Rx^3y^3-y^6)$ respectively. To show that these are the only possibilities, we distinguish several cases.

Case I, $A < -2$. Put $x/y = v$ and consider the equation

$$f(v) \equiv v^{12} + Av^6 + 1 = 0.$$

Let s be a real root of $f(v) = 0$. Then $A = -s^6 - s^{-6}$. The elementary real factors of $f(v)$ are

(A) $v^2 + sv + s^2$	(C) $v^2 + s^{-1}v + s^{-2}$	(E) $v + s$	(G) $v + s^{-1}$
(B) $v^2 - sv + s^2$	(D) $v^2 - s^{-1}v + s^{-2}$	(F) $v - s$	(H) $v - s^{-1}$.

We are to combine these into two polynomial factors of the sixth degree. Evidently if one factor has rational coefficients, so has its complementary factor. Let us then examine the factor containing (A), which can be constructed in 22 ways with real coefficients. We find the following results: (ABEF) has rational coefficients if and only if s^6 is rational; (ABGH), if s^2 is rational; (ACFH), if (s^3+s^{-3}) is; (ADFG), if (s^3-s^{-3}) is; (ACEG), if $(s+s^{-1})$ is; (ADEH), if $(s-s^{-1})$ is; and all the others have rational coefficients only if s is rational. But if s or s^2

is rational, so is s^6; if $(s+s^{-1})$ is rational, so is (s^3+s^{-3}); if $(s-s^{-1})$ is rational, so is (s^3-s^{-3}). Hence we have only three possibilities which, solved for A, give $A=R+R^{-1}$, $R<0$, $R\neq -1$; or $A=2-R^2$, $|R|>2$; or $A=-2-R^2$, $R\neq 0$.

Case II, $A>2$. Let si be a pure imaginary root of $f(v)=0$. $A=s^6+s^{-6}$. As above, the elementary real factors of $f(v)$ are

(A) v^2+s^2	(C) $v^2-\sqrt{3}sv+s^2$	(E) $v^2-\sqrt{3}s^{-1}v+s^{-2}$
(B) v^2+s^{-2}	(D) $v^2+\sqrt{3}sv+s^2$	(F) $v^2+\sqrt{3}s^{-1}v+s^{-2}$

The sixth degree factor containing (A) can be constructed in 10 ways with the following results: (ACD) has rational coefficients if and only if s^6 is rational; (AEF), if s^2 is; and all the others if $s\sqrt{3}$ is. But in any case s^6 is rational and we have $A=R+R^{-1}$, $R>0$, $R\neq 1$.

Case III, $|A|<2$. Put $A=-2\cos 6\theta$, $0<6\theta<\pi$. Then the elementary real factors of $f(v)$ are

(A) $v^2-2v\cdot\cos\theta+1$	(D) $v^2+2v\cdot\cos(\theta+\pi/3)+1$
(B) $v^2+2v\cdot\cos\theta+1$	(E) $v^2-2v\cdot\cos(\theta+2\pi/3)+1$
(C) $v^2-2v\cdot\cos(\theta+\pi/3)+1$	(F) $v^2+2v\cdot\cos(\theta+2\pi/3)+1$.

The sixth degree factor containing (A) can be constructed in 10 ways with the following results: (ADE) has rational coefficients if and only if $\cos 3\theta$ is rational; (ACF), if $\cos\theta$ is rational; (ACE), if $\cos(\theta+\pi/3)$ is; (ADF), if $\cos(\theta+2\pi/3)$ is; all the others require that both $\cos\theta$ and $\cos(\theta+\pi/3)$ be rational. But in all cases $\cos 3\theta$ is rational. Hence we have $A=-4\cos^2 3\theta+2=2-R^2$ where $|R|<2$.

Case IV, When $A=\pm 2$, the factors are obvious.

Thus, although other factors for $f(v)$ may appear, they cannot exist unless A has one of the values stated and then one of the given methods of factoring is also possible.

Editorial Note. It suffices to set $y=1$, and with this change the proposer assumes that

$$f(x)=x^{12}+Ax^6+1=(x^6+ax^4+bx^2+1)^2-(cx^5+dx^3+ex)^2. \tag{1}$$

After equating coefficients and setting $c=2m$, $e=2n$, it turns out that

$$a=2m^2,\qquad b=2n^2,\qquad md=m^4+n^2,\qquad nd=n^4+m^2,$$

and that either $n=m$ or $mn=1$. Hence

$$\begin{aligned} A &= 2-(m^3-3m)^2, & m=n\neq 0,\\ &= \;-(m^6+m^{-6}), & mn=1. \end{aligned} \tag{2}$$

We add the following remarks: If $m=n=0$, $A=2-d^2$, where d is any rational number. If in the first parenthesis on the right of (1) the constant term is -1 instead of $+1$, we find that

$$
(3)\qquad \begin{aligned} A &= -2-(m^3+3m)^2, && a=-b=2m^2\neq 0, \qquad d=m^3-m,\\ &= \quad\ -(m^6+m^{-6}), && mn=-1,\\ &= -2-d^2, && m=n=0, \end{aligned}
$$

where d in the last line is an arbitrary rational number. These are sufficient conditions. More comprehensive sufficient conditions result in a simple manner by observing that a necessary and sufficient condition that the sixth power of any root of $f(x)=0$ be rational, is that

$$
(4)\qquad A = R + R^{-1}, \qquad f(x) = (x^6+R)(x^6+R^{-1}),
$$

where R is any rational number not zero. The necessary and sufficient condition that $f(x)=0$ have a multiple root is that $R=\pm 1$ in (4). If in (4) we replace R by $-R^2$, we get a new form for A, $A=\epsilon 2-(R+\epsilon R^{-1})^2$ where $\epsilon=\pm 1$. This suggests a more general sufficient condition of this type.

$$
(5)\qquad A=\epsilon 2-R^2, \qquad f(x)=(x^6+Rx^3+\epsilon)(x^6-Rx^3+\epsilon), \qquad \epsilon=\pm 1.
$$

It appears to be more difficult to show that, if $f(x)$ can be factored in the required manner, either (4) or (5) must be true. Suppose that $f(x)=\phi_1(x)\phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are polynomials of the sixth degree with rational coefficients. We may take the coefficient of the highest power of x in each as unity without loss of generality. If $\phi_1(x)$ has a rational factor and $\phi_2(x)$ has also such a factor of the same degree, it will be possible to obtain a different form of the desired resolution when the two factors are distinct. They will be distinct if we discard the trivial case of equal roots in $f(x)=0$. If $\phi_1(x)$ is irreducible, then there is only one resolution. For, if there were two distinct forms of resolution, $\phi_1(x)=0$ would have a root in common with a similar equation obtained from the second resolution. In this case, since $\phi_1(x)$ is irreducible, it would have to be identical with this factor of the second decomposition, and the two resolutions are not distinct. For the values of A in the first line of (2) and (3) there are two distinct ways of factoring in each case, the one worked out and the one in (5). Hence here $\phi_1(x)$ is reducible. After selecting $\phi_1(x)$ as follows, we have

$$
(6)\qquad \begin{aligned} \phi_1(x) &= x^6+2mx^5+2m^2x^4+(m^3+\epsilon m)x^3+\epsilon 2m^2x^2+2mx+\epsilon,\\ &= [x^2+mx+\epsilon][x^4+mx^3+(m^2-\epsilon)x^2+\epsilon mx+1]. \end{aligned}
$$

For the value of A in the second line of (2) and (3) the same thing is true and for the same reason. Thus we have

$$
(7)\qquad \phi_1(x)=(x^3+\epsilon m^{-3})(x+m)(x^2+mx+m^2).
$$

Starke's methods may be used in a different manner to determine the form of A necessary for the required resolution by examining the possible rational factors of $\phi_1(x)$. In what follows the term factor means that its coefficients are rational with unity for the coefficient of the highest power of x. If s is a root of $f(x)=0$ all of its roots may be written in terms of s:

$$
(8) \qquad \begin{array}{l}
\text{(a)} \quad s, \quad \omega s, \quad \omega^2 s, \quad -s, \quad -\omega s, \quad -\omega^2 s, \\
\text{(b)} \quad s^{-1}, \quad \omega s^{-1}, \quad \omega^2 s^{-1}, \quad -s^{-1}, \quad -\omega s^{-1}, \quad -\omega^2 s^{-1},
\end{array}
$$

where ω is an imaginary cube root of unity. If any one of s, $\omega^t s^2$, s^3, s^6, $t=0, 1, 2$, is rational, A has the form (4) as is obvious from the form of the equation $x^6+x^{-6}=-A$. If $f(x)$ has a linear factor, a root is rational. If it has a cubic factor, we take for s one of its real roots. If the other two roots are in line (a) they must be $\epsilon_1\omega^t s$, $\epsilon_2\omega^{2t}s$, since their product must be real. Since the product of the three roots is rational, s^3 is rational. In a similar manner, if the other two roots are in (b), or one is in (a) and the other is in (b), s is rational. Suppose now that $f(x)$ has a quadratic factor, then we take one of the roots of this factor for s. If the other root is in (a), then $\omega^t s^2$ is rational. If it is in (b) it must be ϵs^{-1}, and then $s+\epsilon s^{-1}=r$, a rational number, We then have

$$
(9) \qquad \begin{array}{ll}
r^3 = s^3 + \epsilon s^{-3} + \epsilon 3(s + \epsilon s^{-1}), & r^3 - \epsilon 3r = s^3 + \epsilon s^{-3}, \\
(r^3 - \epsilon 3r)^2 = s^6 + s^{-6} + \epsilon 2 = -A + \epsilon 2, & A = \epsilon 2 - (r^3 - \epsilon 3r)^2,
\end{array}
$$

and A has the form (5). If s is imaginary and the other root is in (b), the latter must be s^{-1} and $\epsilon=1$. If s is imaginary with absolute value unity, there is only one case since then the conjugate of s is s^{-1}.

If $A=0$, s is imaginary with absolute value unity. Then (9) shows that a quadratic factor is impossible. There are however two factors x^4+1 and x^8-x^4+1. The first is irreducible since it has neither a linear nor a quadratic factor. If the required resolution were possible, one factor, say $\phi_1(x)$, must have a root in common with x^4+1. It must therefore have x^4+1 for a factor and a quadratic factor. Since the latter is impossible, the required resolution is impossible. If $A=2$ and there is a quadratic factor, the same argument shows that $r=0$ in (9) and then $s^2+1=0$. Hence there is no other quadratic factor than x^2+1. There is another factor x^4-x^2+1. and it must also be irreducible, since x^2+1 is not a factor of it. In this case the resolution is possible, but in only one way. If $A=-2$ there are more than one form of resolution. We now exclude these three cases in the rest of the proof.

If $\omega^t s^4=r$, a rational number, then $r^3+Ar\omega^{2t}s^2+1=0$ and $\omega^{2t}s^2$ is rational since $Ar\neq 0$. Hence A has the form (4). We now consider the roots of $\phi_1(x)$ one of which is s. If all of its roots are in (a) s^6 is rational. If of the roots other than s there are in (b) exactly one or five, then $\omega^t s^4$ is rational. If exactly two or four are in (b), then $\omega^t s^2$ is rational. There remains the case where three are in (a) and three in (b); and we may now suppose that $\phi_1(x)$ is irreducible. The constant term in $\phi_1(x)$ must be ± 1, and one at least of its roots must be the negative of a root of $\phi_2(x)$. For suppose that $\phi_1(x)$ has the roots s, $-s$, $\epsilon\omega^t s$, then $\phi_2(x)$ must have the root $-\epsilon\omega^t s$. Thus $\phi_1(x)$ and $\phi_2(-x)$ have a root in common, but, since $\phi_1(x)$ is irreducible, they must have all their roots in common, and $\phi_1(x)=\phi_2(-x)$. Hence each coefficient of an even power of x in $\phi_1(x)$ is the same as that of the same power of x in $\phi_2(x)$; whereas the coefficients of the odd powers of x in the two factors are the same but with opposite signs. One of the roots of $\phi_1(x)$ must

be the positive or negative reciprocal of some other one of its roots. For, if $\phi_1(x)$ has the root s but not s^{-1}, then the latter must be a root of $\phi_2(x)$. Hence $\phi_1(x)$ must have the root $-s^{-1}$. If $\phi_1(x)$ has the roots s and s^{-1}, then $\phi_1(x)$ and $x^6\phi_1(x^{-1})$ have a root in common. For the same reason as before they must have all their roots in common. Thus $\phi_1(x)=0$ is a reciprocal equation and the constant term is unity. Also $\phi_1(x)=x^6\phi_1(x^{-1})$. The coefficients of x^5 and x are the same; the coefficients of x^4 and x^2 are the same. If $\phi_1(x)$ has the roots s and $-s^{-1}$, we show in the same way that $\phi_1(x)$ and $x^6\phi_1(-x^{-1})$ have all their roots in common, and the constant term in $\phi_1(x)$ is -1. Here we have $\phi_1(x)=-x^6\phi_1(-x^{-1})$; the coefficients of x^5 and x are the same, those for x^4 and x^2 are equal but of opposite sign. We have now enough information to write

$$f(x) = [x^6 + ax^4 + \epsilon(ax^2 + 1)]^2 - x^2[cx^4 + dx^2 + c]^2, \tag{10}$$

where a, c, d are rational. Set $x^2=-\epsilon$, in both members of the equality, and we get $2-\epsilon A=\epsilon[2c-\epsilon d]^2$, or

$$A = \epsilon 2 - (2c - \epsilon d)^2. \tag{11}$$

Here A has the form in (5), but we must have both $a=0$ and $c=0$. For, since $\phi_1(x)$ is irreducible, there is only one form of resolution, and we always have the form in (5) in this case. If either $a\neq 0$, or $c\neq 0$, there would be two distinct forms of resolution. The form that d may have will appear later. This concludes the proof that A must have the form in (4) or (5) if the required resolution is possible.

By reasoning similar to the above we easily prove the following two results: If $\phi_1(x)=x^6+R$, $R\neq 0$, is reducible, then $R=-r^2$, or r^3, where r is any rational number not zero. These are easily seen to be sufficient conditions. If $\phi_1(x)=x^6+Rx^3+\epsilon$ is reducible, then $R=r+\epsilon r^{-1}$, or $r^3-\epsilon 3r$. It is obvious that if R has the first form $\phi_1(x)$ is reducible; in the second case

$$\phi_1(x) = [x^2 + rx + \epsilon][x^4 - rx^3 + (r^2 - \epsilon)x^2 - \epsilon rx + 1]. \tag{12}$$

We can exclude one of these conditions since $-(r^2+r^{-2})=\epsilon 2-(r+\epsilon r^{-1})^2$. We can now write necessary and sufficient conditions that there are more than one form of the required resolution:

$$A = -(r^2 + r^{-2}), \qquad A = r^3 + r^{-3}, \qquad A = \epsilon 2 - (r^3 - \epsilon 3r)^2, \tag{13}$$

where values of r and ϵ are excluded which make $A=2$.

We now consider the possibility of factoring $f(x)$ when A is rational and does not have the form in (4) or (5). The case $A=0$ has already been considered. Also the cases where $f(x)$ has a factor of the first, second, third, or sixth degree have been considered. In all these cases A has the form in (4) or (5). If $f(x)$ has a factor of the fifth degree, $\psi_1(x)$, it will suffice to consider it as irreducible. Such a factor could not have the roots s and $-s$. For, if it has these two roots, it must have as roots the negatives of each of its roots and this is impossible since its degree is odd and there is no zero root. If then $\psi_1(x)$ has the root s the other

factor $\psi_2(x)$ must have the root $-s$; and then $\psi_1(-x)$ and $\psi_2(x)$ have a root in common. Hence $\psi_2(x)$ has the factor $\psi_1(-x)$ and a quadratic factor. This is then an excluded case.

Suppose $f(x)$ has an irreducible factor $\psi_1(x)$ of the fourth degree and a factor $\psi_2(x)$ of the eighth degree which is irreducible, or has two irreducible factors of the fourth degree. If $\psi_1(x)$ has the root s but not $-s$, then $\psi_2(x)$ has the factor $\psi_1(-x)$ and a second factor with four roots two of which are the negatives of the other two. Hence we need only consider a factor of this latter type which we take for $\psi_1(x)$. If $\psi_1(x)$ has no pair of roots one of which is the reciprocal of the other, then $\psi_2(x)$ has a factor whose roots are the reciprocals of those of $\psi_1(x)$. The other factor of $\psi_2(x)$ must then have four roots which are reciprocals in pairs and negatives in pairs. Hence we may finally take as $\psi_1(x)$ an irreducible factor with the roots s, $-s$, s^{-1}, $-s^{-1}$,

$$\psi_1(x) = x^4 + ax^2 + 1, \qquad -a = s^2 + s^{-2}. \tag{14}$$

The other factor must have the roots

$$\begin{array}{llllll} (15) & \text{(a)} & \omega s, & \omega^2 s, & -\omega s, & -\omega^2 s, \\ & \text{(b)} & \omega s^{-1}, & \omega^2 s^{-1}, & -\omega s^{-1}, & -\omega^2 s^{-1}; \end{array}$$

and we easily find that

$$\psi_2(x) = x^8 - ax^6 + (a^2 - 1)x^4 - ax^2 + 1, \qquad A = a^3 - 3a. \tag{16}$$

If $\psi_1(x)$ is reducible, $a = r + r^{-1}$, or $a = \epsilon 2 - r^2$, where r is rational. Hence we exclude from the values of a

$$-1, \qquad r + r^{-1}, \qquad \epsilon 2 - r^2. \tag{17}$$

If $\psi_2(x)$ has a quadratic factor with both roots in (15) (a), then $\omega^t s^2 = r$, a rational number. If $t = 0$, then $s^2 = r$. Since $-a = s^2 + s^{-2} = r + r^{-1}$, $\psi_1(x)$ is reducible. If $t \neq 0$, we have from $\psi_1(s) = 0$, $r^2\omega^t + ar\omega^{2t} + 1 = 0$. Hence $ar = 1$, $2r^2 - ar - 1 = 0$, $r^2 = 1$, $a = \pm 1$. If $a = 1$, $\psi_1(x)$ is reducible, and $a = -1$ is excluded. If one root is in (a) and the other is in (b), the roots must be of the form $\epsilon_1\omega^t s$, $\epsilon_2\omega^{2t}s^{-1}$. Since the sum of the roots is rational, we have

$$\omega^t s + \epsilon\omega^{2t}s^{-1} = r, \qquad s^3 + \epsilon s^{-3} + \epsilon 3r = r^3, \qquad s^3 + \epsilon s^{-3} = r^3 - \epsilon 3r = u,$$

where u is rational. Hence $x^6 - ux^3 + \epsilon = 0$ and $\psi_1(x) = 0$ have a root in common. Hence the first equation has all the roots of $\psi_1(x)$, since the latter is irreducible. Then $u = 0 = r^3 - \epsilon 3r$, and r is irrational or zero. In the latter case, the only one admissible, $s^2 = -\epsilon\omega^t$ and we again get $a = \pm 1$.

If $\psi_2(x)$ has a linear or cubic factor, arguments similar to the above show that it also has a quadratic factor. We now have the final result. If $A = a^3 - 3a$, where a is rational and does not have a value in (17), then $f(x)$ has an irreducible factor $\psi_1(x)$ of the fourth degree and a factor $\psi_2(x)$ of the eighth degree which does not admit a quadratic factor.

3719 [1935, 48]. *Proposed by Morgan Ward, Institute for Advanced Study.*
Prove that

$$\sum_{r=0}^{n} {}_nC_r/(x+r)(x+r+1)\cdots(x+r+n) = 2^n/x(x+2)(x+4)\cdots(x+2n).$$

I. *Solution by H. D. Ruderman, Brooklyn, N. Y.*

The identity will be proved by induction. It is obviously true for $n=0$, 1. Assume that it is true for n; it will be shown that it must then be true also for $n+1$.

$$\begin{aligned}\sum_{r=0}^{n+1} {}_{n+1}C_r \prod_{i=0}^{n+1}(x+r+i)^{-1} \\ = \frac{1}{n+1}\sum_{r=0}^{n+1}\left[{}_nC_{r-1}+{}_nC_r\right]\left[\prod_{i=0}^{n}(x+r+i)^{-1}-\prod_{i=0}^{n}(x'+r+i)^{-1}\right],\end{aligned}$$

where $x'=x+1$. In what follows undefined terms are omitted from the summations.

$$\begin{aligned}\sum_{r=0}^{n+1} {}_{n+1}C_r \prod_{i=0}^{n+1}(x+r+i)^{-1} \\ = \frac{1}{n+1}\Bigg\{2^n\left[\prod_{i=0}^{n}(x+2i)^{-1}-\prod_{i=0}^{n}(x'+2i)^{-1}\right] \\ + \sum_{r'=0}^{n}{}_nC_{r'}\left[\prod_{i=0}^{n}(x'+r'+i)^{-1}-\prod_{i=0}^{n}(x+2+r'+i)^{-1}\right]\Bigg\},\end{aligned}$$

where $r'=r-1$,

$$\begin{aligned}&= \frac{2^n}{n+1}\left[\prod_{i=0}^{n}(x+2i)^{-1}-\prod_{i=0}^{n}(x+2+2i)^{-1}\right] \\ &= \frac{2^n}{n+1}2(n+1)\prod_{i=0}^{n+1}(x+2i)^{-1} = 2^{n+1}\prod_{i=0}^{n+1}(x+2i)^{-1}.\end{aligned}$$

This completes the proof.

II. *Solution by F. Underwood, University College, Nottingham.*

Denote by u_r the expression summed in the given problem and by S_n the value of the sum. By decomposition into partial fractions we have

$$u_r = \sum_{s=0}^{n}\frac{(-1)^s{}_nC_r}{s!(n-s)!(x+r+s)}, \tag{1}$$

$$S_n = \frac{1}{n!}\sum_{r=0}^{n}\sum_{s=0}^{n}\frac{(-1)^s{}_nC_r\,{}_nC_s}{x+r+s}, \quad = \frac{1}{n!}\sum_{t=0}^{2n}\frac{1}{x+t}\sum_{s=0}^{t}(-1)^s{}_nC_{t-s}\,{}_nC_s, \tag{2}$$

where terms which have no meaning are to be omitted; and, if $t>n$, $t-n\leqq s\leqq n$. From the identity $(1-x^2)^n=(1-x)^n(1+x)^n$, we obtain

$$\sum_{s=0}^{2p}(-1)^s{}_nC_{2p-s}\,{}_nC_s=(-1)^p{}_nC_p,\qquad 0\leqq p\leqq n. \tag{3}$$

If $2p>n$, then $2p-n\leqq s\leqq n$. In (2) the terms are zero for odd values of t. Hence

$$S_n=\sum_{p=0}^{n}\frac{(-1)^p}{p!(n-p)!}\,\frac{1}{x+2p}.$$

This is the decomposition into partial fractions of the right member of the equality in the problem. Hence the equality of the problem is true.

Solved also by Frank Ayres, Jr., E. J. McShane, and D. V. Widder.

Editorial Note. If we denote by $f(x, n)$ the left member of the equation of the problem, it is clear that this function has certain terms which are infinite for $x=0, -1, -2, \cdots, -2n$. These values should be excluded during the proof. After the proof $f(x, n)$ can then be defined for the odd values in this sequence. Ayres gave a proof by mathematical induction different from I; and McShane gave one using partial fractions in a manner different from II. Widder set

$$\begin{aligned}f(x, n)&=\sum_{r=0}^{n}\binom{n}{r}\int_0^\infty e^{-(x+r)t}\,\frac{(1-e^{-t})^n}{n!}\,dt\\&=\int_0^\infty e^{-xt}\,\frac{(1-e^{-t})^n}{n!}\sum_{r=0}^{n}\binom{n}{r}e^{-rt}dt=\int_0^\infty e^{-xt}\,\frac{(1-e^{-2t})^n}{n!}\,dt\\&=2^n/x(x+2)\cdots(x+2n).\end{aligned}$$

This solution is closely related to II. For, if x does not have any one of the excluded values, the first integral can be evaluated in two ways one of which gives $f(x, n)$ in the original form and the other gives it in the altered form after decomposition into partial fractions. The following changes of the integral are the equivalent of the use of the identity $(1-x^2)^n=(1-x)^n(1+x)^n$.

This is a good exercise for the methods of finite differences. We shall recall certain elementary facts which are here useful. If we set $1/(x+r)(x+r+1)\cdots(x+r+j-1)=(x+r)^{(-j)}$, we have a rule analogous to that for the mth derivative of $(x+r)^{-j}$

$$\Delta^m(x+r)^{(-j)}=(-1)^m j(j+1)\cdots(j+m-1)(x+r)^{(-j-m)}.$$

If δ denotes the operator such that $\delta f(x)=f(x+2)-f(x)$, then analogous to the rule for $[d^n/dt^n(x+2t)^{-1}]_{t=0}$ we have

$$\begin{aligned}\delta^m f(x)&=(U^2-1)^m f(x)=\Delta^m f(x+2\cdot 0),\\ \delta^n x^{(-1)}&=\Delta^n(x+2\cdot 0)^{(-1)}=(-1)^n n!/x(x+2)\cdots(x+2n).\end{aligned}$$

Thus

$$f(x, n) = \frac{(-1)^n}{n!} \sum_{r=0}^{n} {}_nC_r \Delta^n (x+r)^{(-1)} = \frac{(-1)^n}{n!} [(U+1)^n \Delta^n] x^{(-1)}$$
$$= \frac{(-1)^n}{n!} (U^2 - 1)^n x^{(-1)} = 2^n / x(x+2) \cdots (x+2n).$$

Solution I may be put in a similar form. After setting ${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$, and revising the two summations, we have

$$f(x, n) = \sum_{r=0}^{n-1} {}_{n-1}C_r [(x+r+1)^{(-n-1)} + (x+r)^{(-n-1)}],$$
$$= -\frac{1}{n} \sum_{r=0}^{n-1} {}_{n-1}C_r [(x+r+2)^{(-n)} - (x+r)^{(-n)}],$$
$$= -\frac{1}{n} (U^2 - 1) f(x, n-1).$$

Repeated applications of this formula give

$$f(x, n) = \frac{(-1)^n}{n!} (U^2 - 1)^n f(x, 0) = \frac{(-1)^n}{n!} (U^2 - 1)^n x^{(-1)},$$

with the same result as above.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department of sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The twenty-first annual meeting of the Mathematical Association of America will be held at Duke University, Durham, North Carolina, December 30, 1936–January 1, 1937, in conjunction with the meetings of the American Mathematical Society. The Society will hold sessions Tuesday and Wednesday; the annual informal banquet will be held Wednesday evening; the final session of the Society Thursday morning and the first session of the Association Thursday afternoon will be held at the University of North Carolina at Chapel Hill; and the second session of the Association will be held at Duke University Friday forenoon.

Duke University has placed the dormitories on the men's campus at the disposal of the mathematicians and their guests free of charge; these will be available from Sunday afternoon through Friday. Meals will be served at the Union at very reasonable rates. The programs of these meetings and reservation cards will be sent to members at the usual time.

At the annual meeting of the Society for the Promotion of Engineering Edu-

cation held at Madison, Wisconsin June 23–26, a division of mathematics was formed. The following were elected officers of the division: Chairman, T. C. Fry; Secretary, J. H. Weaver; Members of the Executive Committee, R. S. Burington, W. C. Brenke, R. V. Churchill. The objectives of the division were agreed to be

1. To provide a better mathematical background for modern engineering education by
   a. Discussing the subject matter and methods of presentation of the courses in mathematics required of engineering students;
   b. Making recommendations concerning advanced and elective courses which appear especially valuable for undergraduate and graduate students in the light of current engineering practices;
2. To stimulate research in the field of applied mathematics among teachers of mathematics, and to aid them in their direction of the research of their advanced students by presenting
   a. Papers designed to disseminate information on recent applications of mathematics to engineering problems;
   b. Expository papers giving a general view of mathematical fields which have recently come to have engineering applications;
   c. Research papers involving use of mathematics in problems of physics and engineering;
3. To consider and evaluate criticisms on the teaching of mathematics and to make suggestions regarding proper qualifications of teachers with particular reference to the needs of engineering students;
4. To hold regular meetings in connection with the annual meeting of the S.P.E.E. and occasional joint meetings with other divisions in the S.P.E.E. or with other groups when papers of mutual interest are available.

The Association for Symbolic Logic is now publishing a quarterly journal entitled *The Journal of Symbolic Logic*, the first number of which appeared in May. The editors are Professors Alonzo Church of Princeton University and C. H. Langford of the University of Michigan. Professor H. B. Curry, of Pennsylvania State College, has recently been made Vice-President of the association.

The honorary degree of Doctor of Science was conferred on Professor E. R. Hedrick, of the University of California at Los Angeles, by the University of Michigan on June 20.

The American Association for the Advancement of Science appointed Professor Oswald Veblen, of the Institute for Advanced Study, as its representative at the International Congress of Mathematicians at Oslo, July 13–18.

Guggenheim Fellowships have been awarded to Dr. Solomon Gandz, of New York City, and to Professor M. H. Stone, of Harvard University. Dr. Gandz will study early algebra, especially its connection with ancient Greek, Babylo-

nian, and Egyptian mathematics, and its influence upon medieval European mathematics. Professor Stone will carry on research in the field of theory of linear representation in abstract space.

Professor Solomon Lefschetz, of Princeton University, gave three lectures on *Combinatory topology* at the University of Paris in May.

Dr. Rufus Oldenburger of Armour Institute of Technology visited some of the technical schools in Europe during the past summer, and gave a paper at the algebra section of the International Congress of Mathematicians, Oslo, Norway.

Professor Marston Morse of the Institute for Advanced Study was elected a member of the American Philosophical Society at the annual meeting held in Philadelphia, April 23–25.

Professor J. L. Walsh of Harvard University was elected a member of the National Academy of Sciences at the annual meeting in Washington, April 27–29.

Dr. Felix Bernstein of Columbia University has been appointed to a professorship in biometrics at New York University.

Professor A. L. Candy of the University of Nebraska has been named professor emeritus at that institution.

Associate Professor H. C. Carver of the University of Michigan has been promoted to a professorship.

D. J. Colbert of St. Bonaventure College has been promoted to an assistant professorship.

Assistant Professor C. C. Craig of the University of Michigan has been promoted to an associate professorship.

Dr. Ben Dushnik of the University of Michigan has been promoted to an assistant professorship.

Professor W. B. Fite has been appointed to the newly established Davies professorship at Columbia University.

Dr. C. A. Garabedian has been appointed associate professor of mathematics, organist, and choir director at Wheaton College (Mass.).

Professor B. O. Koopman of Columbia University is teaching at Harvard University during the academic year.

Dr. E. W. Miller of the University of Michigan has been promoted to an assistant professorship.

Professor W. V. Parker of the Georgia School of Technology has been ap-

pointed to an associate professorship in mathematics at the Louisiana State University.

Dr. E. B. Roessler of the University of California at Davis has been promoted to an assistant professorship in mathematics and appointed assistant statistician in the Experiment Station there.

Professor D. M. Smith of the Georgia School of Technology has been appointed head of the department of mathematics.

John Oxtaby and Dr. Stanislaw Ulam have been elected to the Society of Fellows at Harvard University.

Dr. H. E. Spencer of Cornell University has been appointed to a professorship at Presbyterian College, South Carolina.

Assistant Professor Marion E. Stark of Wellesley College has been promoted to an associate professorship.

Dr. Max Zorn of Yale University has been appointed to an associate professorship at the University of California at Los Angeles.

The following appointments to instructorships in mathematics are announced:

Brooklyn College: Dr. James Singer
Univ. of California at Los Angeles: Dr. M. R. Hestenes
Colby College: Dr. I. J. Schoenberg
Cornell University: Dr. J. H. Curtiss, Dr. Saunders MacLane, Dr. J. B. Rosser, Dr. J. K. L. MacDonald, L. L. Lowenstein, Dr. C. E. Rhodes, Dr. S. E. Warschawski
Georgia School of Technology: Dr. W. E. Sewell
Harvard University: R. P. Boas, Jr., Dr. A. E. Pitcher (Benjamin Peirce Instructor)
Massachusetts Institute of Technology: Dr. W. T. Martin
University of Michigan: Dr. Ralph Hull
New York State Teachers College: Dr. C. Grace Shover
Pomona College: Dr. H. J. Hamilton

The following have been appointed National Research Fellows in Mathematics for the year 1936–1937: D. M. Dribin, Aaron Fialkow, Nathan Jacobson. Norman Levinson, and C. B. Tompkins. This list includes renewals.

Dr. Julia T. Colpitts, associate professor of mathematics at Iowa State College, died on August 8 at Southampton, England, on her way home from the Congress at Oslo. She was a charter member of the Association, and was active in the Iowa Section of which she had been Chairman.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N. Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Annual Meeting, Duke University, Durham, N. C., Dec. 31, 1936–Jan. 1, 1937.
Twenty first Summer Meeting, Pennsylvania State College. Sept., 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, Normal, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, Topeka, March 14.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, MAY 9; Washington, Dec. 5.
MICHIGAN, Ann Arbor, March 21; Albion, Nov. 28.
MINNESOTA
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S. C., April 17–18.
SOUTHERN CALIFORNIA, Fullerton, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, Madison, May 9.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 9, NOVEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

Brother J. Leo, A.B.(DePaul Univ.) Instr., St. Mary's Coll., Winona, Minn.
C. W. Moran, A.M.(Loyola Univ., Chicago) Teacher, Lane Tech. High School, Chicago, Ill.
E. N. Oberg, Ph.D.(Minnesota) Instr., Jr. Coll., Rochester, Minn.
Marvin Roberts, A.B.(New Mexico) Teacher, High School, Hachita, N.M.
W. C. Rufus, Ph.D.(Michigan) Asso. Prof., Astr., Univ. of Michigan, Ann Arbor, Mich.
E. N. Shawhan, Ph.D.(Ohio State) Instr., Univ. of Minnesota, Minneapolis, Minn.
Sister M. Laurine Sheeran, A.M.(Catholic Univ.) Head of Dept., Mount St. Joseph Coll., Maple Mount, Ky.
P. I. Speicher, A.M.(Pennsylvania) Asst. Prof., Albright Coll., Reading, Pa.
G. W. Starcher, Ph.D.(Illinois) Asso. Prof., Ohio Univ., Athens, Ohio
Sister M. Clotilda Sullivan, A.M.(Canisius) Instr., Mercyhurst Coll., Erie, Pa.
Nellie R. Swanson, A.M.(Minnesota) Instr., State Teachers Coll., Dickinson, N.D.
T. A. H. Teeter, B.S.C.E.(Purdue) Asso. Prof., Engineering, Univ. of Minnesota, Minneapolis, Minn.
H. P. Thielman, Ph.D.(Ohio State) Prof., Coll. of St. Thomas, St. Paul, Minn.
K. L. Warren, Ph.D.(Michigan State Coll.) Dean, Physical Sciences, Coll. of Applied Sci., Grand Rapids, Mich.

The Trustees voted to accept the invitation of the Pennsylvania State College to meet there in September 1937.

The Trustees approved the organization of a Southwestern Section of the Association and the By-Laws submitted by the temporary officers. The membership of the Section is to comprise the members of the Association resident in Arizona and New Mexico; in addition individual members resident in areas immediately adjacent to these states may affiliate as individual members.

Informal reports were given by the President and the Secretary on current activities not ready for formal reports.

W. D. Cairns, *Secretary-Treasurer*

THE THIRTEENTH MEETING OF THE INDIANA SECTION

The thirteenth annual meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, May 1 and 2, 1936, at Manchester College, North Manchester, Indiana.

Approximately two hundred attended the public lecture on Friday evening and forty registered at the sessions on Saturday, including the following twenty-three members of the Association: W. C. Arnold, J. H. Butchart, H. T. Davis, J. E. Dotterer, W. E. Edington, P. D. Edwards, C. H. Frick, G. H. Graves, H. E. H. Greenleaf, F. H. Hodge, Florence Long, Juna M. Lutz, T. E. Mason, H. A. Meyer, F. R. Moulton, Mary S. Paxton, D. H. Porter, J. A. Reising, C. K. Robbins, L. S. Shively, W. O. Shriner, Anna K. Suter, K. P. Williams.

At the business session on Saturday the following officers were elected for next year: Chairman, W. E. Edington, DePauw University; Vice-Chairman, W. O. Shriner, Indiana State Teachers College; Secretary, P. D. Edwards, Ball State Teachers College. The fourteenth annual meeting will be held at DePauw University, Greencastle, Indiana, in May 1937.

Professor K. P. Williams made a report for the committee appointed to en-

courage and recognize superior preparation for the teaching of secondary mathematics. On the basis of an examination conducted April 18, 1936, a Certificate of Merit was awarded by the Indiana Section to each of three students who completed their requirements this year.

Following the annual dinner on Friday evening the members of the Association were guests at a public lecture in the college chapel. Dr. Moulton, director of the Utilities Power and Light Corporation of Chicago, gave an illustrated lecture on astronomy.

At the sessions on Saturday the following program was presented.

1. "Some examples in mathematics" by Professor H. A. Meyer, Hanover College, retiring chairman.

2. "Mathematics and social phenomena" by Professor H. T. Davis, Indiana University.

3. "An investigation of attitudes of high school students toward mathematics" by Helen Darley, Manchester College, introduced by Professor J. E. Dotterer.

4. "Arithmetic three hundred years ago and now" by Professor T. E. Mason, Purdue University.

5. "Mascheroni's *Geometry of the Compasses*" by Professor L. S. Shively, Ball State Teachers College.

6. "Note on Bieberbach's trisection of an angle" by E. L. Godfrey, Indiana University, introduced by Professor K. P. Williams.

7. "A simple recursion method for solving a system of linear equations" by Professor Cornelius Lanczos, Purdue University, introduced by Professor C. K. Robbins.

8. "Synthetic treatment of cycloids" by Professor J. H. Butchart, Butler University.

9. "A system of equations connected with a comet orbit" by Professor K. P. Williams, Indiana University.

10. "A problem in infinite quadratic forms" by R. L. Deputy, Indiana University, introduced by Professor H. T. Davis.

Abstracts of the papers follow:

1. Professor Meyer discussed some examples from various fields of mathematics which have been constructed to correct certain fallacies of current thinking.

2. In this paper, Professor Davis discussed the general problems of econometrics. and indicated the type of mathematical disciplines most useful in the study of mathematical economics. The paper concluded with a discussion of the problems presented by economic time series. The speaker used lantern slides in the presentation of his paper.

3. Miss Darley recently completed an investigation concerning the attitudes of high-school students toward mathematics. This investigation was conducted by the use of questionnaires which were filled out by 1000 students from four high schools. The conclusions arrived at from these questionnaires included

the fact that 59% of the students do like mathematics. With regard to mathematical magazines and articles, it was found that only 19% of the students had been reading them, the majority of the remainder not being aware of their existence. Some of the suggestions of the students were that the teacher should be better prepared in the subject matter; that the teacher should be more interested and enthusiastic about the subject; and that the classroom work should be varied by the teacher's reading articles concerning the origin and history of mathematics.

4. Professor Mason compared and contrasted, as to contents and methods of presentation, four arithmetics printed in the years 1585, 1631, 1880 and 1925, respectively.

5. Mascheroni published in 1797 his *Geometry of the Compasses* in which he showed the possibilities of compasses alone as an instrument for making geometrical constructions. By means of three "fixed compasses" whose radii are 1, $\sqrt{3}$ and $\sqrt{2}$, he showed how to divide a circle into 2, 3, 4, 6, 8, 12 and 24 equal parts. By the use of two additional fixed compasses the division into 5, 10, 15, 20, 48 and 120 equal parts can be made. Constructions with compasses only were also given for the bisection of an arc, the fourth proportional to three given lines, the intersection of two given lines and of a given line with a given circle. Upon these fundamental constructions rests a proof that any construction which may be made with the ruler and compasses, may be made with the compasses alone. In this paper Professor Shively exhibited these constructions.

6. Mr. Godfrey presented a simple way of showing the connection between the limaçon and Bieberbach's method for trisecting an angle. An exposition of this method was published in *Scripta Mathematica*, vol. 3, 1935, page 326.

7. A new recursion method for the solution of systems of linear equations was discussed by Professor Lanczos. The method of determinants for the solution of linear equations is only of theoretical interest if the number of equations is large. The customary elimination method which reduces successively the order of the system from n to $n-1$, and so on, involves laborious calculations. The method presented builds up a transformation matrix, the elements of which are obtained by successive recursions from the matrix elements of the given system. With the help of this matrix the solution of the system can be obtained by successive recursions. The resulting scheme is analogous to the scheme of the elimination method but the result is obtained in a different sequence and by a much smaller number of operations.

8. Properties of the cycloid, epicycloid and hypocycloid are easily obtained by a purely geometrical treatment. The evolute is generated by a rolling circle homothetic to that which generates the primary curve with respect to the center of the given fixed circle. Professor Butchart obtained relations between the generating circles which lead to expressions for the length of arc of the curve and the area between the curve and the fixed circle.

9. In this paper Professor Williams dealt with the peculiarities of the equations that concern the distance of Encke's comet during its appearance in 1931.

10. Mr. Deputy gave a short treatment of a problem of Hilbert in infinite quadratic forms, approaching it through direct analysis by a new and distinct method. He obtained a solution for a form having a continuous spectrum by the use of the reciprocal of the characteristic matrix of the form.

P. D. EDWARDS, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The Seventeenth Annual Meeting of the Illinois Section of the Mathematical Association of America was held Friday and Saturday, May 8 and 9, 1936, at Illinois State Normal University, Normal, Illinois.

A total of fifty-three persons registered from fifteen colleges and eleven high schools including the following twenty-six members of the Association: Edith I. Atkin, O. K. Bower, Laura E. Christman, C. E. Comstock, J. J. Corliss, H. B. Curtis, D. R. Curtiss, W. M. Davis, Sister Mariola Dobbin, Elinor B. Flagg, R. E. Gadske, A. E. Gault, R. M. Ginnings, M. C. Hartley, Mildred Hunt, E. C. Kiefer, W. C. Krathwohl, A. H. Larsen, J. R. Mayor, H. J. Miles, E. B. Miller, C. N. Mills, G. E. Moore, H. A. Simmons, Norma K. Stelford, E. H. Taylor.

Professor E. C. Kiefer of James Millikin University had charge of the arrangement of the program and presided at the meetings.

At the business session the following officers were elected for next year: C. N. Mills, Illinois State Normal University, Chairman; W. B. Storm, State Teachers College, De Kalb, Vice-Chairman; and Edith I. Atkin, Illinois State Normal University, Secretary-Treasurer. It was decided to hold the next meeting at De Kalb.

An important item of business was the passing of the following resolution submitted by W. C. Krathwohl, E. H. Taylor and H. B. Curtis:

> *Whereas*, the report of the National Committee on the Reorganization of Mathematics in Secondary Education has been of such great value in improving the curriculum and the instruction in secondary mathematics, and
>
> *Whereas*, there exists a need for a similar report on mathematics in the elementary schools by persons competent to conduct such an investigation, therefore be it
>
> *Resolved*, that the Illinois Section of the Mathematical Association of America urge the Mathematical Association of America to join with the National Council of Teachers of Mathematics in an effort to obtain the appointment of a national commission on mathematics in the elementary schools and to obtain means for its financial support.

It was voted to ask the secretary to send letters to two beloved members who are suffering from prolonged illnesses: Professor H. E. Slaught of the University of Chicago, and Professor E. B. Lytle of the University of Illinois.

At the dinner held in Fell Hall the guests were welcomed by President Fair-

child of the University. The meal was arranged and served under the direction of Mr. R. D. Marsh, business manager of the school. The address of the evening was a most interesting illustrated lecture on "Tall buildings" by Professor W. C. Huntington of the Civil Engineering Department of the University of Illinois, by invitation.

At the Friday afternoon and Saturday morning sessions the following papers were presented:

1. "A circular locus" by Dr. G. E. Moore, University of Illinois.
2. "The need for a national commission on mathematics in the elementary schools" by Professor E. H. Taylor, Eastern Illinois State Teachers College.
3. "Derivatives and directional derivatives for functions of a complex variable" by Professor D. R. Curtiss, Northwestern University.
4. "A note on $x^2+ay^2+bz^2+abt^2$" by R. M. Thrall, University of Illinois, introduced by Professor Kiefer.
5. "Some modern trends in the teaching of mathematics" by R. E. Gadske, Carbondale High School.

Abstracts of the papers follow, numbered as their titles.

1. In this paper Dr. Moore discussed a new circular locus associated with two triangles having a common circum-circle. The fact that this circle has a radius equal to half the radius of the circumcircle was proved. Its center relative to the whole configuration was established, and several interesting geometric relationships were pointed out. The center and radius of inversion by means of which the new circle could be obtained from the circumcircle were established and a simple geometric construction for the inversion was shown. Two new harmonic cross-ratios were developed in connection with notable lines and points of the configuration.

2. This paper briefly reviews evidence, presented by Professor Taylor in previous papers, which shows (1) the lack of mastery by college freshmen of the meanings and processes of elementary arithmetic, and (2) the decline in the last eight years in the number of semester hours of arithmetic offered and required by teachers colleges in their curricula for the preparation of elementary teachers. The evidence shows that beginning elementary teachers are not as well prepared in arithmetic as twenty years ago. If mathematics is to hold its place in elementary and secondary education, the mathematics curriculum must be planned by those who believe in the educational value of the subject.

3. Professor Curtiss's paper discussed various limits connected with the difference quotient $\Delta w/\Delta z$ at a point z_0, in whose neighborhood the function $w=f(z)$ has continuous first partial derivatives and maps a neighborhood of z_0 on a neighborhood of $w=w_0$. The limits discussed were (a) that of $\Delta w/\Delta z$ itself as Δz approaches zero; (b) that of its absolute value; (c) that of its angle; (d) that of its real part; and (e) that of its imaginary part. It is well known, though seldom thus stated, that if the limit (b) exists then the limit (a) does also either for w or the conjugate of w, and that if the limit (c) does, so also does the limit (a). It is true also that the existence of the limits (d) and (e) each imply the

existence of the limit (a). These implications, in fact, hold when we replace the requirement that the limits exist no matter how Δz approaches zero by the hypothesis that the limit exists and is the same for three different directions in which Δz approaches zero.

4. In this paper Mr. Thrall gave a theorem relating to representation of integers by the forms $x^2+ay^2+bz^2+abt^2$. The theorem gave enough information to prove the universal character of the named forms for all but one of the sets of values for a and b for which the form is universal.

5. The purpose of Mr. Gadske's paper was to report on some of the trends in the teaching of mathematics, with emphasis upon a certain individualized teaching experiment that he performed. His experiment was an attempt to determine the relative value of an individualized unit method as compared with the conventional classroom method of teaching high school algebra. The experimental group was taught by the unit method in which the instruction and progress of the pupils were strictly individual within each unit, and the assignments were unit assignments. The control group was taught by the conventional group method whereby the pupils progressed as a group through lectures, demonstrations, recitations, daily assignments, tests, and group remedial instruction. Mr. Gadske finds that the differences in the two methods were very pronounced. The algebra achievement of the experimental group was far superior to that of the control group. The experiment is one of the many that must be made in order to reach final conclusions concerning the relative merits of individual instruction versus group instruction.

EDITH I. ATKIN, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The fourth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin, Madison, on Saturday, May 9, 1936. The chairman of the Section, Professor H. H. Conwell, presided.

The attendance was sixty-one, including the following thirty-one members of the Association: Florence E. Allen, Katherine S. Arnold, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, H. H. Conwell, G. S. Cook, L. A. V. DeCleene, Henry Ericson, H. P. Evans, M. L. Hartung, R. C. Huffer, M. H. Ingraham, R. L. Jeffery, Elizabeth E. Knight, Caroline A. Lester, C. C. MacDuffee, J. S. McNair, Morris Marden, Sister Mary Felice, E. A. Nordhaus, R. E. Norris, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, I. S. Sokolnikoff, P. L. Trump, J. A. Ward, Margarete C. Wolf.

Following a luncheon at the University Club, a business meeting was held at which the following officers were elected for the year 1936–1937: Chairman, L. A. V. DeCleene, St. Norbert College; Secretary, G. A. Parkinson, University of Wisconsin Extension Division at Milwaukee; Program Committee, I. S. Sokolnikoff, University of Wisconsin, Henry Ericson, Washington High School,

Milwaukee, Elizabeth E. Knight, Milwaukee State Teachers College. An invitation to hold the next annual meeting at Milwaukee-Downer College was accepted.

The following papers were presented at the morning and afternoon sessions:

1. "Graphical representation of the real and complex roots of cubic equations" by J. A. Ward, University of Wisconsin.

2. "Extension of the concept of length" by Professor R. L. Jeffery, Acadia University.

3. "The use of homogeneous coordinates in the teaching of elementary mathematics" by Professor H. P. Pettit, Marquette University.

4. "Linear algebras" by Professor H. H. Conwell, Beloit College.

5. "Some applications of vector methods to differential geometry" by Professor G. A. Parkinson, University of Wisconsin Extension Division at Milwaukee.

6. "Mathematical placement tests" by Professor Elizabeth E. Knight, Milwaukee State Teachers College.

7. "Some curriculum and evaluation problems related to mathematics" by Professor M. L. Hartung, University of Wisconsin.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. In this paper Mr. Ward gave a graphical method of obtaining the complex (as well as real) roots of cubics. He also showed how to solve any cubic with numerical coefficients by using only two graphs. The quartic was treated similarly. An extension was made to the n-ic, and the equation of the locus of its complex roots was given.

2. In this paper Professor Jeffery started with the concept of "length" as applied to a line segment, and showed how this concept had been extended to include linear sets consisting of an infinite number of line segments, linear sets which contained no line segments, and plane curves. He also gave a brief introduction to the idea of length of a set of points in the plane as developed by Carathéodory.

3. The use of homogeneous coordinates in the teaching of elementary Analytics has worked out in his classes, Professor Pettit reported, to give increased generality and economy of labor in the attack on many topics. It is to be noted that in the hands of a teacher not in sympathy with the method, or one who has inadequate knowledge of its application, the student may be confronted with considerable confusion. If handled sympathetically and with proper care, the method offers distinct advantages in the reduction of the number of special cases, symmetry of forms, and ease of memorization.

4. In this paper Professor Conwell discussed the early work of W. R. Hamilton on couples and triads which led to his discovery of quaternions; the work of C. S. Peirce in the development of Linear Associative Algebras; the later postulational methods in this field; and finally the extension to Linear Associative Algebras of infinite order.

5. In this paper Professor Parkinson showed how the fundamentals of the theory of twisted curves and surfaces in Differential Geometry can be readily developed by vector methods. He developed the Frenet-Serret formulas and the formula of Lancret for twisted curves. The coefficients of the first fundamental form in the theory of surfaces were developed by setting up the unit tangent vectors and the surface normal at a point of a surface. The introductory portion of the paper was devoted to a brief history of the ideas presented.

6. Professor Knight reported the results of a study of placement scores and marks in college mathematics classes for the seven semesters ending in February 1936. The scores which entering Freshmen made in the Iowa Mathematics Placement Tests were used as a basis for advising them what course to enter. Results showed that the percentage of failures among students who did not follow the advice given was much greater than the general percentage among all students. A scatter diagram showed correlations between scores on the tests and college grades in mathematics to such an extent that the tests will continue to be used as a basis for advising Freshmen what course to enter with the greatest chance of success.

7. Mathematics as an element in general education is usually justified by rather general statements concerning the values to be derived from its study. More evidence is needed to prove that these values are being achieved. Achievement of many of the values implies an understanding of certain major generalizations concerning mathematics as a mode of thought. "Mathematics is abstract and symbolic" is an example. Professor Hartung believes that an authoritative list of such statements in precise form is needed; that units should be constructed which aim directly at these generalizations as major, rather than incidental, outcomes; and that means of obtaining evidence that pupils understand the generalizations should be found.

ETHELWYNN R. BECKWITH, *Secretary*

SOLUTIONS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS IN THE VICINITY OF SINGULAR POINTS

By JESSE PIERCE, Heidelberg College

Introduction. J. A. Nyswander has determined the general solution of the system of linear differential equations

$$t\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij}(t)\cdot x_j, \qquad (i = 1, \cdots, n), \tag{1}$$

where the $\theta_{ij}(t)$ are analytic in t in the vicinity of the origin.

The solutions are shown to occur in groups of the form

$$x_i = t^{\lambda_\sigma}[P_{i0}(t) + P_{i1}(t)\log t + \cdots + P_{ih}(t)\log^h t], \qquad (i = 1, \cdots, n;\ h = 0, 1, \cdots, \epsilon), \tag{2}$$

where the $P_{ih}(t)$ are power series in t and λ_σ is a root of the λ-determinant $|\theta_{ij}(0)-\lambda\delta_{ij}|$.*

The author, under the direction of Nyswander, determined the general solution of equations (1) where the $\theta_{ij}(t)$ are expansible in Laurent series in t, for his doctor's dissertation.

The solutions have the same form as in the case treated by Nyswander except that the $P_{ih}(t)$ are Laurent series instead of power series.

In the present paper solutions of systems of linear differential equations are discussed in which the $\theta_{ij}(t)$ are not required to be analytic in t but satisfy certain integrability properties.

The solutions are found in terms of infinite series of definite integrals.

There will be two cases considered.

Case 1. The first system of linear differential equations to be discussed has the form

$$\frac{dx_i}{dt} = \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] x_i + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^\beta} x_j + \frac{\theta_i(t)}{t^\alpha}, \qquad (i = 1, \cdots, n; j \neq i), \tag{3}$$

where α and β are real positive constants, not necessarily integers, which satisfy the inequality

$$\alpha \geqq \beta > 1. \tag{4}$$

The path of integration for the definite integrals occuring in the solution functions will be the straight line drawn from the origin to the point t_0 and

$$0 \leqq \rho \leqq \rho_0, \tag{5}$$

where $|t| = \rho$.

The coefficient functions are assumed to satisfy the following relations:

I. The functions $\theta_{ij}(t)$, $j \neq i$, are integrable (Riemann), but not necessarily analytic, for all values of t on the chosen path of integration and satisfy the inequalities

$$|\theta_{ij}(t)| \leqq A, \tag{6}$$

where A is a real positive constant.

II. The functions $\theta_{ii}(t)$ are bounded and have the indefinite integrals

$$\int \theta_{ii}(t)dt = F_{ii}(t), \tag{7}$$

which satisfy the inequalities

$$e^{|F_{ii}(t)|} \leqq D, \tag{8}$$

where D is a positive constant and t lies on the path of in

III. The function $\theta(t)/t^\alpha$ has the indefinite integral

* American Journal of Mathematics, vol. 51, April, 1929.

$$\int (\theta(t)/t^{\alpha})dt = F(t), \tag{9}$$

which is finite in the deleted vicinity of the origin and satisfies the inequality

$$e^{-M\rho^{1-\alpha}} \leqq \left| e^{F(t)} \right| \leqq e^{-m\rho^{1-\alpha}}, \tag{10}$$

where

$$m > \frac{(n-1)AD^2\rho^{\alpha-\beta}}{\beta-1}, \quad +\infty > M \geqq m, \tag{11}$$

for all values of t on the path of integration except $t=0$.

IV. The function $\theta_i(t)$ has the form

$$\theta_i(t) = \phi_i(t)\cdot e^{-m_1 t^{1-\alpha}} \tag{12}$$

where $m_1 > M$ and

$$\left| \phi_i(t) \right| \leqq E, \tag{13}$$

for all values of t on the chosen path of integration.

Case 2. If any of the conditions imposed in case 1 are not satisfied and the system of differential equations has the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^{\alpha}} x_j + \frac{\theta_i(t)}{t^{\alpha}}, \qquad (i = 1, \cdots, n), \tag{14}$$

where the $\theta_{ij}(t)$, $\theta_i(t)$ are bounded integrable (Riemann) functions of t in the vicinity of the origin the path of integration will be chosen as any curve from t_0 to t that does not pass through the origin and is of finite length. The length of the path of integration will be represented by u and the shortest distance from the origin to the path of integration will be respresented by ρ_2.

The functions $\theta_{ij}(t)$, $\theta_i(t)$ are assumed to be integrable (Riemann) and satisfy the inequalities

$$\left| \theta_{ij}(t) \right| \leqq A, \tag{6}$$

$$\left| \theta_i(t) \right| \leqq A_1, \tag{15}$$

for all values of t on the chosen path of integration.

In article 1 the general solution of a system of homogeneous linear differential equations, case 1, is derived and a particular solution of a system of non-homogeneous linear differential equations is found in article 2.

In articles 3 and 4 solutions of systems of homogeneous and non-homogeneous linear differential equations, case 2, are discussed.

1. *Solutions of Homogeneous Linear Differential Equations*, *Case* 1.

Consider the system of homogeneous linear differential equations

$$\frac{dx_i}{dt} = \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] x_i + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^\beta} x_j, \qquad (i = 1, \cdots, n; j \neq i), \tag{16}$$

where the assumptions I, II, III, are satisfied.

The transformation

$$x_i = \sum_{h=1}^{\infty} y_{ih} \tag{17}$$

reduces equations (16) to the form

$$\sum_{h=1}^{\infty} \frac{dy_{ih}}{dt} = \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] \sum_{h=1}^{\infty} y_{ih} + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^\beta} \sum_{h=1}^{\infty} y_{jh}, \qquad (i = 1, \cdots, n; j \neq i). \tag{18}$$

A formal solution of equations (18) can be found by solving the system of linear differential equations

$$\left\{\begin{aligned} \frac{dy_{i1}}{dt} &= \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] y_{i1} \\ \frac{dy_{ih}}{dt} &= \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] y_{ih} + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^\beta} y_{jh-1}. \end{aligned}\right. \tag{19}$$

Equations (19) have the formal solution

$$\left\{\begin{aligned} y_{i1} &= c_i e^{F(t)+F_{ii}(t)} \equiv \eta_{i1}(t) \cdot e^{F(t)+F_{ii}(t)} \\ y_{ih} &= e^{F(t)+F_{ii}(t)} \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\theta_{ij}(t_1)}{t_1^\beta} \cdot \eta_{jh-1}(t_1) \cdot e^{F_{jj}(t_1)-F_{ii}(t_1)} dt_1 \equiv \eta_{ih}(t) \cdot e^{F(t)+F_{ii}(t)} \end{aligned}\right. \tag{20}$$

where the c_i are arbitrary constants.

It follows from (4), (5), (6), (8), (11) and (20) that

$$\left\{\begin{aligned} |\eta_{i1}(t)| &= |c_i| \leqq R_1 \\ |\eta_{i2}(t)| &\leqq -\int_{\infty}^{\rho} \frac{(n-1)AD^2R_1 d\rho_1}{\rho_1^\beta} = \frac{(n-1)AD^2R_1\rho^{1-\beta}}{\beta - 1} \\ |\eta_{i3}(t)| &\leqq -\int_{\infty}^{\rho} \frac{(n-1)AD^2}{\rho_1^\beta} \cdot \frac{(n-1)AD^2R_1\rho_1^{1-\beta}}{\beta-1} d\rho_1 \\ &= \frac{\left[\dfrac{(n-1)AD^2\rho^{1-\beta}}{\beta-1}\right]^2 R_1}{2!} \\ |\eta_{ih}(t)| &\leqq \frac{\left[\dfrac{(n-1)AD^2\rho^{1-\beta}}{\beta-1}\right]^{h-1} R_1}{(h-1)!}. \end{aligned}\right. \tag{21}$$

Hence

$$\sum_{h=1}^{\infty} \left| \eta_{ih}(t) \right| \leqq R_1 e^{(n-1)AD^2\rho^{1-\beta}/\beta-1}. \tag{22}$$

Therefore the general solution of (16) is

$$x_i = e^{F(t)+F_{ii}(t)} \sum_{h=1}^{\infty} \eta_{ih}(t), \tag{23}$$

which is absolutely convergent for all values of t on the path of integration except, possibly, the point $t=0$.

It follows from (8), (10) and (22) that

$$\left| x_i \right| = \left| e^{F(t)+F_{ii}(t)} \sum_{h=1}^{\infty} \eta_{ih}(t) \right| \leqq DR_1 e^{[(n-1)AD^2\rho^{1-\beta}/(\beta-1)]-m\rho^{1-\alpha}}. \tag{24}$$

It follows from (11) that the exponent of e in (24) is negative and hence

$$\lim_{t\to 0} \left| x_i \right| = \lim_{t\to 0} \left| e^{F(t)+F_{ii}(t)} \sum_{h=1}^{\infty} \eta_{ih}(t) \right| = 0. \tag{25}$$

2. *Particular Solutions of Non-Homogeneous Linear Differential Equations, Case* 1.

Consider the system of non-homogeneous linear differential equations

$$\frac{dx_i}{dt} = \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] x_i + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^\beta} x_j + \frac{\phi_i(t) e^{-m_1^{1-\alpha}}}{t^\alpha}, \quad \begin{pmatrix} i = 1, \cdots, n \\ j \neq i \end{pmatrix}, \tag{26}$$

where the relations (9), (12) and the inequalities (6), (11), (13) and (15) are satisfied.

The transformation equations (17) reduces equations (26) to the form

$$\sum_{h=1}^{\infty} \frac{dy_{ih}}{dt} = \left[\frac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] \sum_{h=1}^{\infty} y_{ih} + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^\beta} \sum_{h=1}^{\infty} y_{jh} + \frac{\phi_i(t)}{t^\alpha} e^{-m_1 t^{1-\alpha}}, \quad \begin{pmatrix} i = 1, \cdots, n \\ j \neq i \end{pmatrix}.$$

A formal solution of equations (27) can be found by solving the system

$$\begin{cases} \dfrac{dy_{i1}}{dt} = \left[\dfrac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] y_{i1} + \dfrac{\phi_i(t)}{t^\alpha} e^{-m_1 t^{1-\alpha}} & (i = 1, \cdots, n) \\ \dfrac{dy_{ih}}{dt} = \left[\dfrac{\theta(t)}{t^\alpha} + \theta_{ii}(t)\right] y_{ih} + \displaystyle\sum_{j=1}^{n} \dfrac{\theta_{ij}(t)}{t^\beta} y_{jh-1}, & (j \neq i;\ h = 2, 3, \cdots). \end{cases} \tag{28}$$

A particular solution of equations (28) is easily found in the form

$$(29)\quad \begin{cases} y_{i1} = e^{F(t)+F_{ii}(t)} \displaystyle\int_{t_0}^{t} \frac{\phi_i(t_1)}{t_1^{\alpha}} e^{-m_1 t_1^{1-\alpha}-F(t_1)-F_{ii}(t_1)} dt_1 \equiv \xi_{i1}(t)\cdot e^{F(t)+F_{ii}(t)} \\ y_{ih} = e^{F(t)+F_{ii}(t)} \displaystyle\int_{t_0}^{t} \sum_{j=1}^{n} \frac{\theta_{ij}(t_1)}{t_1^{\beta}} \xi_{jh-1}(t_1)\cdot e^{F_{jj}(t_1)-F_{ii}(t_1)} dt_1 \equiv \xi_{ih}(t)\cdot e^{F(t)+F_{ii}(t)}. \end{cases}$$

It follows from (6), (8), (12), (13) and (29) that

$$(30)\quad \begin{cases} |\xi_{i1}(t)| \leqq -\displaystyle\int_{\rho_0}^{\rho} \frac{DEe^{(M-m_1)\rho_1^{1-\alpha}}}{\rho_1^{\alpha}} d\rho_1 < \frac{DEe^{(M-m_1)\rho_0^{1-\alpha}}}{(m_1-M)(\alpha-1)} \\ |\xi_{i2}(t)| \leqq -\displaystyle\int_{\infty}^{\rho} \frac{(n-1)AD^3Ee^{(M-m_1)\rho_0^{1-\alpha}}}{(m_1-M)(\alpha-1)}\cdot \rho_1^{-\beta} d\rho_1 \\ \qquad = \dfrac{(n-1)AD^3Ee^{(M-m_1)\rho_0^{1-\alpha}}}{(m_1-M)(\alpha-1)}\cdot\dfrac{\rho^{1-\beta}}{\beta-1} \\ |\xi_{ih}(t)| \leqq \dfrac{DEe^{(M-m_1)\rho_0^{1-\alpha}}}{(m_1-M)(\alpha-1)}\cdot\dfrac{\left[\dfrac{(n-1)AD^2\rho^{1-\beta}}{\beta-1}\right]^{h-1}}{(h-1)!}. \end{cases}$$

It is evident that

$$(31)\quad \sum_{h=1}^{\infty} |\xi_{ih}(t)| \leqq \frac{DEe^{(M-m_1)\rho_0^{1-\alpha}}}{(m_1-M)(\alpha-1)} e^{(n-1)AD^2\rho^{1-\beta}/(\beta-1)}.$$

Hence the general solution of equations (26) is

$$(32)\quad x_i = e^{F(t)+F_{ii}(t)} \sum_{h=1}^{\infty} [\eta_{ih}(t) + \xi_{ih}(t)],$$

where

$$(33)\quad |x_i| \leqq \left[R_1 + \frac{DEe^{(M-m_1)\rho_0^{1-\alpha}}}{(m_1-M)(\alpha-1)}\right] De^{[(n-1)AD^2\rho^{1-\beta}/(\beta-1)]-m\rho^{1-\alpha}},$$

and $x_i(0)=0$ for all finite values of the c_i.

3. *Solutions of Systems of Homogeneous Linear Differential Equations, Case 2.*

Consider the system of homogeneous linear differential equations

$$(34)\quad \frac{dx_i}{dt} = \frac{\theta_{ii}(t)}{t^{\alpha}} x_i + \sum_{j=1}^{n} \frac{\theta_{ij}(t)}{t^{\alpha}}\cdot x_j, \qquad (i = 1, \cdots, n),$$

where the conditions of case 2 of the introduction are satisfied and

$$(35)\quad \int \frac{\theta_{ii}(t)}{t^{\alpha}} dt = G_{ii}(t),$$

which satisfy the inequalities

$$|G_{ii}(t)| \leqq \frac{A}{\rho_2^{\alpha}}, \tag{36}$$

for all values of t on the chosen path of integration.

A formal solution of equations (34) is readily found in the form (17) where

$$\begin{cases} y_{i1} = c_i e^{G_{ii}(t)} \equiv \eta_{i1}(t)\cdot e^{G_{ii}(t)} \\ y_{ih} = e^{G_{ii}(t)} \displaystyle\int_{t_0}^{t}\sum_{j=1}^{n} \frac{\theta_{ij}(t_1)}{t_1^{\alpha}}\eta_{jh-1}(t_1)\cdot e^{G_{jj}(t_1)-G_{ii}(t_1)}dt_1 \equiv \eta_{ih}(t)\cdot e^{G_{ii}(t)}. \end{cases} \tag{37}$$

It follows from (6), (13), (15), (36) and (37) that

$$\begin{cases} |\eta_{i1}(t)| = |c_i| \leqq R_1 \\ |\eta_{i2}(t)| \leqq \displaystyle\int_0^u \frac{(n-1)Ae^{2A\rho_2^{-\alpha}}R_1}{\rho_2^{\alpha}}du = \frac{(n-1)Ae^{2A\rho_2^{-\alpha}}R_1}{\rho_2^{\alpha}}u \\ |\eta_{ih}(t)| \leqq \dfrac{\left[\dfrac{(n-1)Ae^{2A\rho_2^{-\alpha}}u}{\rho_2^{\alpha}}\right]^{h-1}R_1}{(h-1)!}. \end{cases} \tag{38}$$

Hence

$$\sum_{h=1}^{\infty}|\eta_{ih}(t)| \leqq R_1 e^{(n-1)Ae^{2A\rho_2^{-\alpha}}u/\rho_2^{\alpha}} \tag{39}$$

Therefore the general solution

$$x_i = e^{G_{ii}(t)}\sum_{h=1}^{\infty}\eta_{ih}(t), \tag{40}$$

of equations (34) is absolutely convergent and satisfies the inequality

$$|x_i| \leqq R_1 e^{[A+(n-1)Ae^{2A\rho_2^{-\alpha}}u]\rho_2^{-\alpha}}. \tag{41}$$

4. *Solutions of Systems of Non-Homogeneous Linear Differential Equations, Case* 2.

Consider the system of non-homogeneous linear differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n}\frac{\theta_{ij}(t)}{t^{\alpha}} + \frac{\theta_i(t)}{t^{\alpha}}, \qquad (i = 1, \cdots, n), \tag{42}$$

where

$$|\theta_i(t)| \leqq A_1 \tag{15}$$

when $|t| \leqq r$.

A formal solution of equations (42) can be found by the method of article (2) in the form (17) where

$$
(43)\quad \begin{cases} y_{i1} = e^{G_{ii}(t)} \displaystyle\int_{t_0}^{t} \frac{\theta_i(t_1)}{t_1^\alpha} e^{-G_{ii}(t_1)} dt_1 \equiv \xi_{i1}(t) \cdot e^{G_{ii}(t)} \\ y_{ih} = e^{G_{ii}(t)} \displaystyle\int_{t_0}^{t} \sum_{j=1}^{n} \frac{\theta_{ij}(t_1)}{t_1^\alpha} \cdot \xi_{jh-1}(t_1) \cdot e^{G_{jj}(t_1) - G_{ii}(t_1)} dt_1 \equiv \xi_{ih}(t) \cdot e^{G_{ii}(t)} \end{cases}
$$

It follows from (6), (15), (36), (42) and (43) that

$$
(44)\quad \begin{cases} \left| \xi_{i1}(t) \right| \leqq \displaystyle\int_0^u \frac{A_1}{\rho_2^\alpha} e^{A\rho_2^{-\alpha}} du_1 = \frac{A_1 e^{A\rho_2^{-\alpha}} u}{\rho_2^\alpha} \\ \left| \xi_{i2}(t) \right| \leqq \displaystyle\int_0^u \frac{(n-1)Ae^{2A\rho_2^{-\alpha}}}{\rho_2^\alpha} \cdot \frac{A_1 e^{A\rho_2^{-\alpha}}}{\rho_2^\alpha} u_1 du_1 = \frac{(n-1)AA_1 e^{3A\rho_2^{-\alpha}}}{\rho_2^{2\alpha}} \cdot \frac{u^2}{2!} \\ \left| \xi_{ih}(t) \right| \leqq \dfrac{A_1 e^{-A\rho_2^{-\alpha}}}{(n-1)A} \dfrac{\left[\dfrac{(n-1)Ae^{2A\rho_2^{-\alpha}} u}{\rho_2^\alpha}\right]^h}{h!}. \end{cases}
$$

Hence

$$
(45)\quad \sum_{h=1}^{\infty} \left| \xi_{ih}(t) \right| \leqq \frac{A_1 e^{-A\rho_2^{-\alpha}}}{(n-1)A} \left[e^{(n-1)Ae^{2A\rho_2^{-\alpha}} u/\rho_2} - 1 \right].
$$

Therefore a particular solution of equations (42) is

$$
(46)\quad x_i = e^{G_{ii}(t)} \sum_{h=1}^{\infty} \xi_{ih}(t),
$$

which is absolutely convergent and satisfies the inequalities

$$
(47)\quad \left| x_i \right| \leqq \frac{A_1}{(n-1)A} \left[e^{(n-1)Ae^{2A\rho_2^{-\alpha}} u/\rho_2^\alpha} - 1 \right].
$$

It follows from (40) and (46) that the general solution of (42) is

$$
(48)\quad x_i = e^{G_{ii}(t)} \sum_{h=1}^{\infty} \left[\eta_{ih}(t) + \xi_{ih}(t) \right],
$$

which satisfies the inequalities

$$
(49)\quad \left| x_i \right| \leqq \left[R_1 e^{A\rho_2^{-\alpha}} + \frac{A_1}{(n-1)A} \right] e^{(n-1)Ae^{2A\rho_2^{-\alpha}} u/\rho_2^\alpha} - \frac{A_1}{(n-1)A}.
$$

Hence the general solution (48) of equations (42) will converge for all values of t in the deleted neighborhood of the origin provided the length of the path of integration is finite and does not pass through the origin. It will converge at the origin also provided the conditions of articles 1 and 2 are satisfied.

Conclusion: In the foregoing articles it was assumed that $\alpha \geqq \beta > 1$. The same results obtain when $\beta \leqq 1$ which can be proven by a slight modification of the proofs given above.

The solutions obtained by the methods of the present paper are found directly in terms of the coefficient functions of the differential equations and do not depend upon the properties of λ-determinants as is the case with the solutions by Nyswander and the author.

The coefficient functions are not necessarily analytic in t and therefore the differential equations (42) are more general than (1).

By taking the initial value t_0 of t distinct from zero for the definite integrals entering in the solutions it is possible to find the general solution in the deleted neighborhood of the origin in case 2 and to show that every solution of case 1 has the value zero when $t=0$.

The method of successive approximations is not applicable to case 1 as the right-hand members of equations (26) are not integrable with respect to t at the origin.

The solution functions (48) will converge for all values of t for which (6), (15) and (36) are satisfied provided the path of integration is of finite length and does not pass through the origin whereas any solution function given in terms of Laurent series will converge for values of t for which the $|t|$ is less than the distance from the origin to the nearest singular point of the $\theta_{ij}(t)$, $\theta_i(t)$.

Example of Case 1. The system of homogeneous linear differential equations

$$\frac{dx_1}{dt} = \frac{2t^3+3}{t^2}\,x_1 - \frac{2}{t^2}\,x_2$$

$$\frac{dx_2}{dt} = -\frac{1}{t^2}\,x_1 + \frac{2t^3+3}{t^2}\,x_2$$

has the general solution

$$x_1 = e^{t^2-3t^{-1}}\left[c_1 + 2c_2t^{-1} + c_1t^{-2} + \tfrac{2}{3}c_2t^{-3} + \frac{2}{3\cdot 4}\,c_1t^{-4} + \frac{2^2}{3\cdot 4\cdot 5}\,c_2t^{-5} + \cdots\right]$$

$$x_2 = e^{t^2-3t^{-1}}\left[c_2 + c_1t^{-1} + c_2t^{-2} + \tfrac{1}{3}c_1t^{-3} + \frac{2}{3\cdot 4}\,c_2t^{-4} + \frac{2}{3\cdot 4\cdot 5}\,c_1t^{-5} + \cdots\right]$$

which converges for all finite values of t, and $x_i(0)=0$.

Example of Case 2. The system of homogeneous linear differential equations

$$\frac{dx_1}{dt} = \frac{2t^3+t+4}{t^2}\,x_1 + \frac{4t^3-2}{t^2}\,x_2$$

$$\frac{dx_2}{dt} = \frac{3-6t^3}{t^2}\,x_1 + \frac{4t^3+t+3}{t^2}\,x_2$$

has the general solution

$$
\begin{aligned}
x_1 = {} & c_1e^R + 2c_2(e^S - e^R) + 6c_1\left[e^R\{(t^2 + t^{-1}) - (t_0^2 + t_0^{-1}) + 1\} - e^S\right] \\
& + 12c_2\left[e^S\{-(t^2 + t^{-1}) + (t_0^2 + t_0^{-1}) + 2\}\right. \\
& \left. - e^R\{(t^2 + t^{-1}) - (t_0^2 + t_0^{-1}) + 2\}\right] + \cdots \\
x_2 = {} & c_2e^S + 3c_1(e^R - e^S) + 6c_2\left[e^S\{-(t^2 + t^{-1}) + (t_0^2 + t_0^{-1}) + 1\} - e^R\right] \\
& + 18c_1\left[e^R\{(t^2 + t^{-1}) - (t_0^2 + t_0^{-1}) + 2\}\right. \\
& \left. - e^S\{-(t^2 + t^{-1}) + (t_0^2 + t_0^{-1}) + 2\}\right] + \cdots
\end{aligned}
$$

where

$$
\begin{aligned}
R &= t^2 + \log t/t_0 - 4t^{-1} - t_0^2 + 4t_0^{-1} \\
S &= 2t^2 + \log t/t_0 - 3t^{-1} - 2t_0^2 + 3t_0^{-1}.
\end{aligned}
$$

This solution will converge for all finite values of t and t_0 except the origin.

THE TRIANGLE BORDERED WITH SQUARES

By J. R. MUSSELMAN, Western Reserve University

1. One of the most interesting figures in "The Geometry of the Triangle" is that of a triangle with squares constructed internally and externally upon its sides. J. Neuberg in his *Bibliographie du Triangle et du Tétraèdre** has listed the properties of this figure which had been discovered before 1923. More recently Thébault† has published several supplements to *Mathesis* which contain proofs of many more theorems concerning this figure. It is the purpose of this paper to show how some of these results may be easily proved by an analytic method, and to use this method to obtain additional properties of the figure. We choose any pair of perpendicular lines as the axes of a rectangular coordinate system and name as the positive direction of rotation that which keeps the area of a polygon always on the left as $ABCDA$. We shall designate by the single complex numbers a, b, c respectively the coordinates of the vertices of the triangle ABC—in general the letter p will indicate the complex coordinate of the point P. As far as possible we shall use the same notation that prevails among the European writers.

2. At any vertex of a triangle the line which makes with the median an angle whose bisector is likewise the bisector of the angle of the triangle at that vertex is called a symmedian line. The intersection of the three medians is the centroid or center of gravity of the triangle; the intersection of the three symmedian lines is the symmedian point of the triangle.

In any triangle ABC there is one point Ω such that $\angle\Omega AB = \angle\Omega BC = \angle\Omega CA = \omega$; and one point Ω' such that $\angle\Omega' BA = \angle\Omega' CB = \angle\Omega' AC = \omega'$. These points Ω and Ω' are called the Brocard points of the triangle, and the angle ω or its equal ω' is the Brocard angle of the triangle. Let us designate by A',

* Mathesis, vol. 37 (1923), pp. 289–293.

† Mathesis, vol. 47 (1933); vol. 48, (1934).

B', C' the centers of the squares $CBA_2'A_1'$, $ACB_2'B_1'$, $BAC_2'C_1'$ constructed externally on the sides BC, CA, AB of the triangle ABC; by A'', B'', C'' the centers of the squares $BCA_1''A_2''$, $CAB_1''B_2''$, $ABC_1''C_2''$ constructed internally; by G, K and θ the centroid, symmedian point and Brocard angle of the triangle ABC; by W_a', W_b', W_c', M_a', M_b', M_c' the centers of the positive squares constructed on $C_2'B_1'$, $A_2'C_1'$, $B_2'A_1'$, $C_1'B_2'$, $A_1'C_2'$, $B_1'A_2'$; by W_a'', W_b'', W_c'', M_a'', M_b'', M_c'' the centers of the positive squares constructed on $B_1''C_2''$, $C_1''A_2''$, A_1'', B_2'', $B_2''C_1''$, $C_2''A_1''$, $A_2''B_1''$; by A_m, B_m, C_m, A_m', B_m', C_m', A_m'', B_m'', C_m'' the midpoints of the segments BC, CA, AB, $B_1'C_2'$, $C_1'A_2'$, $A_1'B_2'$, $B_1''C_2''$, $C_1''A_2''$, $A_1''B_2''$ and by P_a, P_b, P_c the centers of the positive squares constructed either on $B_2'C_1'$, $C_2'A_1'$, $A_2'B_1'$ or on $C_1''B_2''$, $A_1''C_2''$, $B_1''A_2''$.

3. If $BCA_1''A_2''$ is a positive square the rank* of the matrix

$$\begin{vmatrix} b & c & a_1'' & a_2'' \\ 1 & i & -1 & -i \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 \qquad (i^2 = -1)$$

must be two. Hence

$$a_1'' = (1+i)c - ib, \qquad a_2'' = ic + (1-i)b$$

and the coordinate of A'', the center of the square, is

$$a'' = \frac{(1-i)b + (1+i)c}{2}$$

Also

$$a_1' = (1-i)c + ib, \qquad a_2' = (1+i)b - ic,$$

$$a' = \frac{(1+i)b + (1-i)c}{2}.$$

$$a_m' = \frac{2a - ib + ic}{2}, \qquad a_m'' = \frac{2a + ib - ic}{2},$$

$$m_a' = a - ib + ic, \qquad m_a'' = a + ib - ic,$$

$$w_a' = \frac{4a - (1+i)b - (1-i)c}{2}, \qquad w_a'' = \frac{4a - (1-i)b - (1+i)c}{2}$$

$$p_a = -a + b + c..$$

The coordinates of the points B' and C' can be deduced from these of A' by circular permutations of the letters a, b, c and similarly for all of the other points.

* See J. R. Musselman, this MONTHLY, vol. 40 (1933), p. 159.

4. Since

$$
\begin{aligned}
a+b+c &= a''+b''+c'' = a_m+b_m+c_m = a_m'+b_m'+c_m' \\
&= a_m''+b_m''+c_m'' = m_a'+m_b'+m_c' = m_a''+m_b''+m_c'' \\
&= w_a'+w_b'+w_c' = w_a''+w_b''+w_c'' = p_a+p_b+p_c \\
&= a_1'+b_1'+c_1' = a_2'+b_2'+c_2' = a_1''+b_1''+c_1'' \\
&= a_2''+b_2''+c_2'' = a'+b'+c',
\end{aligned}
$$

we have the theorem that *the centroid G of the triangle ABC is likewise the centroid of the triangles $A'B'C'$, $A''B''C''$, $A_mB_mC_m$, $A_m'B_m'C_m'$, $A_m''B_m''C_m''$, $M_a'M_b'M_c'$, $M_a''M_b''M_c''$, $W_a'W_b'W_c'$, $W_a''W_b''W_c''$, $P_aP_bP_c$, $A_1'B_1'C_1'$, $A_2'B_2'C_2'$, $A_1''B_1''C_1''$*, and *$A_2''B_2''C_2''$*.

5.1 The triangles $A_m'BC$, $A_mB'C'$, $A''B_mC_m$, $A''AA_m$, $AB_m''C_m''$, $A_m'A'A''$, $A_m'A_1'A_2''$, $A_m'A_2'A_1''$ and $W_a'P_aA_m$ have the same centroid G_a' whose coordinate is given by

$$g_a' = g + i(c-b)/6.$$

Hence the segment GG_a' is perpendicular to BC and equal to one-sixth of BC.

5.2 The triangles $A_m''BC$, $A_mB''C''$, $A'B_mC_m$, $A'AA_m$, $AB_m'C_m'$, $A_m''A'A''$, $A_m''A_1'A_2''$, $A_m''A_2'A_1''$ and $W_a''P_aA_m$ have the same centroid G_a'' whose coordinate is given by

$$g_a'' = g + i(b-c)/6.$$

Hence the segment GG_a'' is perpendicular to CB and equal to one-sixth of CB.

5.3 The triangles $M_a'BC$, $M_a'A'A''$, $M_a'A_1'A_2''$, $M_a'A_2'A_1''$, $AM_b''M_c''$ $A_1''AB$, $A_2''AC$, have the same centroid N_a' whose coordinate is

$$n_a' = g + i(c-b)/3.$$

Hence the segment GN_a' is perpendicular to BC and equal to one third of BC.

5.4 The triangles $M_a''BC$, $M_a''A'A''$, $M_a''A_1'A_2''$, $M_a''A_2'A_1''$, $AM_b'M_c'$, $A_1'AB$, $A_2'AC$ all have the same centroid N_a'' whose coordinate is

$$n_a'' = g + i(b-c)/3.$$

Hence the segment GN_a'' is perpendicular to CB and equal to one third of CB.

5.5 By circular permutation of the letters a, b, c we obtain points G_b', G_c', G_b'', G_c'', N_b', N_c', N_b'', N_c'' with properties corresponding to the points G_a', G_a'', N_a', N_a''. Since the vector $\overline{G_a'G_b'} = i\overline{CG}/2$, not only are the sides of the triangle ABC perpendicular to the medians of the triangle $G_a'G_b'G_c'$ and four times their length, but also the sides of the triangle $G_a'G_b'G_c'$ are perpendicular to the medians of the triangle ABC and equal to one-third of their length. Since the vector $\overline{N_a'N_b'} = i\overline{CG}$, not only are the sides of the triangle ABC perpendicular to the medians of the triangle $N_a'N_b'N_c'$ and equal to two times their length, but also the sides of the triangle $N_a'N_b'N_c'$ are perpendicular to the medians of the

triangle ABC and equal to two-thirds of their length. The triangles $G_a''G_b''G_c''$ and $N_a''N_b''N_c''$ are symmetric to $G_a'G_b'G_c'$ and $N_a'N_b'N_c'$ respectively with G as the center of symmetry.

5.6 The Lagrange resolvents* of the hexagon $G_a'G_c''G_b'G_a''G_c'G_b''$ show $V_2=V_3=V_4=0$. This implies that the diagonals of the hexagon meet at a point; the opposite sides of the hexagon are equal and negatively parallel. The component triangles $G_a'G_b'G_c'$ and $G_a''G_b''G_c''$ have their corresponding sides equal and negatively parallel, they are inversely equivalent in area and perspective from G. The area of each triangle is one-twelfth that of ABC while the area of the hexagon is one-sixth that of ABC. A similar situation holds for the hexagon $N_a'N_c''N_b'N_a''N_c'N_b''$, but here the area of each triangle $N_a'N_b'N_c'$ or $N_a''N_b''N_c''$ is one-third that of ABC, while the area of the hexagon is two-thirds that of ABC.

6. The following formula gives the area S of the triangle ABC

$$-4iS = \begin{vmatrix} a & b & c \\ \bar{a} & \bar{b} & \bar{c} \\ 1 & 1 & 1 \end{vmatrix}$$

where $\bar{a}$ is the complex number conjugate to a. For any quadrilateral $ABCD$ the expression $(c-a)(\bar{b}-\bar{d})$ is a complex number whose real part is twice the norm† of the quadrilateral while the coefficient of the imaginary part gives twice the area of the quadrilateral. For a square or a pseudosquare the norm is zero. (Pseudo square is the name given by Neuberg to any quadrilateral whose diagonals are equal and perpendicular to each other. If in addition, the diagonals bisect each other, the quadrilateral is a square.)

7.1 Since

$$\overline{M_a''A_m''} = \overline{A_m''A} = \overline{AA_m'} = \overline{A_m'M_a'} = i(c-b)/2 = i\overline{BC}/2$$

the points M_a'', A_m'', A, A_m', M_a' lie on a line, an altitude of the triangle ABC, such that the distance between any two of the points as ordered above is $\overline{BC}/2$.

7.2 Since $\overline{A'A_m}=\overline{A_mA''}=i\overline{BC}/2$ the points A', A_m, and A'' are collinear with A_m as the mid-point.

7.3 Since

$$\overline{A'A} = \overline{AW_a'} = \frac{2a-(1+i)b-(1-i)c}{2} = \overline{P_aA''} = \overline{A''M_a'}$$

the points A', A and W_a' are collinear with A as the mid-point; the points P_a, A'', and M_a' are collinear with A'' as the mid-point.

7.4 Since

* See J. R. Musselman, Am. Journal of Math., Vol. 57 (1935), p. 503.

† F. Morley and F. V. Morley, *Inversive Geometry*, 1933, p. 211.

$$\overline{A''A} = \overline{AW_a''} = \frac{2a-(1-i)b-(1+i)c}{2} = \overline{P_aA'} = \overline{A'M_a''},$$

we see that A is the mid-point of the segment $A''W_a''$, and A' is the mid-point of the segment P_aM_a''.

7.5 Since

$$\overline{A''A_m'} = \overline{A_m'W_a'} = (2a-b-c)/2 = \overline{P_aA_m} = \overline{A_mA} = \overline{A'A_m''} = \overline{A_m''W_a''},$$

the points A_m', A_m and A_m'' are the mid-points respectively of the segments $A''W_a'$, P_aA, and $A'W_a''$. The line P_aA is a median of the triangle ABC, the other two lines are parallel to it.

8.1 Since $\overline{W_a''W_a'} = i\overline{BC} = i\overline{P_cP_b}/2 = 2\overline{M_a''A_m''}$ we have, in connection with 7.1 and 7.2, three sets of equal parallelograms $M_a''A'A_mA_m''$, $A_m''A'A_mA$, $AA'A_mA_m'$, $A_m'A'A_mM_a'$, $M_a''A_mA''A_m''$, $A_m''A_mA''A$, $AA_mA''A_m'$, $A_m'A_mA''M_a'$; $M_a''W_a''W_a'A$, $A_m''W_a''W_a'A_m'$, $AW_a''W_a'M_a'$; $M_a''A'A''A$, $A_m''A'A''A_m'$, $AA'A''M_a'$. In addition $A'W_a''W_a'A''$ is a parallelogram. The quadrilaterals $BW_a''CW_a'$, $BM_a''CA$, $BA_m''CA_m'$, $BACM_a'$ and $BA'CA''$ are equal pseudosquares; the last named of these is a square with center at A_m and with the area $\overline{BC}^2/2$. Since $\overline{P_cP_b} = 2\overline{BC}$, ABC is the medial triangle of $P_aP_bP_c$.

8.2 Since

$$\overline{A''A_m'} = (2a-b-c)/2 = i\overline{C_2'B_1'}/2 = i\overline{B_1''C_2''}/2,$$

we have, in connection with 7.5 three groups of four equal parallelograms $A''P_aA_mA_m'$, $A''A_mAA_m'$, $A_m'P_aA_mW_a'$, $A_m'A_mAW_a'$; $A''A'A_m''A_m'$, $A''A_m''W_a''A_m'$, $A_m'A'A_m''W_a'$, $A_m'A_m''W_a''W_a'$; $P_aA'A_m''A_m$, $P_aA_m''W_a''A_m$, $A_mA'A_m''A$, $A_mA_m''W_a''A$. In addition $C_2'B_1''C_2''B_1'$ is a parallelogram. Also $A''C_2'W_a'B_1'$, $A''B_1''W_a'C_2''$, $P_aC_2'AB_1'$, $P_aB_1''AC_2''$, $A'C_2'W_a''B_1'$ and $A'B_1''W_a''C_2''$ are equal pseudosquares whose area is $s^2-3\overline{BC}^2/2$. ($s^2=\overline{AB}^2+\overline{BC}^2+\overline{CA}^2$). Of these pseudosquares $A''C_2'W_a'B_1'$ and $A'B_1''W_a''C_2''$ are squares with centers at A_m' and A_m'' respectively. We also notice that the segments $C_2'B_1'$ and $B_1''C_2''$ are perpendicular to the median from A of the triangle ABC and equal to twice its length.

8.3 From the vector equations

$$\overline{A'A} = \overline{A_mA_m'} = \overline{M_a''W_a''} = i\overline{C'B'} = i\overline{C_1'B_2'}/2 = i\overline{M_b'M_c'}/2,$$

together with 7.3, we deduce that the following are five sets of equal parallelograms: $A'P_aA''A$, $A'A''M_a'A$, $AP_aA''W_a'$, $AA''M_a'W_a'$; $A'A_mA_m'A$, $AA_mA_m'W_a'$; $P_aA_mA_m'A''$, $A''A_mA_m'M_a'$; $A'M_a''W_a''A$, $AM_a''W_a''W_a'$; $P_aM_a''W_a''A''$, $A''M_a''W_a''M_a'$. In addition $C_1'M_b'M_c'B_2'$, $A'P_aM_a'W_a'$, $A_mM_a''W_a''A_m'$ are parallelograms. There are two groups of equal pseudosquares, $C'A'B'A$, $C'AB'W_a'$, $C'P_aB'A''$, $C'A''B'M_a'$, $C'A_mB'A_m'$, $C'M_a''B'W_a''$; $C_1'A'B_2'W_a'$, $C_1'P_aB_2'M_a'$, $M_b'A'M_c'W_a'$, $M_b'P_aM_c'M_a'$. Of these $C_1'P_aB_2'M_a'$

and $C'A_mB'A_m'$ are squares with centers at A'' and the mid-point of $M_a'A'$ respectively. Since $M_b'M_c'=2C'B'$, we see that $A'B'C'$ is the medial triangle of $M_a'M_b'M_c'$.

8.4 From $\overline{A''A}=\overline{A_mA_m''}=\overline{M_a'W_a'}=i\overline{B''C''}=i\overline{B_2''C_1''}/2=i\overline{M_c''M_b''}/2$, together with 7.4, we have that the following are five sets of equal parallelograms: $A''P_aA'A, A''A'M_a''A, AP_aA'W_a'', AA'M_a''W_a''$; $A''A_mA_m''A, AA_mA_m''W_a''$; $P_aA_mA_m''A', A'A_mA_m''M_a''$; $A''M_a'W_a'A, AM_a'W_a'W_a''$; $P_aM_a'W_a'A', A'M_a'$-$W_a'M_a''$. In addition $B_2''M_c''M_b''C_1''$, $A''P_aM_a''W_a''$, $A_mM_a'W_a'A_m''$ are parallelograms. There are two groups of equal pseudosquares: $B''A''C''A$, $B''AC''W_a''$, $B''P_aC''A'$, $B''A'C''M_a''$, $B''A_mC''A_m''$, $B''M_a'C''W_a'$; B_2''-$A''C_1''W_a''$, $B_2''P_aC_1''M_a''$, $M_c''A''M_b''W_a''$, $M_c''P_aM_b''M_a''$. Of these $B_2''P_a$-$C_1''M_a''$ and $B''A_mC''A_m''$ are squares with centers at A' and the mid-point of $M_a''A''$ respectively. Since $M_c''M_b''=2B''C''$ we see that $A''B''C''$ is the medial triangle of $M_a''M_b''M_c''$.

8.5 From the equations $\overline{A'A_m'}=\overline{A_mM_a'}=\overline{A_m''W_a'}$, $\overline{A''A_m''}=\overline{A_mM_a''}=\overline{A_m'W_a''}$, $\overline{W_a''A_m}=\overline{A_m''P_a}$, $\overline{W_a'A_m}=\overline{A_m'P_a}$, $\overline{W_a''M_a''}=\overline{M_a'A'}$, $\overline{W_a''M_a'}=\overline{M_a''A''}$, we have two groups of equal parallelograms $A'A_mM_a'A_m'$, $A'A_m''$-$W_a'A_m'$, $A_mA_m''W_a'M_a'$; $A''A_mM_a''A_m''$, $A''A_m'W_a''A_m''$, $A_mA_m'W_a''M_a''$. In addition $W_a''A_mA_m''P_a$, $W_a'A_m'P_aA_m$, $W_a'M_a'A'M_a''$ and $W_a''M_a''A''M_a'$ are parallelograms. By circular permutation of the letters A, B, and C and the subscripts a, b, and c wherever they occur in sections 7 and 8 we will obtain twice as many more theorems as therein stated.

9. The following table gives the areas of the fourteen triangles named in section 4.

Triangle	*Area*	*Triangle*	*Area*
$A'B'C'$	$S+s^2/8$	$A''B''C''$	$S-s^2/8$
$A_m'B_m'C_m'$	$7S/4+s^2/4$	$A_m''B_m''C_m''$	$7S/4-s^2/4$
$M_a'M_b'M_c'$	$4S+s^2/2$	$M_a''M_b''M_c''$	$4S-s^2/2$
$W_a'W_b'W_c'$	$7S+5s^2/8$	$W_a''W_b''W_c''$	$7S-5s^2/8$
$A_1'B_1'C_1'$	$4S+s^2/4$	$A_1''B_1''C_1''$	$4S-s^2/4$
$A_2'B_2'C_2'$	$4S+s^2/4$	$A_2''B_2''C_2''$	$4S-s^2/4$
$P_aP_bP_c$	$4S$	$A_mB_mC_m$	$S/4$

10.1 Since

$$m_a' + 2a' = m_b' + 2b' = m_c' + 2c' = a + b + c$$

the lines $M_a'A'$, $M_b'B'$, $M_c'C'$ meet at the point G; also $M_a'G=2GA'$. These lines are the medians of $A'B'C'$ and of $M_a'M_b'M_c'$.

10.2 As

$$m_a'' + 2a'' = m_b'' + 2b'' = m_c'' + 2c'' = a + b + c$$

the lines $M_a''A''$, $M_b''B''$, $M_c''C''$ meet at the point G; also $M_a''G=2GA''$. These lines are the medians of $A''B''C''$ and of $M_a''M_b''M_c''$.

10.3 Since

$$p_a + 2a = p_b + 2b = p_c + 2c = a + b + c,$$

the lines P_aA, P_bB, P_cC meet at the point G; also $P_aG = 2GA$. The midpoint of the segment P_aA is A_m, hence these lines are the medians of ABC and of $P_aP_bP_c$.

10.4 From 7.1 we have that the lines $M_a''M_a'$, $M_b''M_b'$, $M_c''M_c'$ are the altitudes of ABC and thus meet at the point H, the orthocenter of ABC.

10.5 From 7.2 we see that the lines $A'A''$, $B'B''$, $C'C''$ are the perpendicular bisectors of the sides of ABC and hence meet at the point O.

10.6 The lines $W_a'W_a''$, $W_b'W_b''$, $W_c'W_c''$ meet at a point Y such that $\overline{OH} = \overline{HY}$.

10.7 In any triangle, the circumcenter, the orthocenter, and the centroid lie on a line known as the Euler line of the triangle. The line connecting the points G, O, H, Y is the Euler line of the triangles $A_mB_mC_m$, ABC and $P_aP_bP_c$.

11.1 The lines P_aM_a', P_bM_b', P_cM_c' meet at a point F' which is the orthocenter of $M_a'M_b'M_c'$. The point F' is connected with other points by the vector relation

$$(2 + \cot\theta)\overline{GF'} = 2\overline{GO} + 2\overline{KG}\cot\theta.$$

11.2 The lines $A'W_a'$, $B'W_b'$, $C'W_c'$ meet at a point R' which is the orthocenter of $A'B'C'$ and the circumcenter of $M_a'M_b'M_c'$.

11.3 The lines A_mA_m', B_mB_m', C_mC_m' meet at the point Q' which is the circumcenter of $A'B'C'$.

11.4 The lines $M_a''W_a''$, $M_b''W_b''$, $M_c''W_c''$ meet at a point T' such that R' is the midpoint of $T'F'$ or $T'G = 2GF'$.

11.5 If M_α', M_β', M_γ' are the points determined as follows

$$m_\alpha' + 2w_a'' = 3m_a'', \quad m_\beta' + 2w_b'' = 3m_b'', \quad m_\gamma' + 2w_c'' = 3m_c'',$$

then T' is the orthocenter and F' is the circumcenter of $M_\alpha'M_\beta'M_\gamma'$, which triangle has $M_a'M_b'M_c'$ as its medial triangle.

11.6 The points F', R', Q', T, G are collinear with $F'G = 2GR'$ and $R'G = 2GQ'$. This line is the Euler line of the triangles $A'B'C'$, $M_a'M_b'M_c'$ and $M_\alpha'M_\beta'M_\gamma'$.

12.1 The lines P_aM_a'', P_bM_b'', P_cM_c'' meet at a point F'' which is the orthocenter of $M_a''M_b''M_c''$. The point F'' is connected with other points by the vector relation

$$(2 - \cot\theta)\overline{GF''} = 2\overline{GO} + 2\overline{GK}\cot\theta.$$

12.2 The lines $A''W_a''$, $B''W_b''$, $C''W_c''$ meet at a point R'' which is the orthocenter of $A''B''C''$ and the circumcenter of $M_a''M_b''M_c''$.

12.3 The lines A_mA_m'', B_mB_m'', C_mC_m'' meet at a point Q'' which is the circumcenter of $A''B''C''$.

12.4 The lines $M_a'W_a'$, $M_b'W_b'$, $M_c'W_c'$ meet at a point T'' such that R'' is the midpoint of $T''F''$ or $T''G = 2GF''$.

12.5 If M_α'', M_β'', M_γ'' are the points determined as follows

$$m_\alpha'' + 2w_a' = 3m_a', \quad m_\beta'' + 2w_b' = 3m_b', \quad m_\gamma'' + 2w_c' = 3m_c',$$

then F'' is the circumcenter and T'' is the orthocenter of $M_\alpha''M_\beta''M_\gamma''$, which triangle has $M_a''M_b''M_c''$ as its medial triangle.

12.6 The points F'', R'', Q'', T'', G are collinear with $R''G=2GQ''$ and $F''G=2GR''$. This line is the Euler line of the triangles $A''B''C''$, $M_a''M_b''M_c''$ and $M_\alpha''M_\beta''M_\gamma''$.

13. The lines $W_a'A''$, $W_b'B''$, $W_c'C''$ meet at a point M' whose coordinate is given by $m'=g+s^2(g-k)/4S$. The lines $W_a''A'$, $W_b''B'$, $W_c''C'$ meet at a point M'' whose coordinate is given by $m''=g-s^2(g-k)/4S$. Since $\overline{GM'}=\overline{KG}\cot\theta$ and $\overline{GM''}=\overline{GK}\cot\theta$, the points M', M'', K, G are collinear and G is the midpoint of $M'M''$.

14.1 The lines $W_a'A_m''$, $W_b'B_m''$, $W_c'C_m''$ meet at a point E_1', such that

$$(7 + 4\cot\theta)\overline{GE_1'} = 14\overline{OG} + 7\overline{GK}\cot\theta.$$

The lines $A'A_m'$, $B'B_m'$, $C'C_m'$ meet at a point E_2' and the lines A_mM_a', B_mM_b', C_mM_c' meet at a point E_3' such that $\overline{E_1'E_2'}=2\overline{E_2'E_3'}$ and $2\overline{E_2'G}=\overline{GE_3'}$. Hence E_1', E_2', E_3', G are collinear, and this line is the Euler line of $A_m'B_m'C_m'$ as the midpoint of $E_2'E_3'$ is the circumcenter of the triangle $A_m'B_m'C_m'$.

14.2 The lines $W_a''A_m'$, $W_b''B_m'$, $W_c''C_m'$ meet at a point E_1'' such that

$$(7 - 4\cot\theta)\overline{GE_1''} = 14\overline{OG} + 7\overline{KG}\cot\theta.$$

The lines $A''A_m''$, $B''B_m''$, $C''C_m''$ meet at a point E_2'' and the lines A_mM_a'', B_mM_b'', C_mM_c'' meet at a point E_3'' such that $\overline{E_1''E_2''}=2\overline{E_2''E_3''}$ and $2\overline{E_2''G}=\overline{GE_3''}$. Hence the points E_1'', E_2'', E_3'', G are collinear, and this line is the Euler line of the triangle $A_m''B_m''C_m''$ as the midpoint of $E_2''E_3''$ is the circumcenter of $A_m''B_m''C_m''$.

15.1 The lines A_mW_a', B_mW_b', C_mW_c' meet at a point J_1' such that

$$(13 + 4\cot\theta)\overline{GJ_1'} = \overline{GO} + 2\overline{KG}\cot\theta.$$

The lines P_aA_m', P_bB_m', P_cC_m' meet at a point J_2' such that $3\overline{GJ_1'}=\overline{J_1'J_2'}$.

15.2 The lines A_mW_a'', B_mW_b'', C_mW_c'' meet at a point J_1'' such that

$$(13 - 4\cot\theta)\overline{GJ_1''} = \overline{GO} + 2\overline{GK}\cot\theta.$$

The lines P_aA_m'', P_bB_m'', P_cC_m'' meet at a point J_2'' such that $3\overline{GJ_1''}=\overline{J_1''J_2''}$.

15.3 The lines $W_a'M_a''$, $W_b'M_b''$, $W_c'M_c''$ meet at a point P' such that

$$(2 + \cot\theta)\overline{GP'} = 6\overline{OG} + 2\overline{GK}\cot\theta;$$

the lines $W_a''M_a'$, $W_b''M_b'$, $W_c''M_c'$ meet at a point P'' such that

$$(2 - \cot\theta)\overline{GP''} = \overline{6OG} + \overline{2KG}\cot\theta.$$

15.4 The lines P_aW_a', P_bW_b', P_cW_c' meet at a point S' such that

$$(14 + 3\cot\theta)\overline{GS'} = \overline{2GO} + \overline{6KG}\cot\theta;$$

the lines P_aW_a'', P_bW_b'', P_cW_c'' meet at a point S'' such that

$$(14 - 3\cot\theta)\overline{GS''} = \overline{2GO} + \overline{6GK}\cot\theta.$$

16. Numerous relations connect these vectors; the simplest are $(2+\cot\theta)\overline{GR'} = \overline{OM''}$ and $(2-\cot\theta)\overline{GR''} = \overline{OM'}$. These state that not only are the segments GR' and OM'' parallel, but that a fixed relation exists between their lengths. A similar statement holds for GR'' and OM'. Corresponding relations can be written for all the points located on the lines 11.6 and 12.6. Other types of identities among these vectors are illustrated by the following four relations:

$$(13 + 4\cot\theta)\overline{GJ_1'} + (13 - 4\cot\theta)\overline{GJ_1''} = \overline{2GO},$$

$$(13 + 4\cot\theta)\overline{GJ_1'} - (13 - 4\cot\theta)\overline{GJ_1''} = \overline{4GM'}.$$

$$(2 + \cot\theta)(\overline{GP'} + \overline{GF'}) = (2 - \cot\theta)(\overline{GP''} + \overline{GF''}).$$

$$(7 + 4\cot\theta)\overline{GE_2'} - (2 + \cot\theta)\overline{GR'} = (7 - 4\cot\theta)\overline{GE_2''} - (2 - \cot\theta)\overline{GR''}.$$

17.1 The perpendicular erected at A_2' to the line $A_2'B_1'$ meets the perpendicular erected at A_1' to the line $A_1'C_2'$ at a point R_a'. The perpendicular erected at B_2' to the line $B_2'C_1'$ meets the perpendicular erected at B_1' to the line $B_1'A_2'$ at a point R_b'. The perpendicular erected at C_2' to the line $C_2'A_1'$ meets the perpendicular erected at C_1' to $C_1'B_2'$ at a point R_c'. We have

$$\overline{R_b'R_c'} = 2ia + (1 - i)b - (1 + i)c = \overline{B_2'C_1'}.$$

Consequently with the aid of 7.3 and 8.3 we see that $R_b'B_2'C_1'R_c'$ and $R_b'M_c'M_b'R_c'$ are parallelograms and $R_b'W_a'R_c'A'$ and $R_b'M_a'R_c'P_a$ are equal pseudosquares. The circumcenter and orthocenter of the triangle $R_a'R_b'R_c'$ are F' and R' respectively, hence the line 11.6 is also the Euler line of $R_a'R_b'R_c'$.

17.2 The perpendicular erected at A_2'' to the line $A_2''B_1''$ meets the perpendicular erected at A_1'' to the line $A_1''C_2''$ at a point R_a''. The perpendicular erected at B_2'' to the line $B_2''C_1''$ meets the perpendicular erected at B_1'' to the line $B_1''A_2''$ at a point R_b''. The perpendicular erected at C_2'' to the line $C_2''A_1''$ meets the perpendicular erected at C_1'' to the line $C_1''B_2''$ at a point R_c''. We have

$$\overline{R_b''R_c''} = -2ia + (1 + i)b - (1 - i)c = \overline{B_2''C_1''}.$$

Hence, with the aid of 7.4 and 8.4 we see that $R_b''B_2''C_1''R_c''$ and $R_b''M_c''M_b''R_c''$ are parallelograms, and $R_b''A''R_c''W_a''$ and $R_b''P_aR_c''M_a''$ are equal pseudosquares. The circumcenter and orthocenter of the triangle $R_a''R_b''R_c''$ are F'' and R'' respectively, hence the line 12.6 is also the Euler line of the triangle $R_a''R_b''R_c''$.

17.3 The perpendicular erected at A_1' to the line $A_1'B_2'$ meets the perpendicular erected at A_2' to the line $A_2'C_1'$ at a point S_a'. The perpendicular erected at B_1' to the line $B_1'C_2'$ meets the perpendicular erected at B_2' to the line $B_2'A_1'$ at a point S_b'. The perpendicular erected at C_1' to the line $C_1'A_2'$ meets the perpendicular erected at C_2' to the line $C_2'B_1'$ at a point S_c'. Since $\overline{GS_a'}=i\overline{CB}$, then $\overline{S_b'S_c'}=3i\overline{GA}$. Using 8.2 and 7.5, we can show that $S_b'B_1'C_2'S_c'$ and $S_b'C_2''B_1''S_c'$ are parallelograms and $S_b'A''S_c'W_a'$, $S_b'P_aS_c'A$ and $S_b'A'S_c'W_a''$ are equal pseudosquares. The centroid and circumcenter of $S_a'S_b'S_c'$ are G and M' respectively, hence the line 13 is the Euler line of the triangle $S_a'S_b'S_c'$.

17.4 The perpendicular erected at A_1'' to the line $A_1''B_2''$ meets the perpendicular erected at A_2'' to the line $A_2''C_1''$ at a point S_a''. The perpendicular erected at B_1'' to the line $B_1''C_2''$ meets the perpendicular erected at B_2'' to the line $B_2''A_1''$ at a point S_b''. The perpendicular erected at C_1'' to the line $C_1''A_2''$ meets the perpendicular erected at C_2'' to the line $C_2''B_1''$ at a point S_c''. Since $\overline{GS_a''}=i\overline{BC}$, then $\overline{S_b''S_c''}=3i\overline{AG}$. Using 8.2 and 7.5 we see that $S_b''C_2'B_1'S_c''$ and $S_b''B_1''C_2''S_c''$ are parallelograms and $S_b''W_a'S_c''A''$, $S_b''AS_c''P_a$ and $S_b''W_a''S_c''A'$ are equal pseudosquares. The centroid and circumcenter of the triangle $S_a''S_b''S_c''$ are G and M'' respectively, hence the line 13 is the Euler line of the triangle $S_a''S_b''S_c''$. Moreover, this triangle is the reflection in G of the triangle $S_a'S_b'S_c'$; its area is three times that of ABC. The hexagon $S_a''S_c'S_b''S_a'S_c''S_b'$ is of the type described in 5.6.

18.1 If we designate by V_a', V_b', V_c' the points of intersection of the lines $C_1'A_2'$ and $A_1'B_2'$, $A_1'B_2'$ and $B_1'C_2'$, $B_1'C_2'$ and $C_1'A_2'$ respectively, the angles formed at M' by the intersecting lines $W_a'A''$, $W_b'B''$, $W_c'C''$ are equal to the angles of the triangle $V_a'V_b'V_c'$. Moreover, A_m', B_m', V_c', M' are concyclic.

18.2 If we designate by V_a'', V_b'', V_c'' the points of intersection of the lines $C_1''A_2''$ and $A_1''B_2''$, $A_1''B_2''$ and $B_1''C_2''$, $B_1''C_2''$ and $C_1''A_2''$ respectively, the angles formed at M'' by the intersecting lines $W_a''A'$, $W_b''B'$, $W_c''C'$ are equal to the angles of the triangle $V_a''V_b''V_c''$. Also A_m'', B_m'', V_c'', M'' are concyclic.

18.3 If we designate by U_a', U_b', U_c' the points of intersection of the lines $C_2'A_1'$ and $A_2'B_1'$, $A_2'B_1'$ and $B_2'C_1'$, $B_2'C_1'$ and $C_2'A_1'$ respectively, the angles formed at F$'$ by the intersecting lines P_aM_a', P_bM_b', P_cM_c' are equal to the angles of the triangle $U_a'U_b'U_c'$. Moreover, A'', B'', U_c', F' are concyclic.

18.4 If we designate by U_a'', U_b'', U_c'' the points of intersection of the lines $C_2''A_1''$ and $A_2''B_1''$, $A_2''B_1''$ and $B_2''C_1''$, $B_2''C_1''$ and $C_2''A_1''$ respectively, the angles formed at F'' by the intersecting lines P_aM_a'', P_bM_b'', P_cM_c'' are equal to the angles of the triangle $U_a''U_b''U_c''$. Finally, A', B', U_c'', F'' are concyclic.

CONFIGURATIONS INSCRIPTIBLE IN A PLANE CUBIC CURVE

By J. M. FELD, New York, N. Y.

1. *Introduction.* A set of p points and l lines in a plane, such that λ lines pass through each point and π points lie on each line, form a configuration designated by the symbol p_λ, l_π. Since $p\lambda = l\pi$, this symbol may be contracted, when $p=l$, to p_λ. Two configurations, containing the same number of elements of each kind, are said to be equivalent if a pair of incidence-preserving one-to-one correspondences exist, one between their points and another between their lines. It is known that a number of non-equivalent configurations bearing the same symbol are possible. S. Kantor* determined all configurations 8_3, 9_3 and 10_3, showing that for 9_3 three real configurations are realizable and for 10_3 ten. Further investigations into the existence and properties of configurations p_3 were made by V. Martinetti† who determined all configurations 11_3, and A. Schoenfliess‡ who investigated the regular configurations p_3. In another paper Schoenfliess showed that most of the regular configurations p_3 that he had determined lie on cubic curves.§ That the three configurations 9_3 lie on a C_3 is self-evident. It is, however, interesting to inquire with what degree of freedom such configurations may be inscribed in a given non-singular C_3. It is the purpose of this paper to answer this question for that configuration 9_3 known as the *Pascal*‖ configuration. Moreover, several theorems regarding inscribed Pascal configurations will be proved, and it will be shown how larger configurations bearing the symbols 12_4, 16_3 and 36_7, 84_3 can be built up from Pascal configurations and inscribed in non-singular cubics.

2. *Inscribed Pascal Configurations.* Let u_i, $i=1, 2, \cdots, 9$ represent the nine points of a Pascal configuration. The configuration may be visualized as a hexagon with the property that the point of intersection of each pair of opposite sides is collinear with the pair of opposite vertices not on these sides. Let u_7, u_8, u_9 be the respective points of intersection of the pairs of sides (u_2u_3, u_5u_6), (u_3u_4, u_6u_1), (u_4u_5, u_1u_2). The following matrix displays the nine sets of three collinear points:

$$\begin{matrix} u_1 & u_7 & u_4 \\ u_3 & u_9 & u_6 \\ u_5 & u_8 & u_2 \end{matrix} \tag{1}$$

* Wiener Berichte, vol. 84, 2 (1881), p. 915, 1291.

† Ann. di matem. (2), vol. 15 (1887) p. 1.

‡ Math. Ann. vol. 31 (1888), *Über die regelmässigen Configurationen n_3.*

§ Gött. Nachr. (1889), p. 334.

‖ The Pascal configuration (also known as the Pappus configuration) arises from the application of Pascal's theorem to a hexagon inscribed in a degenerate conic composed of two distinct lines. Its points are the six vertices of the hexagon and the three collinear points in which the pairs of opposite sides of the hexagon intersect. An interesting and elementary discussion of the three configurations 9_3 is given in Hilbert and Cohn-Vossen's *Anschauliche Geometrie*, Berlin, 1932, p. 91.

The points in each row of (1) are collinear and the points in each set of three taken diagonally, as in expanding a determinant of the third order, are collinear. The points in each column, however, are non-collinear. We shall refer to these three columnar sets of points as the triangular sets.

We proceed to determine conditions under which a real Pascal configuration may be inscribed in a real non-singular cubic C_3. Let C_3 be given in the Weierstrass canonical form. Then the coordinates of any point on C_3 can be expressed parametrically in the form $x=\wp u$, $y=\wp' u$, where $\wp u$ is the Weierstrass *$\wp$-function*. We let u represent indiscriminately points on the cubic or the corresponding values of the parameter u. Under these conditions, in order that $3n$ points, $u_1, u_2, \cdots, u_{3n}$ on C_3, form a complete set it is necessary and sufficient that $u_1+u_2+\cdots+u_{3n}\equiv 0 \pmod{\omega, \omega'}$, where ω and ω' are the periods of $\wp u$.* We can now express the requirement that nine points u_i lie in sets of three on nine lines by means of the following nine congruences.

(2) $$u_1+u_4+u_7\equiv 0$$

(3) $$u_2+u_5+u_8\equiv 0$$

(4) $$u_3+u_6+u_9\equiv 0$$

(5) $$u_1+u_2+u_9\equiv 0$$

(6) $$u_3+u_4+u_8\equiv 0$$

(7) $$u_5+u_6+u_7\equiv 0$$

(8) $$u_2+u_3+u_7\equiv 0$$

(9) $$u_1+u_6+u_8\equiv 0$$

(10) $$u_4+u_5+u_9\equiv 0$$

Adding (2), (7) and (8) we obtain

$$3u_7\equiv -(u_1+u_2+u_3+u_4+u_5+u_6).$$

Adding (3), (6) and (9) we get the same value for $3u_8$; and from the sum of (4), (5) and (10) the same value for $3u_9$. Therefore

(11) $$3u_7\equiv 3u_8\equiv 3u_9.$$

In like manner it can be shown that

$$3u_1\equiv 3u_3\equiv 3u_5$$

and that

$$3u_2\equiv 3u_4\equiv 3u_6.$$

We note that $3u_7\not\equiv 0$, for if $3u_7\equiv 0$, u_7, u_8, u_9 are points of inflection. But u_7, u_8, u_9 are real, and since the real points of inflection on a cubic are collinear,

* Halphen, *Traité des fonctions elliptiques*, 1888, II, p. 413.

u_7, u_8, u_9 would have to be collinear, contrary to our hypothesis that u_7, u_8, u_9 form a triangular set. Likewise $3u_1 \not\equiv 0$ and $3u_2 \not\equiv 0$.

We shall consider two cases: (a) C_3 unipartite and (b) C_3 bipartite. Let C_3 be unipartite. From (11) we obtain

$$u_8 = u_7 + (m\omega + n\omega')/3.$$

Consequently if u_7 designates a fixed real point on C_3, not a point of inflection, a point, u_8, distinct from u_7, can be chosen in eight ways. But on a unipartite cubic, in order that u_8 be real, m and n must be equal.* Thus u_8 can have only the two real values $u_7+\omega/3+\omega'/3$, and $u_7+2\omega/3+2\omega'/3$, except for periods. If we let u_8 be congruent to one of these u_9 must be congruent to the other. Let

$$u_8 \equiv u_7 + \omega/3 + \omega'/3, \qquad u_9 \equiv u_7 + 2\omega/3 + 2\omega'/3.$$

Since $u_7+u_8+u_9 \equiv 3u_7 \not\equiv 0$, u_7, u_8, u_9 form a triangular set as required. Likewise if u_1, not a point of inflection, and not congruent to u_7, u_8, u_9, is a given real point on C_3, a pair of real points u_3 and u_5 not collinear with u_1 and distinct from u_7, u_8, u_9 is uniquely determined. Moreover, since u_4 is collinear with u_1 and u_7, u_4 is real and determined; and by the preceding argument u_2 and u_6 are also real and determined. The nine points are therefore determined by two of them, viz., u_7 and u_1 as follows:

$$u_8 \equiv u_7 + \omega/3 + \omega'/3, \qquad u_9 \equiv u_7 + 2\omega/3 + 2\omega'/3$$
$$u_5 \equiv u_1 + \omega/3 + \omega'/3, \qquad u_3 \equiv u_1 + 2\omega/3 + 2\omega'/3$$
$$u_4 \equiv -u_1 - u_7, \qquad u_2 \equiv u_4 + \omega/3 + \omega'/3, \qquad u_6 \equiv u_4 + 2\omega/3 + 2\omega'/3.$$

That the nine points so determined satisfy conditions (2) to (10) is easily verified. They therefore constitute the points of a real Pascal configuration.

THEOREM. *A doubly infinite number of real Pascal configurations can be inscribed in a non-singular unipartite cubic. Any two points of the cubic which are not points of inflection and such that thrice the difference of their arguments is not congruent to zero may be selected as a pair of vertices of the configuration, and the remaining points are then uniquely determined.*

Let C_3 be bipartite. In this case, real points on the odd branch are given by real values of their parameters, while real points on the even branch are given by real values of $u-\omega'/2$. Let a fixed real point, not a point of inflection, on the odd branch be denoted by u_7, wherefore u_7 is real. Consequently the argument $u_8 \equiv u_7+(m\omega+n\omega')/3$ will denote a real point if and only if $m=1$, 2 and $n=0$. Therefore, as in the unipartite case, u_8 may be either of two points and u_9 must be the other. Let us write

* For a bipartite cubic ω and ω'/i are real and positive. For a unipartite cubic ω and ω' are conjugate complex. Points on the even branch of a bipartite cubic are given by values of u such that $u-\omega'/2$ is real. The odd branch of either type of cubic is given by real values of u. See Hilton, H., *Plane Algebraic Curves*, 1932, p. 252.

$$u_8 \equiv u_7 + \omega/3, \qquad u_9 \equiv u_7 + 2\omega/3 \qquad (u_7 \text{ real}).$$

If a fixed real point on the even branch be denoted by u_7, $u_7 - \omega'/2$ must be a real number; and in order that u_8 and u_9 represent real points there must obtain the congruences

$$u_8 \equiv u_7 + \omega/3, \qquad u_9 \equiv u_7 + 2\omega'/3 \qquad (u_7 - \omega'/2 \text{ real}).$$

Therefore u_8 and u_9 also lie on the even branch. Thus the points of the triangular set u_7, u_8, u_9 must lie on the same branch of C_3. Likewise the points of each of the other triangular sets must lie on one branch. As in the case of the unipartite C_3, u_1 may be chosen anywhere except at a point of inflection or coincident with u_7, u_8, u_9, and the entire configuration is determined. The three triangular sets of the configuration all lie on the odd branch or else two of them lie on the even branch and one on the odd branch.

THEOREM. *A two-fold infinity of real Pascal configurations can be inscribed in a real bipartite cubic. The points of the configuration are completely determined by two points of cubic provided* (1), *that neither is a point of inflection and* (2), *that thrice the difference of the arguments is not congruent to zero. The nine points of the configuration fall in three triangular sets of which either two or none lie on the even branch.*

Let a line l intersect C_3 in three distinct real points a_1, a_2, a_3, not points of inflection. Let b_1, b_2, b_3 be the three real points of inflection. Let b_1a_1, b_2a_1, b_3a_1 meet C_3 in u_1', u_3', u_5' respectively. Then

$$b_1 + a_1 + u_1' \equiv b_2 + a_1 + u_3' \equiv b_3 + a_1 + u_5' \equiv 0$$

and since $3b_1 \equiv 3b_2 \equiv 3b_3 \equiv 0$ we obtain

$$3u_1' \equiv 3u_3' \equiv 3u_5' \equiv -3a_1.$$

Likewise if b_1a_2, b_2a_2, b_3a_2 meet C_3 in u_7', u_8', u_9' respectively and if b_1a_3, b_2a_3, b_3a_3 meet C_3 in u_2', u_4', u_6' respectively, we have

$$3u_7' \equiv 3u_8' \equiv 3u_9' \equiv -3a_2$$
$$3u_2' \equiv 3u_4' \equiv 3u_6' \equiv -3a_3.$$

In order that nine points u_i' $(i=1, 2, \cdots, 9)$ satisfying the preceding congruences be real their arguments must be the following:*

$$\begin{array}{llll} & u_1' \equiv -a_1, & u_7' \equiv -a_2, & u_4' \equiv -a_3 \\ (12) & u_3' \equiv -a_1 + \omega/3, & u_9' \equiv -a_2 + \omega/3, & u_6' \equiv -a_3 + \omega/3 \\ & u_5' \equiv -a_1 + 2\omega/3, & u_8' \equiv -a_2 + 2\omega/3, & u_2' \equiv -a_3 + 2\omega/3. \end{array}$$

The u_i' in any column of (12) evidently form a triangular set, while the u_i' in each row and the u_i' in each "diagonal" are collinear. The nine points, u_i',

* The cubic is assumed to be bipartite. For a unipartite cubic ω should be replaced by $\omega + \omega'$.

are therefore the vertices of a real Pascal configuration. We shall speak of this configuration as the configuration derived from l. Each point of intersection of l with C_3 gives rise to a triangular set of the configuration. It will be noted that a conic which osculates C_3 at a_i can osculate the cubic again at each of the points of the triangular set to which a_i gives rise.

If l is the harmonic polar of b_1 the a_i are the points of contact of the three tangents from b_1. Consequently $b_1+2a_i\equiv 0$, and therefore $6a_i\equiv 0$. Thus it follows that the derived configuration of an harmonic polar has for its vertices the nine real sextactic points.* We shall call this the real *sextactic configuration.*

THEOREM. *The configuration derived from a real harmonic polar of a bipartite cubic has for its vertices the nine real sextactic points.*

If the points of an inscribed real Pascal configuration, other than the real sextactic configuration, are projected upon C_3 from one of its real points of inflection a second real Pascal configuration is obtained. The same configuration is obtained irrespective of the real point of inflection chosen. Projecting the new configuration we get back the original configuration. We shall refer to such pairs of configurations as *conjugate configurations*. The real sextactic configuration is evidently self-conjugate. The conjugate of configuration (12) is obtained by replacing a_i by $-a_i$. It is now easy to verify the two following theorems

THEOREM. *If two inscribed Pascal configurations are conjugate to each other either is the derived configuration of each line of the other.*

THEOREM. *If two points, belonging to different triangular sets of an inscribed Pascal configuration, are coincidence points,*† *all the points of the configuration are coincidence points. The conjugate of a configuration of coincidence points is also a configuration of coincidence points.*

3. *Larger Configurations Inscriptible in a Cubic.* Let K represent the real sextactic configuration inscribed in a bipartite cubic, C_3. The points of K are given by (12) if $a_1\equiv\omega/2$, $a_2\equiv(\omega+\omega')/2$, $a_3\equiv\omega'/2$. Dropping the primes, let u_3u_5, u_5u_1, u_1u_3 meet C_3 again at u_{10}, u_{11}, u_{12} respectively. Then since it follows that $u_{10}\equiv 0$, $u_{11}\equiv 2\omega/3$, $u_{12}\equiv\omega/3$, these points must be the real points of inflection, and therefore collinear. Likewise lines u_9u_8, u_8u_7, u_7u_9 meet C_3 again at u_{10}, u_{11}, u_{12} respectively and so do lines u_6u_2, u_2u_4, u_4u_6. These statements can be summed up as follows: The three triangular sets are perspective in pairs and have a common axis of perspectivity $u_{10}u_{11}u_{12}$. The twelve points u_i, $i=1, 2, \cdots, 12$, lie by threes on sixteen lines and these lines pass by fours through the twelve points. The sixteen lines are the six "diagonals" of (12), the nine sides of the three triangular sets of K and the line $u_{10}u_{11}u_{12}$.

* On the unipartite cubic l intersects C_3 in one real point and two conjugate imaginary points and the derived sextactic configuration in this case has only three of its points real.

† A coincidence point on C_3 is one at which tangent non-singular cubics can have contact of the eighth order.

THEOREM. *A real configuration bearing the symbol* 12_4, 16_3 *can be inscribed in a bipartite cubic. The points of the configuration are the nine real sextactic points and the three real inflection points.*

Of the four inflectional triangles of C_3, one has a real side and two imaginary sides. In that triangle let l designate the real side, l' and l'' the imaginary sides. The derived configuration of l is, if C_3 is bipartite, the real sextactic configuration and, if C_3 is unipartite, a sextactic configuration in which only three points are real; the configuration in either case to be represented by K. The derived configurations of l' and l'' are also sextactic configurations which we shall call K' and K'' respectively. The points of K, K' and K'' together with the points of l, l' and l'' respectively determine three inscribed configurations with the symbol 12_4, 16_3. Let these configurations be called Q, Q' and Q'' respectively. Letting (k_i, k_j) represent point $k_i\omega/6+k_j\omega'/6$, the following matrices represent K, K' and K''.

	(1, 0)	(1, 3)	(4, 3)
K	(3, 0)	(3, 3)	(0, 3)
	(5, 0)	(5, 3)	(2, 3)
	(1, 2)	(1, 5)	(4, 5)
K'	(3, 2)	(3, 5)	(0, 5)
	(5, 2)	(5, 5)	(2, 5)
	(1, 4)	(1, 1)	(4, 1)
K''	(3, 4)	(3, 1)	(0, 1)
	(5, 4)	(5, 1)	(2, 1)

The points in each column form a triangular set, as before. The sides of the three triangular sets of K meet C_3 again in the inflection points on l, viz., (2, 0), (4, 0), (0, 0), which together with the points of K constitute the points of Q. Likewise the sides of the triangular sets of K' meet C_3 again in the inflection points on l', viz., (2, 2), (4, 2), (0, 2), which in conjunction with the points of K' constitute the points of Q'. Finally the sides of the triangular sets of K'' meet C_3 again in the points of inflection on l'', viz., (2, 4), (4, 4), (0, 4), which with the points of K'' comprise the points of Q''. The sides of each Q are the six "diagonals," the nine sides of the triangular sets of its constituent K and its constituent l. The three configurations Q, Q', Q'' have no points and no lines in common. They form, however, part of a larger inscribed configuration of thirty-six points and eighty-four lines with the symbol 36_7, 84_3. The thirty-six points have already been accounted for; indeed they are the points of Q, Q' and Q''. Since each of these configurations contains sixteen lines, there are thirty-six more lines to account for. These are the remaining nine inflectional lines, the nine harmonic

polars and eighteen lines determined as follows. Let u_{ij}, u_{ij}', u_{ij}'' represent the point in row i and column j of K, K', K'' respectively. Then evidently points u_{ij}, u_{ik}', u_{il}'', $j\neq k\neq l$, are collinear. Let λ_{jkl}^{i} represent the line on which they lie. For each value of $i=1, 2, 3$ there are six λ's, thus accounting for the eighteen lines. Each of the eighty-four lines intersects C_3 in three points of the configuration and through each point of the configuration pass seven of its lines.

THEOREM. *On every non-singular cubic there are thirty-six points which lie by threes on eighty-four lines. These lines in turn pass by sevens through each of the thirty-six points. The thirty-six points are composed of the twenty-seven sextactic points and the nine inflection points.*

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

NOTE ON ELEMENTARY VECTOR ANALYSIS AND ON AN APPLICATION TO DIFFERENTIAL GEOMETRY

By MORRIS KLINE, New York University

Textbooks on vector analysis and textbooks on differential geometry which use vector methods are silent on questions of method and rigor which are essential to the conclusions there arrived at. This paper attempts to supply some details in elementary vector theory and to give one application in differential geometry which is not, to my mind, clearly treated in current textbooks.

Most textbooks on vector analysis fail to give a definition of the limit of a vector function of one scalar variable. Three definitions which have been used are here presented with a proof of their equivalence.

DEFINITION I. *The variable vector* $\mathbf{r}(s)$ *approaches the constant vector* $\mathbf{A}$ *as a limit for* $s\to s_0$ *if the magnitude of the difference* $\mathbf{r}(s)-\mathbf{A}$ *approaches zero as* s *approaches* s_0. *This definition holds whether or not* $\mathbf{A}$ *is a zero vector, except that in the case where* $\mathbf{A}$ *is zero, the definition imposes no restriction on the limiting direction of* $\mathbf{r}(s)$.

DEFINITION II. $\text{Lim}_{s\to s_0}\mathbf{r}(s)=\mathbf{A}$ *if, given any* $\epsilon>0$, *there exists a* δ_ϵ *such that* $(\mathbf{r}(s)-\mathbf{A})^2<\epsilon$ *for* $|s-s_0|<\delta_\epsilon$.

DEFINITION III. *The vector* $\mathbf{r}(s)=x(s)\mathbf{i}+y(s)\mathbf{j}+z(s)\mathbf{k}$ *has the limit* $A=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ *for* $s\to s_0$ *if the scalar functions* $x(s)$, $y(s)$, *and* $z(s)$ *have the limits* a_1, a_2, a_3, *respectively, for* $s\to s_0$. *This definition assumes the vector function is expressible in terms of components with fixed directions. Here the components need not be rectangular Cartesian.*

We now prove that these definitions are equivalent. We do so by proving (A) that if a limit exists in the sense of the third definition then it exists in the sense of the second; (B) that if a limit exists in the sense of the second definition then it exists in the sense of the first; and (C) that if a limit exists in the sense of the first definition, then it exists in the sense of the third.

Proof (A). Definition III implies definition II. For suppose a limit exists in the sense of definition III. Then, given any $\epsilon>0$, we select $\epsilon'=\sqrt{\epsilon/3}$ and there exists a δ such that for $|s-s_0|<\delta$, we have

$$|x(s)-a_1|<\epsilon'$$
$$|y(s)-a_2|<\epsilon'$$
$$|z(s)-a_3|<\epsilon'.$$

By adding, we have $(x(s)-a_1)^2+(y(s)-a_2)^2+(z(s)-a_3)^2<\epsilon$, or

$$(\mathbf{r}(s)-\mathbf{A})^2<\epsilon \quad \text{for} \quad |s-s_0|<\delta.$$

Proof (B). Definition II implies definition I. Suppose a limit exists in the sense of definition II. Then given any $\epsilon>0$, take $\epsilon'=\epsilon^2$. There exists a δ such that

$$(\mathbf{r}(s)-\mathbf{A})^2<\epsilon' \quad \text{for} \quad |s-s_0|<\delta,$$

or

$$|\mathbf{r}(s)-\mathbf{A}|^2<\epsilon' \quad \text{for} \quad |s-s_0|<\delta,$$

or

$$|\mathbf{r}(s)-\mathbf{A}|<\epsilon \quad \text{for} \quad |s-s_0|<\delta.$$

Proof (C). Definition I implies definition III. Suppose a limit exists in the sense of definition I. Then for any $\epsilon>0$, there exists a δ such that

$$|\mathbf{r}(s)-\mathbf{A}|<\epsilon \quad \text{for} \quad |s-s_0|<\delta.$$

Then

$$|\mathbf{r}(s)-\mathbf{A}|^2<\epsilon^2 \quad \text{for} \quad |s-s_0|<\delta,$$

or

$$(x(s)-a_1)^2+(y(s)-a_2)^2+(z(s)-a_3)^2<\epsilon^2 \quad \text{for} \quad |s-s_0|<\delta,$$

or

$$|x(s)-a_1|^2+|y(s)-a_2|^2+|z(s)-a_3|^2<\epsilon^2 \quad \text{for} \quad s-s_0|<\delta.|$$

Then, surely,

$$|x(s)-a_1|^2<\epsilon^2 \quad \text{for} \quad |s-s_0|<\delta,$$

and

$$|x(s) - a_1| < \epsilon \quad \text{for} \quad |s - s_0| < \delta.$$

Likewise

$$|y(s) - a_2| < \epsilon \quad \text{for} \quad |s - s_0| < \delta,$$

and

$$|z(s) - a_3| < \epsilon \quad \text{for} \quad |s - s_0| < \delta.$$

We may now define continuity of a vector function by requiring that $\mathbf{A}$ in any of the above definitions be $\mathbf{r}(s_0)$.

The notion of a derivative, defined as usually, is now in order. Since the derivative is a limit, if it exists in one sense it exists in the other two senses. Of course, where the vector function is expressed in terms of components, the derivative of the function is obtained by differentiating the scalar components.

Taylor's Theorem for vector functions of a single scalar variable is most easily proven by resorting to component representation and expanding each component separately by Taylor's Theorem for scalar functions. The result

$$\begin{aligned} \mathbf{r}(s) = \mathbf{r}(s_0) + \mathbf{r}'(s_0)\Delta s + \cdots \\ + \mathbf{r}^{(n-1)}(s_0)\frac{\Delta s^{n-1}}{(n-1)!} + x^{(n)}(s_1)\frac{\Delta s^n}{n!}\mathbf{i} + y^{(n)}(s_2)\frac{\Delta s^n}{n!}\mathbf{j} + z^{(n)}(s_3)\frac{\Delta s^n}{n!}\mathbf{k}, \end{aligned}$$

has the slight peculiarity that the components of the nth derivative term have different values of s, though each s_i lies between s and s_0. This peculiarity causes no difficulty in using the theorem.

The problems of the total differential of a vector function of two scalar variables, change in the order of partial differentiation, and Taylor's theorem for a vector function of two scalar variables are handled by methods exactly analogous to the methods used in elementary calculus textbooks on scalar functions.

We turn now to a topic of differential geometry. Blaschke, Bieberbach, and Weatherburn* base their vector treatments of the osculating circle and osculating sphere of a space curve on the notion of the number of coincident points which the circle and curve, or the sphere and curve, have in common. This approach is obviously the analogue of a method used for scalar functions and frequently found in elementary calculus textbooks.† However, no textbook I have seen builds up the theory for vector functions which justifies the procedure used by the writers mentioned above. A good deal of the reasoning used in the rigorous treatments of the corresponding theory for scalar functions cannot be used for vector functions, for the analogue has no meaning in many

* Blaschke, W., *Vorlesungen über Differentialgeometrie*, 1921, page 17.
Bieberbach, L., *Differentialgeometrie*, 1932, pp. 12 and 35.
Weatherburn, C. E., *Differential Geometry*, 1931, vol. 1, page 21.

† See, for example, Granville, Smith & Longley: *Elements of the Differential and Integral Calculus*, 1927, p. 170; and Fine, H. B., *Calculus*, 1927, p. 106.

cases. For example, the method used in the calculus textbooks referred to above depends ultimately upon a theorem on implicit functions which has no analogue for vector functions. I present below an elementary approach for vector functions to the derivation of the equations of the osculating circle and the osculating sphere of a space curve. The treatment is suggested by a non-vector treatment given by Scheffers.*

DEFINITION. *Two curves are said to have contact of the nth order at a common point, P, if it is possible to associate to each point A of C, the first curve, a point α of γ, the second curve, so that as A approaches P, it is true that* $\lim (\alpha P/AP)$ *and* $\lim \{\alpha A/(AP)^{n+1}\}$ *are finite and not zero. This definition presupposes that P is not a singular point. It is not hard to prove—since the proof has no relation to vector theory I omit it here—that if another law for associating α to A is used so that* $\lim (\alpha P/AP)$ *is finite and not zero then it will be true for this new law of association that* $\lim \{\alpha P/(AP)^{n+1}\}$ *will also be finite and not zero.*

In the theorem which follows we suppose that the independent variables s and t represent arc lengths on the respective curves. We can always introduce such variables if they are not present.

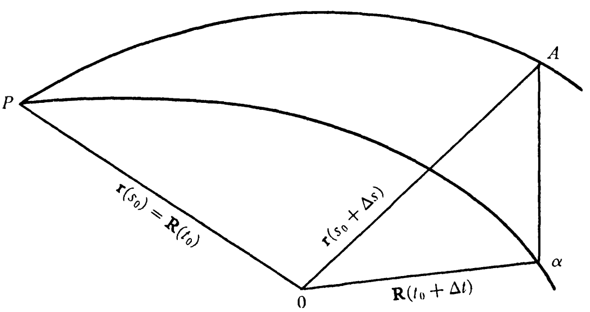
THEOREM. *A necessary and sufficient condition that two curves,* $\mathbf{r}(s)$ *and* $\mathbf{R}(t)$, *have contact of the nth order at a common point is that*

$$\mathbf{r}^{(i)}(s_0) = \mathbf{R}^{(i)}(t_0) \quad \text{for} \quad i = 0, 1, 2, \cdots, n$$

and

$$\mathbf{r}^{(n+1)}(s_0) \neq \mathbf{R}^{(n+1)}(t_0),$$

provided that all derivatives exist.



sufficiency of the condition. Let us choose as our law for associating A on $\mathbf{r}(s)$ to α on $\mathbf{R}(t)$, the relationship $\Delta s = \Delta t$. Now vector $\alpha P = \mathbf{R}(t_0 + \Delta t) - \mathbf{R}(t_0) = \mathbf{R}'(t_0)\Delta t +$ terms of second order in Δt. Therefore

$$\lim_{\Delta t \to 0} \frac{\text{vector } \alpha P}{\Delta t} = \mathbf{R}'(t_0).$$

* Scheffers, G.: *Anwendung der Differential- und Integral-Rechnung auf Geometrie*, vol. 1.

Since the magnitude of a vector approaches the magnitude of its limit we have

$$\lim_{\Delta t \to 0} \frac{\alpha P}{\Delta t} = 1.$$

Likewise

$$\lim_{\Delta s \to 0} \frac{AP}{\Delta s} = 1.$$

Also Δt approaches zero with Δs. Hence

$$\lim_{\Delta s \to 0} \frac{\alpha P}{AP} = \lim_{\Delta t \to 0} \frac{\alpha P}{\Delta t} \cdot \lim_{\Delta s \to 0} \frac{\Delta t}{\Delta s} \cdot \lim_{\Delta s \to 0} \frac{\Delta s}{AP} = 1,$$

and our first condition for contact is fulfilled. We now prove that $\lim \{\alpha A/(PA)^{n+1}\}$ is finite and not zero.

$$\begin{aligned}
\text{Vector } \alpha A &= \mathbf{r}(s_0 + \Delta s) - \mathbf{R}(t_0 + \Delta t) \\
&= \mathbf{r}(s_0) - \mathbf{R}(t_0) + \mathbf{r}'(s_0)\Delta s - \mathbf{R}'(t_0)\Delta t + \cdots \\
&\quad + \mathbf{r}^n(s_0) \frac{\Delta s^n}{n!} - \mathbf{R}^n(t_0) \frac{\Delta t^n}{n!} + \mathbf{r}^{n+1}(s_0) \frac{\Delta s^{n+1}}{(n+1)!} - \mathbf{R}^{n+1}(t_0) \frac{\Delta t^{n+1}}{(n+1)!} \\
&\quad + \text{terms of order } n + 2 \text{ in } \Delta s \text{ and } \Delta t.
\end{aligned}$$

Since $\Delta s = \Delta t$, and by our hypothesis on the derivatives

$$\text{vector } \alpha A = \{\mathbf{r}^{n+1}(s_0) - \mathbf{R}^{n+1}(t_0)\} \frac{\Delta s^{n+1}}{(n+1)!} + \text{terms of order } n + 2 \text{ in } \Delta s,$$

we have

$$\lim_{\Delta s \to 0} \frac{\text{vector } \alpha A}{\Delta s^{n+1}} = \frac{\mathbf{r}^{n+1}(s_0) - \mathbf{R}^{n+1}(t_0)}{(n+1)!}.$$

Therefore

$$\lim_{\Delta s \to 0} \frac{\alpha A}{\Delta s^{n+1}} = \frac{1}{(n+1)!} \left| \mathbf{r}^{n+1}(s_0) - \mathbf{R}^{n+1}(t_0) \right|.$$

From a previous step we have directly that

$$\lim_{\Delta s \to 0} \left(\frac{AP}{\Delta s}\right)^{n+1} = 1.$$

Since

$$\lim_{\Delta s \to 0} \frac{\alpha A}{(AP)^{n+1}} = \lim_{\Delta s \to 0} \frac{\alpha A}{\Delta s^{n+1}} \cdot \lim_{\Delta s \to 0} \frac{\Delta s^{n+1}}{(AP)^{n+1}},$$

$$\lim_{\Delta s \to 0} \frac{\alpha A}{(AP)^{n+1}} = \frac{1}{(n+1)!} \left| \mathbf{r}^{n+1}(s_0) - \mathbf{R}^{n+1}(t_0) \right| \neq 0$$

and is finite by our hypothesis on the derivatives.

Proof of the necessity of the condition. Since contact of the nth order exists it must be that some law of correspondence associates α to A so that $\lim (\alpha P/AP)$ is finite and not zero. Hence Δt must approach zero with Δs, else αP would not approach zero. Now

$$\lim_{\Delta s\to 0} \frac{\alpha P}{AP} = \lim_{\Delta t\to 0} \frac{\alpha P}{\Delta t} \cdot \lim_{\Delta s\to 0} \frac{\Delta t}{\Delta s} \cdot \lim_{\Delta s\to 0} \frac{\Delta s}{AP} \cdot$$

In the first part of this proof we showed that $\lim (\alpha P/\Delta t) = \lim (\Delta s/AP) = 1$. It follows then that $\lim (\Delta t/\Delta s)$ is finite and not zero. We may then set

$\Delta t = \lambda\Delta s + \epsilon\Delta s$ where λ is constant and ϵ approaches zero with Δs.

We have by hypothesis that $\lim \{\alpha A/(AP)^{n+1}\}$ is finite and not zero. But

$$\lim \frac{\alpha A}{(AP)^{n+1}} = \lim \frac{\alpha A}{(\Delta s)^{n+1}} \cdot \lim \frac{(\Delta s)^{n+1}}{(AP)^{n+1}} \cdot$$

Since the first and third limits are neither zero nor infinite, the second limit is neither zero nor infinite. Using the expression for Δt, we get:

$$\begin{aligned} \text{Vector } \alpha A &= \mathbf{r}(s_0 + \Delta s) - \mathbf{R}(t_0 + \Delta t) \\ &= \mathbf{r}(s_0) - \mathbf{R}(t_0) + \Delta s[\mathbf{r}'(s_0) - \mathbf{R}'(t_0)(\lambda + \epsilon)] + \cdots \\ &\quad + \frac{\Delta s^{n+1}}{(n+1)!}[\mathbf{r}^{n+1}(s_0) - \mathbf{R}^{n+1}(t_0)(\lambda + \epsilon)^{n+1}] \\ &\quad + \text{terms of order } n + 2 \text{ in } \Delta s. \end{aligned}$$

If we divide through by Δs^{n+1}, let Δs approach 0, and then consider the magnitude of this limit, it must be that the coefficients of terms involving Δs to powers lower than $n+1$ vanish and that the coefficient of Δs^{n+1} does not vanish, for if this did not occur we would have contact of order other than n contrary to hypothesis.

Since ϵ approaches zero with Δs, we have then:

$$(A) \qquad \begin{aligned} \mathbf{r}(s_0) - \mathbf{R}(t_0) &= 0 \\ \mathbf{r}'(s_0) - \lambda\mathbf{R}'(t_0) &= 0 \\ \cdots\cdots&\cdots \\ \mathbf{r}^n(s_0) - \lambda^n\mathbf{R}^n(t_0) &= 0 \\ \mathbf{r}^{n+1}(s_0) - \lambda^{n+1}\mathbf{R}^{n+1}(t_0) &\neq 0. \end{aligned}$$

From the second of these relations:

$\mathbf{r}'(s_0) = \lambda\mathbf{R}'(t_0)$ and since the vectors are unit vectors, $\lambda = \pm 1$. If λ is -1,

we may interchange the positive and negative directions for measuring arc lengths on one of the curves and consider λ as equal to $+1$. The relations (A) are then the desired relations.

We may now apply the above theorem to the question of the osculating circle of a space curve. Consider the circle $(\mathbf{R}(t)-\mathbf{a})^2-s^2=0$ where $\mathbf{a}$ is the position vector of the center and s is the magnitude of the radius. Suppose this circle lies in the osculating plane and has contact of at least second order with the curve $\mathbf{r}(s)$ at $s=s_0$ for the value of $t=t_0$. We have for any t:

$$(\mathbf{R}(t) - \mathbf{a})^2 - s^2 = 0 \tag{1}$$

$$2(\mathbf{R}(t) - \mathbf{a})\mathbf{R}'(t) = 0, \tag{2}$$

$$2(\mathbf{R}(t) - \mathbf{a})\mathbf{R}''(t) + 2 = 0. \tag{3}$$

Because of contact with the curve $\mathbf{r}(s)$ at $s=s_0$, we may substitute in (1), (2), and (3) which hold for $t=t_0$ and we get:

$$(\mathbf{r}(s_0) - \mathbf{a})^2 - s^2 = 0 \tag{4}$$

$$(\mathbf{r}(s_0) - \mathbf{a})\mathbf{r}'(s_0) = 0 \tag{5}$$

$$(\mathbf{r}(s_0) - \mathbf{a})\mathbf{r}''(s_0) + 1 = 0. \tag{6}$$

By using the Serret-Frenet formulas and by reasoning similar to Blaschke's* we determine the position vector of the center of the circle and the radius. The circle so determined is defined to be the osculating circle to the curve. It is possible to reverse the steps of this reasoning to prove that the circle with that center and radius has contact of at least second order.

We consider next the problem of the osculating sphere. We start with a definition of contact between curve and surface.

DEFINITION. *A curve, C, is said to have contact of nth order with a surface, S, at a point P, if there exists a curve, γ, on S which has contact of the nth order with C at P, and if no curve exists on S which has contact of higher order with C at P.*

Since a surface is given in vector form by $\mathbf{R}(u, v)$, a curve on the surface is determined by a relation $\phi(u, v)=0$; we suppose ϕ to be such that it is expressible by $u=f(t)$, $v=g(t)$, where t represents arc length of the curve since such a parameter may be introduced.

We will now determine a sphere which we suppose has contact of at least third order with a curve $\mathbf{r}(s)$ at $s=s_0$. Let $\mathbf{R}(u, v)$ be the position vector of points on the sphere; then the equation of the sphere is given by $(\mathbf{R}(u, v)-\mathbf{a})^2-S^2=0$ where $\mathbf{a}$ is the position vector of the center and S is the magnitude of the radius. Since this sphere is to have contact of at least third order with $\mathbf{r}(s)$, there exists a curve on this sphere given by $u=f(t)$, $v=g(t)$ which has contact

* *Loc. cit.*, p. 17.

of third order with $\mathbf{r}(s)$. Let $\mathbf{R}(t)$ indicate $\mathbf{R}(u, v)$ where we have substituted for u and v in terms of t.

We have for all values of t determining points on the curve:

$$(\mathbf{R}(t) - \mathbf{a})^2 - S^2 = 0 \tag{1}$$

$$2(\mathbf{R}(t) - \mathbf{a})\mathbf{R}'(t) = 0 \tag{2}$$

$$(\mathbf{R}(t) - \mathbf{a})\mathbf{R}''(t) + 1 = 0 \tag{3}$$

$$(\mathbf{R}(t) - \mathbf{a})\mathbf{R}'''(t) + \mathbf{R}'(t)\mathbf{R}''(t) = 0. \tag{4}$$

These relations hold also for $t=t_0$ where t_0 gives the point of contact with $\mathbf{r}(s)$, and since there is contact of at least third order, we may substitute the derivatives of $\mathbf{r}(s)$ for $s=s_0$ for the derivatives of $\mathbf{R}(t)$ for $t=t_0$. We may then use the Serret-Frenet formulas and proceed as in Blaschke* to determine the center and radius of the sphere. The sphere so determined is defined to be the osculating sphere of the curve at the point $s=s_0$.

The suggestions made above for filling in the gaps in elementary vector analysis and in one of its applications may not be the simplest possible. Perhaps, however, these remarks will call attention to the need for more rigorous treatment without which vector analysis is hardly satisfactory to the mathematician.

On the Highest Common Factor of Two Polynomials

By M. M. Flood, Princeton University

In a recent issue of this Monthly, W. V. Parker† considered the resultant of two polynomials in the form due to Frobenius‡ and T. A. Pierce§ and gave a method for determining the degree of their highest common factor. It is the purpose of this note to correct a slight error in the proof of Parker's theorem. In fact, it will be convenient to do this by proving a somewhat more general result.

Theorem. *If M is a non-derogatory‖ matrix whose characteristic function is $f(x)$ and if $g(x)$ is any other polynomial, then the degree of the highest common factor of $f(x)$ and $g(x)$ is the nullity‖ of $g(M)$.*

---

* *Loc. cit.*, p. 17.

† Parker, W. V., *The Degree of the Highest Common Factor of Two Polynomials*, this Monthly, vol. 42 (1935), p. 164.

‡ Frobenius, *Ueber lineare Substitutionen und bilineare Formen*, Journal für Mathematik, vol. 84 (1878), p. 11.

§ Pierce, T. A., *The Practical Evaluation of Resultants*, this Monthly, vol. 39 (1932), p. 161.

‖ Sylvester has called a matrix "non-derogatory" if its characteristic function and reduced characteristic function are the same. The "nullity" of a matrix is the difference between its order and rank.

Proof: Let the distinct zeros of $f(x)$ be $r_1, r_2, \cdots, r_\alpha$ and set

$$f(x) = \prod_{k=1}^{\alpha} (x - r_k)^{\alpha_k}.$$

Now if $d(x)$ is a divisor of $f(x)$ we may set

$$d(x) = \prod_{k=1}^{\alpha} (x - r_k)^{\beta_k},$$

where $\beta_k \leqq \alpha_k$. It is easily seen* that the nullity of $(M-r_k)^{\beta_k}$ is β_k when M is replaced by its classical canonical form and account is taken of the fact that the elementary divisors of M all have different zeros since M is a non-derogatory matrix. Also, since the factors $(x-r_k)^{\beta_k}$ for $k=1, 2, \cdots, \alpha$ are relatively prime we know† that the nullity of $d(M)$ is the sum of the nullities of $(M-r_k)^{\beta_k}$, or simply the degree of $d(x)$. In particular, if $d(x)$ is the highest common factor of $f(x)$ and $g(x)$, there exist polynomials $a(x)$ and $b(x)$ such that $d(x) \equiv a(x)f(x) + b(x)g(x)$. If we set $x=M$ in this identity we obtain $d(M) \equiv b(M)g(M)$. By a familiar theorem on nullities it follows that nullity $\{d(M)\} \geqq$ nullity $\{g(M)\}$. From $g(x) \equiv q(x)d(x)$ it also follows that nullity $\{g(M)\} \geqq$ nullity $\{d(M)\}$ and so nullity $\{g(M)\}=$ nullity $\{d(M)\}=$ degree $\{d(x)\}$ as was to be shown.

The error which Parker made in his proof was the assumption that $q(M)$ was non-singular. This is not the case unless $q(x)$ and $f(x)$ are relatively prime. The theorem proved includes the one of Parker since the matrix A of his note is non-derogatory. The methods used here are essentially those of Frobenius and Weyr.

The further question arises: Is it necessary as well as sufficient for the truth of our theorem that the matrix M be non-derogatory? This question is correctly answered in the affirmative if one requires that it be true for *every* $g(x)$ and a particular M. In general, for a *particular* $g(x)$ and $f(x)$ there will exist several derogatory matrices M such that nullity $\{g(M)\}=$ degree $\{[f(x), g(x)]\}$. The conditions which the elementary divisors of M must satisfy that this should be true are easily set down, however.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent to Review Editor, American Mathematical Monthly, Low Memorial Library, Columbia University, New York, N. Y.

NEW BOOKS RECEIVED

Algebra for College Students. By H. R. Willard and N. R. Bryan. New York, Scott, Foresman and Company, 1936. viii+383 pages. \$2.00.

* Wedderburn, J. H. M., *Lectures on Matrices*, (1934), p. 42.

† Weyr, *Zur Theorie der bilinearen Formen*, Monatsh, f. Math. u. Phys., vol. 1 (1890), p. 179.

College Algebra. By H. C. Carter. New York, Prentice-Hall, Inc., 1936. xii+234 pages. $1.50.

First Year College Mathematics. By M. A. Hill and J. B. Linker. New York, Henry Holt and Company, 1936. xviii+436+155 pages. $2.60.

Introduction to Mathematical Analysis. Revised Edition. By F. L. Griffin. Boston, Houghton Mifflin Company, 1936. x+546 pages. $2.75.

Textbook of Trigonometry. By W. H. H. Cowles and J. E. Thompson. New York, D. Van Nostrand Company, 1936. x+373 pages. $2.50.

Analytic Geometry. By J. W. Young, Tomlinson Fort, and F M. Morgan. Boston, Houghton Mifflin Company, 1936. x+347 pages. $2.25.

An Invitation to Mathematics. By Arnold Dresden. New York, Henry Holt and Company, 1936. xiv+453 pages.$2.80.

Elementary Differential and Integral Calculus. By G. L. Parsons. Cambridge, England, the University Press, 1936. xxxii+220+128 pages. $2.75.

Lessons in Elementary Analysis. By G. S. Mahajani. Poona, Aryabhushan Press, 1934. Second Edition. xii+264 pages.

Examples in Finite Differences, Calculus and Probability. By Harry Freeman. Cambridge, England, The University Press, 1936. 86 pages. $2.50.

Graphical Solutions. By C. O. Mackey, New York, John Wiley & Sons, 1936. viii+130 pages, with chart. $2.50.

Leitfaden der Planimetrie. By F. Gonseth and P. Marti. Zurich, Orell Füssli Verlag. Erster Teil, 1933; 175 pages. Zweiter Teil, 1936; 190 pages.

Analytic Geometry. By P. H. Graham, F. W. John, and H. R. Cooley. New York, Prentice-Hall, Inc., 1936. xx+294 pages. $2.35.

The Study of the History of Mathematics. By George Sarton. Cambridge, Mass., The Harvard University Press, 1936. 112 pages. $1.50.

Plane Trigonometry. By E. S. Allen. New York, McGraw-Hill Book Company, 1936. xii+152 pages, $1.50. With tables, xii+152+xxiv+156 pages. $2.25.

Introductory College Mathematics. By F. E. Johnston. New York, Farrar and Rinehart, 1936. x+314 pages. $2.60.

REVIEWS

Osiris. Studies on the History and Philosophy of Science and on the History of Learning and Culture. (The David Eugene Smith Presentation Volume.) Vol. I, 1936. Bruges (Belgium). 777 pages. $6.00 unbound ($5.00 for members of the History of Science Society).

"Wer vieles bringt, wird jedem etwas bringen"—if I remember Goethe correctly. This might well be used as motto both for the volume of Osiris, dedicated to David Eugene Smith on his 76th birthday (Jan. 21, 1936), and for the life work of the jubilar.

A larger amount of space than is usual is devoted to this review, in order to give an impression of the richness of the volume which should be in every

college library. Nearly all of the thirty-eight articles have some connection with mathematics or with mathematicians. None of the work is technical. A considerable number of illustrations and facsimiles add to the interest of the volume.

The first three articles are devoted directly to Professor Smith.

Following a four page dedication by George Sarton, a complete bibliography of 564 titles by D. E. Smith, from 1894 to 1935, is given by Bertha M. Frick. The range of interest is amazing, including practically all aspects of mathematics except technical research in the narrow sense. It is not possible to allow one's eye to travel over this list of 59 pages without gaining a glimpse of the influence of Smith's work in raising the level of mathematics in this country. It brings forcibly to one's mind the fact that there are vast fields for mathematicians besides technical research, and a realization of the great debt we all owe to such men as David Eugene Smith and Herbert Ellsworth Slaught.

Talents of their kind are far more rare than ability to do the ordinary run of research, since a greater breadth of vision and broader interests are involved. It is becoming increasingly clear that we need a stronger leavening by men and women who have clear ideas concerning the future development of mathematical pedagogy, together with ability to appreciate the trends of mathematical research and to acquire more than a bird's eye view of mathematics as a whole. It must be possible to open for such rare specimens rich and valuable careers within the fold of mathematics, instead of forcing them into the field of professional "education." Fortunately, attempts in this direction are being considered.

All readers of the MONTHLY know that D. E. Smith has ever been a faithful contributor to this journal. From the list referred to, I take the following impressive list of publications by Professor Smith in our periodical: 1894: vol. 1, p. 74–5. 1895: vol. 2, 87, 164, 247, 331, 348–51. 1896: vol. 3, 29–33, 60. 1898: vol. 5, 58. 1910: vol. 17, 118. 1911: vol. 18, 73–80. 1916: vol. 23, 277–9. 1917: vol. 24, 64–71, 221–3. 1918: vol. 25, 99–108. 1920: vol. 27, 120–7, 263–6. 1921: vol. 28, 10–15, 62–3, 64–5, 121–2, 123, 166, 166–8, 207–8, 208–9, 254, 255, 296–9, 303, 303–5, 339–49, 353–4, 368–9, 369–70, 430–2, 433–4, 434–5, 450–1. 1922: vol. 29, 14–15, 15–16, 62–3, 114–5, 115–6, 123–4, 157, 157–8, 209–10, 252–5, 297–9, 299–303, 394–5, 410. 1923: vol. 30, 107–13, 166–74. 1924: vol. 31, 252–3, 299–300, 491–2. 1925: vol. 32, 87–88, 135, 255, 287–94, 393–7, 421–2, 440–50. 1926: vol. 33, 28–31, 383, 384, 502–6, 518, 518–9. 1927: vol. 34, 152–3. 1928: vol. 35, 138–9, 140, 210–2, 280–2, 375–6. 1931: vol. 38, 224–5, 278–9, 331–4, 402–5. 1932: vol. 39, 42–4, 425. 1934: vol. 41, 38–9. I add that, since 1915, Professor Smith has served as Associate Editor of the MONTHLY.

The third article (pp. 79–84), also by Bertha M. Frick, contains a description of the remarkable collection of ca. 20,000 items (ca. 10,000 printed books and pamphlets, over 4,000 autograph letters—many will recall the interesting causeries "Among my autographs" in the MONTHLY of 1921 and 1922—; 3,000 portraits of mathematicians, 160 medals and medallions in honour of mathe-

maticians, 270 mathematical and astronomical instruments, etc.), which was presented to Columbia University in 1931.

The remaining articles of the volume will be given by title, and in most cases characterized in the briefest possible manner.

4. Raymond Clare Archibald (Providence, R. I).—*Unpublished letters of James Joseph Sylvester and other new information concerning his life and work* (2 facsimiles); pp. 85–154.

Curriculum vitae giving main facts of his life (1814–97). New information filling out lacunae from 1841–4. Ten pages on Sylvester's poetry. 30 letters hitherto unpublished.

5. Sherman B. Barnes (New York).—*The editing of early learned journals*; pp. 155–72.

6. Charlotte H. Boatner (New Orleans, La.).—*Certain unpublished letters from French scientists of the revolutionary period taken from the files of Joseph Lakanal* (1 facsimile); pp. 173–83.

7. Ettore Bortolotti (Bologna).—*L'algebra nella storia e nella preistoria della scienza* (2 facsimiles); pp. 184–230.

On Muhammed ben Musa's Algebra, Diophant's Arithmetic, Algebra of ancient Egypt, Babylonian Arithmetic and Algebra, etc.

8. Julian L. Coolidge (Cambridge, Mass.).—*Origins of analytic geometry* (1 fig.); pp. 231–50.

"My thesis then is that the essence of plane analytic geometry is the study of loci by means of their equations and that this was known to the Greeks and was the basis of their study of conic sections." The author uses the spelling Leibnitz. I am under the impression that Leibniz is now universally preferred.

9. George O. S. Darby (Greenwich, Conn.)—*The mysterious Abolays*; pp. 251–9.

Attempts to identify an early Spanish astronomer and astrologist. Possibly corrupted from the Arab Abul-Ais or Abul-Anbas.

10. H. Gray Funkhouser (Exeter, N. H.).—*A note on a tenth century graph* (1 facsimile); pp. 260–2.

11. Solomon Gandz (New York).—*The sources of Al-Khowarizmi's algebra*; pp. 263–77.

12. F. H. Getman (Stanford, Conn.).—*Samuel Morey, a pioneer of science in America* (6 ills. on 3 plates); pp. 278–302.

1793, Steamship; patent 1795, seven years earlier than Fulton. p. 295: "One has only to read a few paragraphs of Morey's description to discover that this early American inventor had anticipated a mechanism which is almost the counterpart of the modern gasoline engine . . . ."

13. Benjamin Ginzburg (Long Island City, N. Y.).—*The scientific value of Copernican induction*; pp. 303–13.

"In any attempt to appreciate the characteristic differences between ancient and modern science, the promulgation of the Copernican theory of the heavens may serve as a sort of experimentum crucis, since as it happens no new empirical facts were involved in the passage from the Ptolemaic to the Copernican theory

and the differences in the theories must be ascribed entirely to the difference in systematic approach." Discussion along these lines.

14. C. Doris Hellman (New York).—*Legendre and the French reform of weights and measures* (2 facsimiles); pp. 314–40.

15. Theodore Hornberger (Ann Arbor, Mich.).—*Samuel Lee* (1625–91), *a clerical channel for the flow of new ideas to XVIIth century New England*; pp. 341–55.

16. S. A. Ionides (Denver, Colo.).—*Caesar's astronomy* (*astronomicum caesareum*) *by Peter Apian, Ingolstadt 1540* (9 ills.); pp. 356–89.

17. Francis R. Johnson (Palo Alto, Cal.).—*The influence of Thomas Digges on the progress of modern astronomy in XVIth century England;* pp. 390–410.

"Into this translation of Copernicus' De Revolutionibus there was inserted a most significant passage, by Thomas Digges himself, setting forth the idea that the universe should be conceived as infinite, with the stars located at varying distances through endless space . . . ." "This orbe of starres fixed infinitely up extendeth hit self in altitude sphericallye, and therefore immouable the pallace of foelicitye garnished with perpetuall shininge glorious lightes innumerable, far excellinge our sonne both in quantitye and qualitye."

18. L. C. Karpinski (Ann Arbor, Mich.).—*The first printed arithmetic of Spain.*

Francesch Sanct Climent: *Suma de la art de arismetrica*, Barcelona 1482 (3 facsimiles); pp. 411–20.

19. Gino Loria (Genoa).—*Michel Chasles e la teoria delle sezzione coniche* (1 portrait); pp. 421–450.

Contains at the end a bibliographic list of 28 papers (1828–71) by Chasles on conic sections.

20. J. Millás i Vallicrosa (Barcelona).—*Una obra astronomica desconocida de Johannes Avendant Hispanus*; pp. 451–75.

21. U. G. Mitchell and Mary Strain (Lawrence, Kan.).—*The number e* (2 facsimiles); pp. 476–96.

I. Introduction: (1) The numbers e and π. II. First approximations to the number e: (2) First natural logarithms. (3) Relations of logarithms to the hyperbola. (4) Development of the exponential series. (5) Association of logarithms and exponents. III. Recognition of the existence of e: (6) The work of Euler. (7) Representation as a limit. (8) Representation by a symbol. (9) Later approximations to e. IV. The nature of e: (10) Proof of irrationality. (11) Proof of transcendence.

On page 493, e is given to 346 decimal places (J. W. Boorman, Artemas Martin's Math. Mag., 1884). Page 495, lines 10 ff., must read: "Legendre suggested that numbers might exist which could not be roots of algebraic equations *with integral coefficients* and in 1844 Liouville succeeded in proving that this is the case." The critical three words are omitted in two other places, pp. 495–6. The statement, p. 479, " . . . after the discovery of transcendental numbers, it (e) was the first number definitely identified as belonging to that class" is misleading, since Liouville himself proved that every infinite decimal, $.c_1\gamma_1c_2\gamma_2 \cdots c_i\gamma_i \cdots$,

where $c_i=1, 2, \cdots, 9$ and γ_i stands for a group of $i!$ zeros, is transcendental.

22. J. Pelseneer (Bruxelles).—*Une lettre inédite de Newton à Pepys* (23 déc., 1693); pp. 497–9.

From a correspondence covering six letters between Newton and Pepys (Nov.–Dec., 1793) on probabilities, 1, 2, 4, 5 have been previously printed; 3 is lost; 6 is here published.

23. A. Pogo (Washington, D. C.).—*Three unpublished calendars from Asyut* (10 pls., 3 figs., 1 table); pp. 500–9.

Ancient Egyptian calendars.

24. Vera Sanford (Oneonta, N. Y.).—*François Legendre* (sic!), *arithméticien*; pp. 510–8.

Spelled le Gendre, throughout the article. Died between 1672 and 1678.—From an arithmetic printed in 1672 by another author, F. Barrême, a remark is quoted which strikes a painfully familiar modern note: "Barrême is outspoken about verbal problems. People have made too much of the Rule of False and they have wasted time on many questions which they call pleasing and curious, but which he calls ridiculous 'because in the whole lifetime of a business man, he would not twice have need of these.' "

25. George Sarton (Cambridge, Mass.).—*Montucla* (1727–99) (2 pls., 11 facsimiles); pp. 519–67.

Of interest to mathematicians: Discussion of his *Histoire des Recherches sur la quadrature du cercle*, 1754; *Histoire des mathématiques* · · · , 1758; Montucla's edition of Ozanam's *Recréations* · · · , 1778.

26. P. Sergescu (Cluj).—*Les Mathématiques dans le journal des Savants. Première période*, 1666–1701; pp. 568–83.

This interesting sketch whets one's appetite for the lecture of the original volumes.

27. Lao G. Simons (New York).—*Short stories in colonial geometry*; pp. 584–605.

Geometry at Harvard College; at Yale College; at Columbia College; at the University of Pennsylvania.

28. A. N. Singh (Lucknow, India).—*On the use of series in Hindu mathematics*; pp. 606–28.

(1) Arithmetic and geometric series are said to occur in Hindu mathematics around 2000 B.C.

(2) Āryabhatā (599 A.D.), Brahmagupta (628) considered the sum of the squares and of the cubes of the first n natural numbers.

(3) Mahāvīra (850) gives the formula $S=a(r^n-1)/(r-1)$.

(4) Mahāvīra gives

$$a^2+(a+b)^2+\cdots+(a+(n-1)b)^2=n[(\overline{2n-1}\cdot b^2/6+ab)(n-1)+a^2].$$

(5) Āryabhatā: $\sum r^3=(\sum r)^2$; let $N_n=1+2+\cdots+n$, then $\sum_1^n N_r=n(n+1)(n+2)/6$ (Figurate numbers).

(6) Binomial series, Pascal Δ: Pingala (ca. 200 B.C.):

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n.$$

After Neugebauer's amazing discoveries concerning early Babylonian mathematics one is prepared for almost anything; yet, some data given about infinite series are startling. Most of all, perhaps, on p. 626 ff.: "Sometime during the first half of the 15th century they discovered what is now known as Gregory's series ($\pi/4=1-\frac{1}{3}+\frac{1}{5}-+\cdots$)"; and (1432 A.D.):

$$r\theta=\frac{r\sin\theta}{1\cos\theta}-\frac{r\sin^3\theta}{3\cos^3\theta}+\frac{r\sin^5\theta}{5\cos^5\theta}-+\cdots$$

(equivalent to arctan $x=x-x^3/3+-\cdots$).

The text of the article might even imply that convergence and divergence were considered.

Verification of these claims will be awaited with interest.

29. Lynn Thorndike (New York).—*Coelestinus' summary of Nicolas Oresme on marvels*; pp. 629–635.

30. J. Tropfke (Berlin).—*Die Siebeneckabhandlung des Archimedes* (11 figs.); pp. 636–51.

Discussion of a Ms. by Ṯābit ibn Qurrah (836–901, Bagdad), part of which contains discussions on the regular heptagon, and which Ṯābit expressly attributes to Archimedes.

31. D. Uvanovic (Zagreb, Jugoslavia).—*The Indian prelude to European mathematics*; pp. 652–7.

"Archimedes' anticipation of integration was matched by Bhaskara's of differentiation. For Bhaskara not only used the equivalent formula $d\sin\phi=\cos\phi\cdot dt$ (sic!) but also in computing what he called the instantaneous motion of a planet, he measured the differential of the planet's longitude $\cdots$." "A century later (i.e. in the 14th century), an Indian manuscript notes a value of π correct to about 30 decimal places."

32. A. Van de Vyver (Gand).—*Les plus anciennes traductions latines médiévales (X^e et XI^e siècles) de traités d'astronomie et d'astrologie;* pp. 658–91.

33. Q. Vetter (Prague).—*Quatre notes sur les mathématiques babyloniennes* (7 figs.); pp. 692–702.

34. K. Vogel (München).—*Bemerkungen zu den quadratischen Gleichungen der babylonischen Mathematik* (5 figs.); pp. 703–17.

Follows the lines of Neugebauer's fundamental investigations of early Babylonian algebra (solution of quadratic equations ca. 2000 B.C., or earlier). The author apparently arrives at the conclusion that the early Babylonians were not familiar with the fact that the quadratic equation has two solutions.

35. J. A. Vollgraff (Leiden).—*Snellius' notes on the reflection and refraction of rays* (2 facsimiles); pp. 718–725.

36. Helen M. Walker (New York).—*An unpublished hydraulic experiment of Roberval* 1668 (1 facsimile); pp. 726–32.

"How to make the bellows of a forge by means of water."

37. P. P. Wiener (New York).—*The tradition behind Galileo's Methodology*; pp. 733–46.

"I wish to show that Galileo's methodology was opposed not to the intellectual traditions of Greek thought but to a specious Aristotelianism current in his day; that wherever Galileo diverges from Aristotle, it is not in method but in content; finally, that the innovations in the contents of his physical doctrines were made by Galileo within the framework of a Platonic conception of the physical world."

38. E. Tinner (Bamberg).—*Die Tafeln von Toledo*; pp. 747–74.

Discussion of astronomical tables going back to an Arab Scientist of the 11th century in Toledo. The existing manuscripts date from the 12th to 14th centuries.

In concluding, I should like to ask why large unbound books of this kind are issued uncut by the publishers? It cost me a lot of time and temper to cut the nearly 800 pages crosswise and lengthwise.

A. J. KEMPNER

Business Mathematics. By C. L. Richtmeyer and J. W. Foust. New York, McGraw-Hill Book Company, 1936. xii+250 pages. $2.00.

This book is a rather elementary presentation of business mathematics intended, as the authors state in the preface, for "college classes that include students who are preparing to teach arithmetic in secondary schools, students who are planning to enter various commercial fields, or students who wish a general course in practical mathematics." The book seems well adapted to the needs of such students since it contains, in addition to work on Simple Interest and Discount (Ch. III), Compound Interest (Ch. VI) and Annuities (Ch. VII), chapters on Operations with Integers and Fractions, Percentage, Measurement, Denominate Numbers, Geometric Figures, Logarithms and Slide Rule, Graphs, Equations, and an Introduction to Statistics.

The work on simple interest and discount and on compound interest follows conventional lines. Chapter VI contains an account of progressions in addition to compound interest. The introduction of progressions at this point is in line with the general policy of the book of providing the students with the mathematical tools necessary for an understanding of later material, in this case mainly the work on annuities which follows in Chapter VII. Under annuities in Chapter VII one finds paragraphs on amortization, depreciation, perpetuities and capitalized cost and bonds. This arrangement brings out the fact that the subject of annuities is a feature common to these subjects and results in simplification. The chapters on Graphs and Statistics are instructive and contain many practical illustrations.

On the whole, the book is well written and contains abundant exercises and adequate tables.

R. G. PUTNAM

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be set to F. W. Owens, 463 East Foster Ave., State College, Pa.

NOTICE TO CLUB SECRETARIES

Many clubs have already sent in their reports for 1935–36 but there are still more to be heard from. All such reports must be received before January 1, 1937 to be acceptable for publication.

This department invites each club to submit, before June 15, 1937, the best paper presented to it by an undergraduate during the year 1936–37. It is hoped that each club will make sure that its entry reaches this office as early as may be. These papers will be listed in this department with special mention of those judged to be the best.

A MATHEMATICAL RECREATION

A recreation made up of items like the examples given below was arranged and used by the Mathematics Club of Oshkosh State Teachers College, Oshkosh, Wisconsin. It is suggested that half the pleasure of such an entertainment lies in the construction, and each club may well modify the terms used to suit the individual interests.

Find the mathematical term represented by:

1. A color + slang word for gentlemen (= tangent).
2. Word used with either + noise + devoured (= ordinate).
3. Abbreviation for advertisement + a bird + a small coin (= adjacent).

CLUB PROGRAMS AND OFFICERS, 1935–36.

The Mathematics Club of Oshkosh State Teachers College

President, T. Dohlke; Vice President, G. Beard; Secretary, Dorothy Lindgren; Treasurer, J. Lentz; Faculty Advisers, Dr. May M. Beenken, Dr. Irene Price. Monthly meetings included a mathematical party and a picnic on Lake Winneconne. In addition there was the annual party given by the faculty to the senior members of the club. Papers and speakers were: "Pythagoras" by Jane Engel; "Proofs of the Pythagorean theorem" by H. Wetak; "Newton's Principia" by A. Womaski; "Hyperbolic functions" by T. Dohlke; "The origin of typical problems" by C. Swiston; "Theory of probability" by K. Dornstreich; "Men who made mathematics" by J. Oldfield; "Non-Euclidean geometry" by H. Gorwitz; "Different number bases" by Jeanne McVacar; "History of trigonometry" by R. Sloan; "Nomography" by A. Womaski; "Applications of mathematics in everyday life" by L. Dobyns.

Pi Mu Epsilon of the University of Alabama

Director, J. P. Gill; Vice Director, T. P. Petruska; Treasurer, H. S. Thurston; Librarian, Professor W. P. Ott; Secretary, W. F. Adams. Besides two social meetings, i.e. a bridge party and a picnic, the following papers were read and discussed at the regular meetings: "Matrix methods" by Dr. P. Hummel; "A system of concentric circles will invert into a system of coaxial circles" by A. J. Pyka; "The method of indefinite descent" by G. N. Carmichael; "The oscilloscope and Lissajous figures" by R. E. Lake and F. Mitchell; "Mathematics in modern schools" by J. P. Gill; "On a collineation group of order 192" by W. E. Koss; "Polar reciprocal of a conic with respect to

an auxiliary conic" by T. P. Petruska; "The conic generated by the intersections of the polars of a range of points with respect to two fixed circles" by R. Tipton.

The Mathematics Club of Mount Mary College

Besides cooperating with the Intercollegiate Mathematical Association of Milwaukee the club listened to the following papers: "Mary Somerville" by Adele Pideck; "Interesting facts about triangles" by Muriel Nevins; "Non-Euclidean geometry" by Doris Beringer; "The story of numbers" by Eleanor Herbst; "Weights and measures" by Margaret Kreiziger; "Early arithmetic" by Florence Winking.

The Mathematics Club of Massachusetts State College

At the six regular meetings the following papers were given and subsequently discussed by club members: "Bertrand Russell's contribution to mathematics" by J. Freedman; "Review of D. E. Smith's Religio Mathematici" by C. N. Clark; "Hyperbolic functions" by O. R. Putnam; "Computation of π" by R. Minzner; "Duodecimal number system" by H. D. Boutelle; "Functions without finite derivatives" by Ruth Blassberg.

Delta Chi of the University of Kansas City

President, G. Milne; Vice President, R. Magovern; Secretary, Margaret Hopper; Treasurer, C. Poolivich; Faculty Advisers, M. Luby and D. T. Sigley. This new club of twenty-five members holds two meetings each month.

Shuttleworth Mathematical Society, University of Saskatchewan

Honorary President, Dean R. Ling; President, N. Sharp; Vice President, W. Thomas; Secretary, J. Dickert; Recording Secretary, N. MacDonald; Faculty Adviser, Dr. H. H. Ferris. Regular programs included: "Magic squares and cubes" by J. Dickert; "π" by N. MacDonald; "Congruent numbers" by N. Sharp; "The probability curve" by H. Forest. At two social meetings mathematical games and catch problems furnished amusement. A banquet was held with Dr. Herzberg as guest speaker. Dr. Herzberg is visiting professor from Germany. His subject was "The story of the atom."

The Mathematical and Physical Society of the University of Toronto

This club holds meetings on alternate Thursday afternoons. Lectures are given on many subjects and prizes are offered at each meeting for the solution of problems in mathematics. Lectures during the year included, "Some remarks on modern social and economic life" by Professor M. A. Mackenzie, Honorary President of the Society; "The problem of three bodies" by Professor A. F. C. Stevenson; "Science in the Antartic" by Professor T. Griffith Taylor, member of Scott's last expedition to the Antartic.

Michigan State College Mathematics Club

President, Elizabeth Blythe; Vice President, L. Lusenring; Secretary-Treasurer, Marie Paulie. Even at the annual picnic there was a talk on mathematics by Dr. Kimball, while at the ten regular meetings the following papers were read and discussed: "Magic and magic squares" by Professor Crowe assisted by E. Blythe and D. Sexmich; "Number sevens" by Professor Grove, Mary Michel and L. Wiesner; "Pythagoras and his works" by Professor Plant, R. Slack, Frances Davis, M. Ahmes and R. Edwards; "Calculus and its variations" by Professor Powell; "Astronomy" by Professor Kimball; "Statistics" by M. Henry and R. Calvin, R. Dressel, H. Shaw and L. Heyda; "Diophantine equations" by R. Zimmer; "Graphical solutions of equations" by Melba Case, Frances Davis, Peg Harrison and Marie Paulie.

Delta Nabla of Westminster College

President, R. Lake; Vice President, J. Veazey; Secretary and Treasurer, A. Oelslager; Faculty Adviser, Dr. H. C. Black. The annual award made by the club to "the most outstanding mathe-

matics student of the freshman class" was won by J. Heinrich. New members were pledged at the annual College Tap Day, and at the initiation banquet the pledges gave short historical biographies of famous mathematicians. Papers presented and discussed at the regular meetings included: "Vacuum pumps" by Dean A. C. Burr; "Graduate study of mathematics" by H. Lawton; "Curve-fitting on semi-log charts" by H. Meyer; and "Mechanical aids to computation" by W. Shira.

The Yale Undergraduate Mathematics Club

Chairman, H. Brooks; Vice Chairman, R. Ryder; Treasurer, J. Eaton. Besides awarding prizes for the best solutions of a set of problems proposed by the faculty, and issuing a *Bulletin* containing original work of undergraduates, the club held regular meetings for the presentation and discussion of the following papers: "Approximation polynomials" by Professor E. G. Miles; "Theory of equations" by Professor O. Ore; "Primes" by M. Hall; "Some applications of center of mass" by R. A. Rosenbaum and R. S. Bender; "Number systems in mathematics" by Professor H. T. Engstrom; "Gaussian logarithms" and "Method of finding day, given date" by T. D. Woolsey and T. Rosenthal; "The historical and mathematical development of world maps" by Professor B. H. Brown of Dartmouth College; "Vector analysis" by F. Carter and H. Brooks; "Theory of errors" by A. Leiner; "Continued fractions" by W. Leighton of Harvard University; "Theories of nuclear structure" by Professor H. Margenau.

The Mathematics Club of the University of Buffalo

President, Evelyn Jackle; Secretary, R. W. Card. The Wilfred H. Sherk Memorial Prize in mathematics was presented to R. Schafer, a freshman, for his paper on "The derivation of certain formulae in analytic geometry for finding the area of a triangle or the volume of a tetrahedron." Programs at the monthly meetings were as follows: "Comparison of methods in analytic geometry and descriptive geometry" by Evelyn Jackle and Professor C. E. Harrington; "Code arithmetic" by Anna Bronstein, D. Lauer and F. Hull; "Methods of mathematics" by Dr. Harriet F. Montague; "Foundations of mathematics" by H. Wideman; "Roman architecture and engineering" by J. W. Greenwood of Technical High School; "Secondary school teaching" by Sara C. Walsh of East High School. The Philosophy Club, Classic Club and Senior Unit in Education were guests at the meetings when subjects presented were of interest to those groups.

Pi Mu Epsilon of the University of Georgia

Director, Iris Callaway; Vice Director, E. C. Mallory; Second Vice Director, P. C. Stephens; Secretary-Treasurer, C. C. Bolger. All meetings, save business meetings and annual banquet, were open to all students. Much interest was shown at each meeting in the solutions of problems in geometry, copies of which had been given out at the meeting before the discussion. The list of papers presented follows: "Arithmetic, the queen of the sciences" by Dr. R. P. Stephens; "The development of arithmetic" by M. Melvin; "History of early geometry, philosophy and lives of geometers in the fifth and sixth centuries" by H. Ballew; "Theorems proposed and solved in fifth and sixth centuries" by L. Kirkwood; "The development of geometry, 1450–1800" by H. Stanley; "Possibilities of future development in geometry and its application to science" by Dr. D. C. Barrow; "Mathematics in the curriculum" by Dr. R. P. Stephens. Current issues of the MONTHLY were discussed by P. Hill and H. Beckwith.

Students Mathematical Round Table, University of Illinois

First Semester-President, J. M. Dobbie; Secretary, M. A. Norval; Program Committee, R. M. Thrall and P. R. Halmos. Second Semester-President, F. L. Dennis, Secretary D. R. Shreve; Program Committee, J. R. F. Kent and Dorothy Gibson. Two picnics were held and twelve regular meetings. Papers presented and discussed were: "Modern synthentic geometry in space" by F. C. Gentry; "Napier's logarithms" by D. R. Shreve; "The use of inverse probability" by J. R. F. Kent; "The mechanical solution of ordinary differential equations" by F. P. Welch; "The mathematical game of Nim" by F. L. Dennis; "A study of $d^2y/dx^2+x^2y=0$" by H. C. Fryer; "Composite power series" by A. C. Burdette; "An heuristic derivation of a formula for the distribution of twin

primes" by D. L. Netzorg; "Some fundamental concepts of differential geometry" by Vera Modesitt; "The axiom of Zermelo and its applications in the theory of cardinal numbers" by D. E. Kibbey; "Uses of the Hollireth tabulating machine in statistical methods" by P. R. Halmos.

Pi Mu Epsilon of Syracuse University

Director, Dr. Floyd Decker; Vice Director, Evelyn Albright; Secretary, Karthryn Bonsted; Treasurer, Roger Coney. Monthly meetings were held with the following papers and speakers: "Elementary vectors" by J. Day; "The uses of mathematics in astronomy" by Dr. E. Cherrington; "The uses of calculus in chemistry" by Dr. H. Essex; "The relation of mathematics and philosophy" by Dr. R. Hoople. A Christmas party and an initiation banquet were added to the regular meetings.

The Mathematics Club of Boston University

President, R. Johanson; Vice President, Eleanor Martin; Secretary, Virginia M. Brigham; Treasurer, Barbara Young. The year opened with a supper social in order to interest prospective members. In April a joint meeting was held with the Mathematics Club of Wellesley College. Papers and speakers at the regular meetings were: "Cycloids" by B. Young; "Simple facts about relativity" by P. Doe; "Art and mathematics" by Professor Mode; "Albert Einstein" by J. Miller; "Mathematical stimulants" by E. Martin; "Algebra of matrices" by R. Johanson.

Pi Mu Epsilon and Alpha Mu Epsilon of St. Lawrence University

These two clubs have distinct officers and requirement for membership but hold their regular open meetings for the discussion of papers together. The papers and speakers for the year were as follows: "Puzzles" by Doris Berry; "New numbers" by N. Brandt; "Punchcard technique" by J. Palmer; "Mathematical approach in beating gambling machines" by G. Hazzard; "History of mathematics" by T. Whelley; "Mathematical prodigies" by Dorothy Clements; "Impossible geometrical constructions" by Ruth Husing; "Mathematics and the universe" by G. Moos.

Mathematics Club of Haverford College

President, W. B. Morgan, Jr.; Secretary, P. M. Whitman. Papers read and discussed at the monthly meetings were as follows: "Numerical interpolation" by H. V. Gunmere; "Fibonacci numbers" by Dr. J. H. Redfield of Villanova; "Numerical square roots" by K. A. Beck; "Indeterminate forms" by R. G. Kelly; "Transcendental numbers" by P. M. Whitman; "Applications of group theory to combination analysis" by Dr. J. H. Redfield; "Logarithms" by A. W. Gilmour and D. C. Frystinger.

Kappa Mu Epsilon of Mississippi State College for Women

President, Mabel Sherrill; Vice President, Ruby Nell Gooch; Secretary-Treasurer, Juanita Armes. Besides the regular meetings this group has an annual meeting with the chapter at Mississippi State Agricultural and Mechanical College. This year the program included these papers: "The projective construction of the general cubic and its analytic function" by Blanche Matkison; "The role of numbers in the development of mathematics" by Mary L. Eckles. In March a picnic was held. Contact is kept as far as possible with alumnae members of the chapter.

Mathematics Club of the University of Nevada

President, C. Allen; Vice President, Inez MacGillivray; Secretary-Treasurer, Margaret Jensen; Faculty Adviser, Professor H. C. Ayres. A social hour follows each regular monthly program, while a Christmas party at the home of Professor and Mrs. F. Wood, and picnic at summer home of Emily Ross at Lake Tahoe completed the entertainment part of the year. Papers read and discussed were the following: "Homogeneous coordination" by Professor H. C. Ayres; "Algebraic and geometric fallacies" by C. Allen; "History of mathematics in America" by W. Bowrin; "Early methods of counting" by Inez MacGillivray; "Space curves" by Professor H. C. Ayres;

"The applications of mathematics in electrical engineering" by Professor I. Sandorf; "Non-Euclidean geometry" by Emily Ross.

Mathematics Club of New Jersey College for Women

President, Elsie Matheke; Vice President, Helen Allen; Secretary, Jeanette Simpson; Treasurer, Margaret Matthews; Faculty Adviser, Professor C. Nelson. Two social meetings were held and the club sponsored an exhibition of models and a motion picture designed to show in non-technical terms, the development from the time of Antiphon to the time of Newton, of the technique in computing areas bounded by curves. Three meetings were devoted to reports from Arnoux "Essai de géométrie modulaire à deux dimensions" by Elsie Matheke, Helen Allen, Kathleen Lawrence, Margaret Matthews, Barbara Van Baskirk and Gladys Crenning. At other meetings were read and discussed the following papers: "Report of Commission on training and utilization of advanced students of mathematics" by Evelyn Spencer; "Isotropic lines" by Sabina Marczewski and Jeannette Simpson; "Mathematics of photography" by Professor C. Nelson; "Graphical solutions" by Janet Thompson; "Central difference interpolation" by Professor N. Walter. A joint meeting with the Mathematics Club of Rutgers University was addressed by Dr. Paul Bernays, visiting lecturer at the Institute for Advanced Study. Dr. Bernay's topic was "What is the best proof?"

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 238. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Three circles are drawn having the sides of a triangle as their diameters. A second set of circles are drawn, centered at the vertices of the triangle, each orthogonal to the circle whose diameter is the side opposite its center. Determine the radical center of this second set of circles, and show that they are orthogonal in pairs.

E 239. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Construct the triangle ABC, given c, angle C, and the ratio of b^2+a^2 to the area of the triangle.

E 240. *Proposed by A. Gloden, Luxembourg.*

Find the largest perfect square of five digits, such that the sum of its digits is a cube.

E 241. *Proposed by J. Rosenbaum, Hartford Federal College.*

On the sides A_0B_0, B_0C_0, and C_0A_0 of a triangle $A_0B_0C_0$, the points A_1, B_1, and C_1 are taken respectively, such that $A_0A_1/A_0B_0=B_0B_1/B_0C_0=C_0C_1/C_0A_0$

$=1/n$, where n is a positive integer. A triangle $A_2B_2C_2$ is now obtained from triangle $A_1B_1C_1$ in the same manner that $A_1B_1C_1$ was obtained from $A_0B_0C_0$. We now consider the sequence of triangles, $T_0, T_1, T_2, \cdots, T_n$, formed in this manner. Let R_n be the ratio of the area of T_n to T_0, and determine the limit of R_n as n increases through integer values without limit.

E 242. *Proposed by V. Thébault, Le Mans, France.*

Find a palindromic number whose digits are all even, and whose square contains the ten digit-symbols once each. Show that the solution is unique.

E 243. *Proposed by J. E. Trevor, Cornell University.*

The diameter of the tread of a driving wheel of a locomotive is five feet, and the diameter of its flange is five and a half feet. When the locomotive moves from west to east, a point on the rim of the flange moves in a curved path which at times runs from east to west. Disregarding the up and down motion on this retrograde path, the point is displaced each time a certain distance towards the west. When the locomotive is going sixty miles an hour, what is the sum of these westward displacements in one minute? (Give the answer to the nearest foot.)

E 244. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Find by elementary methods, as n increases without limit, the limit of

$$[1^3a^{1/n} + 2^3a^{2/n} + \cdots (n-1)^3a^{(n-1)/n} + n^3a]/n^4.$$

SOLUTIONS

E 199 [1936, 185]. *Proposed by W. E. Buker, Leetsdale High School, Pa.*

From an arbitrary point within a triangle, draw three lines to the sides of the triangle which shall trisect its area. (Although this problem is not new, it seems to be missing from the more readily accessible texts.)

Solution by D. L. MacKay, Evander Childs High School, New York

As the above problem has an unlimited number of solutions, we impose the further condition that one of the lines of division is drawn to a given point Q on AB, and that the required areas be in the ratio $k:l:m$.

From C draw parallels to PA and PB, meeting AB produced in D and E respectively. Then triangles ABC and PDE have the same area because triangle PAC equals triangle PAD, and triangle PBC equals triangle PBE. Produce QE to F so that QF equals DE, and cut QF into three segments in the ratio $k:l:m$ by the points G and H.

If G lies within the segment QB, call it R, and if H also lies in this segment, call it S. If G lies in the segment BE, project it parallel to BP onto BC and call that point R. If H lies in the segment BE, project it parallel to BP onto BC and call that point S. If G lies in the segment EF, locate K in DA so that DK equals EG, and project K parallel to AP onto AC at R. Similarly, if H lies in EF, locate L in DA so that DL equals EH, and project L parallel to AP onto

AC at S. Then the rays PQ, PR and PS divide the triangle ABC into regions whose areas are in the desired ratios.

When k, l and m are equal, we have the given problem.

It may be of interest to note that the problems of cutting off given fractions from the area of a triangle by lines from a given point inside or outside that triangle appear in the *Woepcke MS.*, a treatise in Arabic on the division of figures, and according to Archibald were contained in Euclid's *Division of Figures*. The same problems are given in the *Practica Geometrica* of Leonardo Pisano in 1220, and one of them was proposed by Ferrari to Tartaglia in 1547. See Archibald's *Euclid's Division of Figures* pages 9–12, 54.

Also solved in less general form by W. B. Clarke, Gilbert Hart, Leon Recht, C. W. Trigg and the proposer.

E 200 [1936, 185]. *Proposed by N. A. Court, University of Oklahoma.*

The square of the distance between a fixed point F and a variable point P is equal to the sum (or the difference) of the powers of the two points with respect to a given sphere S. Find the locus of P. (The corresponding problem in the plane was solved analytically in the Educational Times, Reprints, vol. I (1864), p. 6, by A. Cayley.)

Solution by C. W. Trigg, Cumnock College, Los Angeles

Let the center of the sphere be O, the radius be r, FO be a, FP be x and PO be y. Now the power of a point with respect to a sphere is equal, in both magnitude and sign, to d^2-r^2, where d is the distance of the point from the center of the sphere. So we have

$$x^2 = (a^2 - r^2) + (y^2 - r^2), \quad \text{or} \quad x^2 - y^2 = a^2 - 2r^2,$$

so that the locus of P is a plane perpendicular to FO. In the case of the difference of the powers we have

$$x^2 = \pm \left[(a^2 - r^2) - (y^2 - r^2)\right].$$

That is, either $x^2+y^2=a^2$, in which case the locus of P is a sphere on FO as diameter, or else $y^2-x^2=a^2$ and the locus is again a plane perpendicular to FO. The two planes are distinct except when the sphere is a point-sphere, with center at F.

Also solved by L. M. Kelly, Simon Vatriquant and the proposer.

E 201 [1936, 185]. *Proposed by A. A. Bennett, Brown University.*

Show that

$$\sum_0^3 \operatorname{arccot}(3n - 1) = 0.$$

Solution by C. V. L. Smith, Boston, Massachusetts

In solving this problem, we shall use the identity

$$\text{arccot}\, u + \text{arccot}\, v = \text{arccot}\left(\frac{uv - 1}{u + v}\right)$$

which is readily obtained from the standard formula for the cotangent of the sum of two angles.

The expression which we wish to show vanishes, may be written out as arccot (-1)+arccot (2)+arccot (5)+arccot (8), and if we pair the first and fourth terms by our formula, and the second and third, the expression becomes arccot $(-9/7)$+arccot $(9/7)$. A second use of our formula reduces our expression to arccot (∞) which is zero.

Unless we restrict ourselves to the principal values of these arccotangents, the given expression may equal any integral multiple of π.

Also solved by W. E. Buker, Fred Discepoli, Samuel Kramer, C. A. Murray, J. E. Trevor, C. W. Trigg and Simon Vatriquant.

E 202 [1936, 185]. *Proposed by W. B. Clarke, San Jose, California.*

While three auto mechanics were eating their lunch, they noticed that the center of the trademark on a tire of the nearest automobile was directly over the center of an oil spot on the floor. Measurement showed that it was two inches above the floor, and ten inches below the center of the hub.

A discussion arose as to how far they would have to roll the car straight ahead so that the centers of the trademark, hub and oil spot would again be in line. To settle it, they moved the machine and made measurements, but an argument arose. Bill maintained that the machine had been rolled exactly four feet six inches; Slim claimed it was just short of that, but Doc insisted it was a trifle more.

So they decided to roll the car ahead again until the three points were once more in line and see if they couldn't agree on that. This time Bill's measurement was seven feet nine inches from the oil spot. As before, Slim thought it was a trifle less and Doc the least bit more.

Who was right each time, and what were the distances, to the nearest sixteenth of an inch?

Solution by J. E. Trevor, Cornell University

Taking the center of the oil spot as the origin, and the track of the wheel as x-axis, the center of the trademark describes the cycloid

$$x = 12\theta - 10 \sin \theta, \qquad y = 12 - 10 \cos \theta$$

where θ is the angle swept out by the radius initially connecting the centers of the hub, trademark and oil spot. The hub center is at the point $(12\theta, 12)$.

The condition for the points $(0, 0)$, $(12\theta, 12)$ and (x, y) to be collinear is that $x/y = 12\theta/12$, which reduces by using the equations of the cycloid to $\theta = \tan \theta$. Taking the first two positive roots of this last equation from the tabulation in Jahnke and Emde's *Funktionentafeln*, page 3, we have $\theta = 4.4934$

and 7.7253, whence the distances $12\theta = 53.9208$ inches and 92.7036 inches respectively, or 4 ft. 5 15/16 in. and 7 ft. 8 11/16 in. Consequently Slim was right.

Also solved by C. W. Trigg and M. L. Turner.

E 203 [1936, 186]. *Proposed by J. F. Berry, New York, N. Y.*

Determine a point in the plane of two given circles, from which tangents to the circles shall form a given angle and have a given ratio.

Solution by Wm. Douglas, Courtenay, B. C.

At any point E on the line of centers MN erect a perpendicular to MN. At any point C on this perpendicular as vertex, construct an angle FCE equal to the given angle. (This may be drawn on either side of EC.)

Draw the tangent AB, parallel to FC, tangent to the circle centered at M at the point A, and meeting EC at B. (Either of the two tangents parallel to FC will do.) Locate D on the arm of the given angle ABE so that AB and BD are in the given ratio.

Locate R on MN so that RM equals the radius of the circle centered at N. (R may be within or without the segment MN.) With MN as radius and R as center, describe an arc to meet AD at G. (Either intersection will do.) Through G draw GK parallel to DB and meeting AB at K. Then AK and KG are in the given ratio, and AKG is the given angle.

From G draw a parallel to MN. With M as center and MN as radius, draw an arc cutting this parallel at H. Since $GHMR$ is a parallelogram (for proper choice of H), the circle centered at H and through G is equal to the given circle centered at N, and the same distance from the circle centered at M. Moreover, the point K bears the desired relationships to the circles centered at M and H. Hence we need only to rotate the triangle AKG about M through the angle HMN, into the position $A'K'G'$ to complete our solution. Then K' is the desired point, and $A'K'$ and $K'G'$ are in the desired ratio.

Because of the double choices indicated by the parenthetical notes in the first three paragraphs of this solution, there are really sixteen possible positions for the point K in the general case, with MN as an axis of symmetry. These will coincide by pairs after the rotation, giving at most eight different positions for K'.

D. L. MacKay notes that question 2 in Mathesis, vol. 1 (1881), pp. 12, 43, gives many properties of the figure of this problem. Among them is the statement that the locus of the vertex of a constant angle whose sides are tangent to two given circles is composed of several limaçons of Pascal. He further refers to L'Intermédiaire des Mathématiciens, vol. 9 (1902), pp. 117, 286.

E 204 [1936, 186]. *Proposed by J. E. Trevor, Cornell University.*

Let $abcd$ be any four-digit number, with a not zero. The sum of the digits in the product, $99 \cdot abcd$, is either 18, 27 or 36. Find the necessary and sufficient conditions on a, b, c and d, under which each of these three values is produced.

Solution by W. E. Buker, Leetsdale High School, Pa.

The product $99(abcd)$ is equal to $100(abcd)-abcd$. If the relative sizes of a and c, and of b and d are known, each digit of the required product can be represented.

Thus if $d>b$, $c\geqq a$, the successive digits of the product, from right to left, are $10-d$, $9-c$, $d-b-1$, $c-a$, b and a, whose sum is 18; if $d>b$, $c<a$, the digits are $10-d$, $9-c$, $d-b-1$, $c+10-a$, $b-1$ and a, whose sum is 27; if $d\leqq b$, $c>a$, the digits are $10-d$, $9-c$, $d+9-b$, $c-a-1$, b and a, whose sum is again 27; while if $d\leqq b$, $c\leqq a$, the digits are $10-d$, $9-c$, $d+9-b$, $c+9-a$, $b-1$ and a, whose sum is 36.

Also solved by M. L. Constable, Samuel Kramer, C. A. Murray, C. W. Trigg, Simon Vatriquant and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3801. *Proposed by D. H. Lehmer, Lehigh University.*

Show that

$$\operatorname{arccot} 1=\operatorname{arccot} 2+\operatorname{arccot} 5+\operatorname{arccot} 13+\operatorname{arccot} 34+\cdots,$$

where these integers constitute every other term of the Fibonacci series and satisfy the recurrence $u_{n+1}=3u_n-u_{n-1}$.

3802. *Proposed by D. H. Lehmer, Lehigh University.*

Let $0, u_1, u_2, u_3, , \cdots$ be a sequence of numbers satisfying the recurrence $u_{n+1}=au_n+bu_{n-1}$, and consider the function

$$f(x) = \sum_{1}^{\infty} u_n x^n/n!.$$

Show that $f(x)=-e^{ax}f(-x)$.

3803. *Proposed by Frank Ayres, Jr., Dickinson College.*

Prove that

$$\text{(a)} \qquad \sum_{s=0}^{n} (-1)^s 2^{2s}\binom{n}{s}\binom{2n-2s}{n-s} = (-1)^n\binom{2n}{n},$$

$$\text{(b)} \qquad \sum_{s=0}^{n-1} (-1)^s 2^{2s}\binom{n}{s}\binom{2n-2s}{n+1-s} = (-1)^{n-1}\binom{2n}{n+1}.$$

3804. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres with non-coplanar centers, their six spheres of similitude are grouped into three pairs so that each pair involves all four given spheres. If the spheres in two of these pairs are orthogonal, the two spheres of the third pair are also orthogonal to each other. (See, for instance, *Modern Pure Solid Geometry* by the proposer. The Macmillan Co., 1935.)

3805. *Proposed by R. E. Gaines, University of Richmond.*

Determine a point P on $b^2x^2-a^2y^2=a^2b^2$ such that the tangent and normal lines at that point shall be normal and tangent respectively to $b^2x^2-a^2y^2=-a^2b^2$, and, hence, that if the hyperbola and its conjugate be together considered as a single curve, $(b^2x^2-a^2y^2)^2-a^4b^4=0$, a rectangle may be drawn which is both an inscribed and a circumscribed figure.

3806. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Through the vertices A, B, C of a triangle are drawn three parallels α, β, γ. The symmetricals of BC, CA, AB with respect to β, γ, α form a triangle $A_1B_1C_1$ and those of the same sides with respect to γ, α, β form a triangle $A_2B_2C_2$. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar to ABC, the ratio being

$$\frac{1}{4S}\left(a^2 \sin 2\theta_a + b^2 \sin 2\theta_b + c^2 \sin 2\theta_c\right),$$

a, b, c and S being the sides and the area of the triangle ABC, and θ_a, θ_b, θ_c the angles of α, β, γ with BC, CA, AB.

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are symmetrical with respect to a point P on the nine point circle of the triangle ABC, and if H_1 and H_2 are their orthocenters, H_1 and H_2 have, with respect to the triangles $A_1B_1C_1$ and $A_2B_2C_2$ respectively, the same polar, a straight line passing through P.

SOLUTIONS

3716. *Proposed by J. D. Hill and H. J. Hamilton, Brown University.*

Let, $E_1, E_2, \cdots, E_n, \cdots$ be an infinite sequence of measurable sets in the interval (a, b) such that $mE_n \geqq k > 0$ for $n=1, 2, \cdots$. Does there necessarily exist some infinite sequence of indices $1 \leqq r_1 < r_2 < \cdots < r_i < \cdots$ for which the measure of $E_{r_1} \cdot E_{r_2} \cdots E_{r_i} \cdots$ is greater than zero?

Solution by the Proposers

The answer is no; for let $E_n (n=1, 2, \cdots)$ be the set of points whose abscissae in the binary scale are given by

$$0 \cdot \alpha_{n1}\alpha_{n2} \cdots \alpha_{n,n-1} 1 \alpha_{n,n+1} \cdots \tag{1}$$

where accordingly $\alpha_{nj}=0$ or 1. To each of the 2^{n-1} ways of choosing the first $n-1$ digits of (1) there corresponds a largest and a smallest number, these

being obtained, respectively, by setting $\alpha_{nj}(j \geqq n+1)$ equal to 1 or 0. The difference between these extreme numbers is clearly

$$\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^n},$$

and thus by varying the digits beyond the nth place all the points of an interval of length $1/2^n$ are obtained. Furthermore, for two distinct choices of the initial $n-1$ digits, the corresponding intervals are *non-overlapping*, since the abscissae of the left end points of two such intervals differ by at least $1/2^{n-1}$. Hence E_n consists of the sum of 2^{n-1} non-overlapping intervals, each of length $1/2^n$ and the measure of E_n is therefore 1/2. Moreover, each E_n lies in the interval (0, 1).

Consider *any* infinite product

$$G \equiv E_{\nu_1} \cdot E_{\nu_2} \cdots E_{\nu_i} \cdots \qquad (1 \leqq \nu_1 < \nu_2 < \cdots < \nu_i < \cdots)$$

formed from sets of the sequence $\{E_n\}$ and let

$$G_i \equiv E_{\nu_1} \cdot E_{\nu_2} \cdots E_{\nu_i}.$$

The abscissa of each point in G_i has the form

$$(2) \qquad 0 \cdot \alpha_1\alpha_2 \cdots \alpha_{\nu_1-1}1\alpha_{\nu_1+1} \cdots \alpha_{\nu_2-1}1\alpha_{\nu_2+1} \cdots \alpha_{\nu_i-1}1\alpha_{\nu_i+1} \cdots .$$

Corresponding to each fixed choice of the α's preceding the ν_ith place, the totality of the numbers (2) constitutes an interval of length $1/2^{\nu_i}$. There are altogether $2^{(\nu_i-1)-(i-1)} = 2^{\nu_i - i}$ such intervals in G_i and these are, as before, non-overlapping. Hence

$$mG_i = 2^{\nu_i - i} . \frac{1}{2^{\nu_i}} = \frac{1}{2^i}.$$

But since G is a subset of G_i for every $i = 1, 2, \cdots$, it follows that $mG = 0$.

Solved also by H. D. Ruderman and Hassler Whitney.

3720 [1935, 49]. *Proposed by C. J. Coe, University of Michigan.*

In transforming coordinates from the rectangular system $OX_1Y_1Z_1$ to the congruent rectangular system $OX_2Y_2Z_2$ we have the following table of cosines:

	X_1	Y_1	Z_1
X_2	λ_1	μ_1	ν_1
Y_2	λ_2	μ_2	ν_2
Z_2	λ_3	μ_3	ν_3

the determinant of the array having the value unity. Prove that

$$\lambda_1 + \mu_2 + \nu_3 \geqq -1.$$

Solution by E. T. Browne, University of North Carolina.

Consider the following equation:

$$\begin{vmatrix} \lambda_1 - x & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 - x & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 - x \end{vmatrix} = 0.$$

This is immediately recognized to be the characteristic equation of a real orthogonal matrix, so that the roots all have modulus unity.* Two of the roots may, therefore, be denoted by $\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$, while the third, necessarily real, is 1 or -1, in this case 1, since the product of the three roots equals the determinant.

Now the sum of the three roots is equal to the *trace* of the matrix, i.e., the sum of the terms in the principal diagonal. Hence,

$$\lambda_1 + \mu_2 + \nu_3 = 2\cos\theta + 1 \geqq -1.$$

If the value of the determinant were -1, instead of $+1$, the third root above would be -1, and it follows in an entirely similar manner that in this case

$$\lambda_1 + \mu_2 + \nu_3 \leqq 1.$$

Solved also by J. A. Greenwood and E. J. McShane.

3721 [1935, 114]. *Proposed by Albert Whiteman, Philadelphia, Pa.*

Prove by Fermat's method of infinite descent that an odd prime p of the form $3n+2$ has the quadratic non-residue -3.

Solution by the Proposer

If -3 is not a non-residue of all primes of the form $3n+2$, then let p be the smallest odd prime of this form for which the congruence $x^2 \equiv -3 \bmod p$ is satisfied. Let $x = e$ be a solution of this congruence where $e < p$. We may suppose e to be even since if e were odd then $p - e$, another solution of the congruence, would be even. We shall first consider the case where $e^2 \equiv 1 \bmod 3$.
Let us write

$$e^2 \equiv -3 \bmod p$$

or

(1) $$e^2 = -3 + fp, \qquad f < p, \qquad f \text{ odd}.$$

Hence

$$pf = e^2 + 3 \equiv 4 \bmod 3.$$

Since

$$p \equiv 2 \bmod 3,$$

we get

$$f \equiv 2 \bmod 3.$$

* Hilton, *Homogenous Linear Substitutions*, Oxford 1914, Ex. 14, p. 37.

Now f, which is odd and of the form $3n+2$, must have an odd prime factor q of the form $3n+2$, since, if its factors were all of form 3 or $3n+1$, their product would be of the form $3n$ or $3n+1$. From (1) we have

$$e^2 \equiv -3 \bmod f,$$

so that

$$e^2 \equiv -3 \bmod q.$$

But this last congruence contradicts the hypothesis that p is the smallest prime of the form $3n+2$ for which -3 is a quadratic residue.

We shall now consider the case where $e^2 \equiv 0 \bmod 3$. Let us write

$$e = 3^a k, \qquad k \not\equiv 0 \bmod 3.$$

Since

$$e^2 \equiv -3 \bmod p,$$

we have

$$3^{2a}k^2 \equiv -3 \bmod p;$$

whence

$$3^{2a-1}k^2 \equiv -1 \bmod p,$$

or

$$(2) \qquad 3^{2a-1}k^2 + 1 = ph, \qquad h < p, \qquad h \text{ odd}.$$

Hence

$$ph \equiv 1 \bmod 3.$$

But

$$p \equiv 2 \bmod 3,$$

so that

$$h \equiv 2 \bmod 3.$$

Then h, which is odd and of the form $3n+2$, must have an odd prime factor r of the form $3n+2$. From (2) we have

$$3^{2a-1}k^2 \equiv -1 \bmod h.$$

Hence

$$3^{2a-1}k^2 \equiv -1 \bmod r,$$

and

$$3^{2a}k^2 \equiv -3 \bmod r.$$

Since we are again led to a contradiction there can be no smallest odd prime of the form $3n+2$ for which -3 is a quadratic residue. Hence the theorem is proved. From this it follows that any quadratic non-residue of a prime p of the form $3n+2$ is congruent, mod p, to a number of the form $-3a^2$, $a=1, 2, \cdots, (p-1)/2$.

Solved also by C. G. Latimer.

3722 [1935, 114]. *Proposed by E. P. Starke, Rutgers University.*

Let the points z of the complex plane correspond to those of the sphere by stereographic projection. (See Townsend, *Functions of a Complex Variable*, pp. 184–190.) Find the transformation on z which results from an arbitrary rotation of the sphere.

Editorial Note. The proposer's solution used the methods of the theory of functions of a complex variable and the analytic formulas for the projection, with several references to the text cited in the problem. The problem may be treated in a more elementary manner by regarding it as an example in inversion. In the inversion (N, d), where N is the pole and d is the radius of the sphere of inversion, it is well known that a sphere, or a plane, inverts into a sphere, or a plane; and that the angle between two curves is unaltered by the transformation. All these theorems admit very elementary geometric proofs. Let N and S be the north and south poles of a sphere of diameter d; then in the inversion (N, d) the sphere inverts into the tangent plane π at S, and to a point Q on this sphere (S) corresponds a single point P in π, and conversely. There is one exception; to N itself correspond the infinite points in π. Now consider another diameter N_1S_1 as an axis l_1. To N_1 and S_1 correspond in π the points n_1 and s_1. Since $s_1N n_1$ is a triangle with a right angle at N, we have

$$s_1S\cdot Sn_1 \equiv d^2. \tag{1}$$

By the above theorems the meridian circles of l_1 invert into a system of circles in π passing through n_1 and s_1; and conversely any such circle must invert into a circle on (S) through N_1 and S_1, and hence it must be a meridian circle of l_1. Similarly, the parallel circles of l_1 invert into a second system of circles in π such that each circle is orthogonal to each one of the first system. We may think of the second system as the set of circles with centers on n_1s_1 orthogonal to the circle with the diameter n_1s_1. If a circle of the second system cuts n_1s_1 in A and B, then the latter two points separate n_1 and s_1 harmonically. Also, if P is any point on the circle with diameter AB, we have at once

$$\frac{n_1P}{s_1P} = \frac{n_1A}{s_1A} = \text{constant}. \tag{2}$$

Consider now two points Q and Q' on a parallel circle of l_1, and let the meridian circles through these two points make the angle γ with one another. Then in the map of (S) on π we have two circles cutting at the angle γ in n_1 and s_1, and a

circle of the second system cutting orthogonally the first two in P and P', the inverses of Q and Q'. In the two triangles s_1Pn_1 and $s_1P'n_1$, let the angles at P and P' be respectively ψ and ψ'. Then the difference of these two angles is γ.

Consider S as the origin of complex variables in π. Let z, z', s_1, n_1 denote the complex numbers for P, P', s_1, n_1. Then from (1) we have at once

$$n_1\bar{s}_1 = -d^2, \tag{3}$$

where $\bar{s}_1$ is the conjugate of s_1. Also, from the triangles s_1Pn_1 and $s_1P'n_1$, we have

$$\frac{z'-n_1}{z'-s_1} = \frac{n_1P'}{s_1P'}e^{i\psi'}, \qquad \frac{z-n_1}{z-s_1} = \frac{n_1P}{s_1P}e^{i\psi}. \tag{4}$$

Then from (2) we have

$$\frac{z'-n_1}{z'-s_1} = e^{i\gamma}\frac{z-n_1}{z-s_1}, \qquad n_1\bar{s}_1 = -d^2. \tag{5}$$

This is the desired result since it gives the transformation of π due to a rotation of (S) about l_1 through the angle γ.

In order to write (5) in the explicit form of z' as a function of z, set $z'=Z'd$, $z=Zd$, $n_1=nd$, $s_1=sd$, and we have

$$\frac{Z'-n}{Z'-s} = e^{i\gamma}\frac{Z-n}{Z-s}, \qquad n\bar{s} = -1. \tag{6}$$

After solving for Z' we obtain

$$Z' = \frac{AZ+B}{-\bar{B}Z+\bar{A}}, \qquad \begin{aligned} A &= s\bar{s}e^{i\gamma/2}+e^{-i\gamma/2}, \\ B &= s(e^{i\gamma/2}-e^{-i\gamma/2}), \end{aligned} \tag{7}$$

$$A\bar{A}+B\bar{B} = (s\bar{s}+1)^2, \qquad n\bar{s} = -1.$$

The corresponding expression for z' can be easily obtained from this result.

3724 [1935, 115]. *Proposed by R. E. Gaines, University of Richmond.*

At any point A of a conic a normal chord AB is drawn. The chord BC is drawn perpendicular to AB. Tangents at B and C meet the tangent at A in P and P'. Secants PQR and $P'Q'R'$ are drawn, and the lines AQ and AR meet the lines BR' and BQ' in O and O'. Prove that OO' is parallel to PP'.

Solution by Otto J. Ramler, The Catholic University of America

Let OO' meet PP' at M, and BA at N. Then since PP', RQ, and the tangent at B meet in P, R and Q are corresponding elements in an involution on the conic with A and B as double elements.

Hence $(RQ, AB) = -1 = A(O'O, MN) = B(Q'R'C'A)$ where C' is the second intersection of the conic with BM. Since PA, $R'Q'$ and the tangent at C meet in P', we have a second involution with A and C as double elements.

Hence $(Q'R', CA) = (Q'R', C'A)$, and $C = C'$. It follows that BC meets PP' in M. But BC is parallel to PP'. Therefore M is at infinity, and OO' is parallel to PP'.

Solved also by J. W. Clawson and A. S. Householder.

Editorial Note. The solution above proves the more general theorem that, if ABC is any triangle inscribed in the conic, the lines PP', BC, OO' meet in a point. The given problem is a special case in which BC is parallel to PP'. Householder proved this general theorem using homogeneous coordinates with ABC as the triangle of reference. Clawson proved the theorem of the problem for any chord AB by considering the projection of the conic into a circle and the line OO' into the line at infinity.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbia, Ohio.

The following mathematicians received honorary degrees from Harvard University on September 18 in connection with the Tercentenary Celebration: Professors E. J. Cartan, of the University of Paris; L. E. Dickson, of the University of Chicago; G. H. Hardy, of the University of Cambridge, England; and Tullio Levi-Civita, of the University of Rome.

An honorary degree has been conferred on Professor Stepan Timoshenko, of the University of Michigan, by Lehigh University.

Dr. R. Lucile Anderson has been promoted to an assistant professorship at Hunter College.

Dr. L. M. Blumenthal, formerly National Research Fellow and recently an assistant at the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Missouri.

At Wayne University Dr. W. M. Borgman has been promoted to an assistant professorship, and Dr. K. W. Folley and Dr. D. C. Morrow to associate professorships.

Associate Professor C. T. Bumer of Kenyon College has been promoted to a professorship in mathematics.

Dr. J. H. Butchart of Butler University has been appointed professor and head of the department of mathematics at Phillips University, Enid, Oklahoma.

Dr. J. L. Dorroh of Johns Hopkins University has been appointed an associate professor at Judson College, Marion, Alabama.

H. F. Fehr of the State Teachers College, Montclair, N. J., has been promoted to an assistant professorship.

Dr. R. D. James has been promoted to an assistant professorship at the University of California.

D. H. Leavens of the Harvard Business School has joined the staff of the Cowles Commission for Research in Economics, Colorado Springs.

Professor Anna D. Lewis of Lake Erie College retired in June 1936, and is now living in St. Paul, Minn. Dr. Ruth M. Peters has been appointed her successor at Lake Erie College.

Dr. W. I. Miller of the University of Pittsburgh has been appointed an assistant professor at Bucknell University Junior College, Wilkes-Barre, Pa.

C. N. Mills of Illinois State Normal University has been appointed acting assistant professor of the teaching of mathematics at the University of Wisconsin for the year 1936–37.

The following appointments to instructorships have been announced
Brown University: Dr. J. S. Frame, Dr. G. B. Price
Harvard University: Dr. H. M. MacNeille (Benjamin Peirce Instructor)
University of Maine: Dr. S. H. Kimball
Michigan State College: Dr. J. D. Hill
New Jersey State Teachers College: E. R. Stabler
Ohio State University: Dr. T. H. Southard
University of Pennsylvania: Dr. J. A. Clarkson
Purdue University: Dr. M. W. Keller
Sophie Newcomb Memorial College: Dr. M. Gweneth Humphreys
Trinity College, Hartford, Conn.: Dr. T. L. Downs, Jr., J. F. Wyckoff
Wayne University: Dr. Max Coral
University of Wisconsin at Milwaukee: G. S. Cook, Dr. Louise A. Wolf
Yale University: Dr. Nelson Dunford.

Dr. C. H. Ashton, professor of mathematics and for many years chairman of the department at the University of Kansas, died early in August at the age of seventy. He was a charter member of the Association, and took an active part in the affairs of the Kansas Section.

Dr. E. B. Lytle, associate professor at the University of Illinois, where he had taught for twenty-eight years, died September 8, 1936, at the age of sixty-one. He was a charter member of the Association.

Dr. Ganesh Prasad, the Life-President of the Benares Mathematical Society, died March 9, 1935.

Dr. A. A. Titsworth, professor of mathematics at Rutgers University, died on August 15 at the age of eighty-three. He was for many years a member of the Association.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, Low Memorial Library, Columbia University, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Annual Meeting, Duke University, Durham, N. C., Dec. 31, 1936–Jan. 1, 1937.

Twenty-first Summer Meeting, Pennsylvania State College, Sept., 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, Normal, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, Topeka, March 14.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov. 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, May 9; Washington, Dec. 5.
MICHIGAN, Ann Arbor, March 21; Albion, Nov. 28.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S.C., April 17–18.
SOUTHERN CALIFORNIA, Fullerton, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, Madison, May 9.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

INDEX TO VOLUME XLIII 1936

THE AMERICAN MATHEMATICAL MONTHLY

By R. G. SANGER, The University of Chicago

PAPERS, REPORTS OF MEETINGS

QUESTIONS, DISCUSSIONS, AND NOTES

RECENT PUBLICATIONS—NEW BOOKS RECEIVED

RECENT PUBLICATIONS—REVIEWS

MATHEMATICAL CLUBS—TOPICS

MATHEMATICAL CLUBS—ACTIVITIES

PROBLEMS—AUTHORS

Numbers refer to pages, black face type indicating a problem solved and solution published; italics, a problem solved but solution not published; ordinary type, a problem proposed.

PROBLEMS—SOLUTIONS

Numbers in black-face type refer to problems, those in light face to pages.

NEWS AND NOTICES

PERSONAL MENTION

This section contains the names of members attending meetings, persons taking any active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

NECROLOGY

CORRIGENDA

Volume XLIII, 1936.

P. 77, line 1, for "e^u," read "e^{-u}."

P. 77, equation (25), for "$(c+q)/\sqrt{c^2-q^2}$," read "$(c-q)/\sqrt{c^2-q^2}$."

P. 78, equation 33, for "$(dq/dt)/\sqrt{c^2-q^2}$." read "$c(dq/dt)/(c^2-q^2)$."

P. 80, line 16, for "$2m$ cosh w sinh u," read "-2cm cosh w sinh u."

P. 80, line 3 from bottom, for "Equation (48) $\cdots$," read "For a stationary target, equation (48) $\cdots$."

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLIII, 1936

NUMBER 10, DECEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The twentieth regular meeting of the Rocky Mountain Section was held at the University of Denver, Denver, Colorado, on April 17–18, 1936. There were three sessions. Professor A. W. Recht presided at each.

The attendance was forty-one, including the following twenty-five members of the Association: L. A. Aroian, Jack Britton, Sister Rose Margaret Cook, W. D. Dickinson, Jr., J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, I. L. Hebel, C. A. Hutchinson, L. Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, W. V. Lovitt, S. L. Macdonald, A. E. Mallory, A. S. McMaster, W. K. Nelson, Greta Neubauer, E. D. Rainville, O. H. Rechard, A. W. Recht, C. H. Sisam, W. M. Stewart.

Professor A. E. Mallory, Colorado State College of Education, was elected chairman for the coming year. Professor C. A. Hutchinson, University of Colorado, was elected vice-chairman.

The following papers were read:

1. "Thiele's semi-invariants and their application to problems in statistics" by Professor L. A. Aroian, Colorado State College.

2. "The separation of the roots of the trinomial equation" by Professor A. J. Lewis, University of Denver.

3. "Theorems on a classical heat problem" by E. D. Rainville, United States Bureau of Reclamation.

4. "On the determination of the coefficients of the Kreisteilungs-Gleichung" by Professor A. J. Kempner, University of Colorado.

5. "Four notes on the solution of systems of linear matrix equations in two and three unknowns" by Professor O. H. Rechard, University of Wyoming.

6. "The mathematics of ancient China" by Professor I. L. Hebel, Colorado School of Mines.

7. "Pohlke's theorem in four dimensions" by Professor C. H. Sisam, Colorado College.

8. "The analytic discussion of the locus of the radical center of three circles related to a triangle" by W. M. Stewart, University of Wyoming.

9. "The projective generation of curves and surfaces" by Professor Claribel Kendall, University of Colorado.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles:

1. The main properties of semi-invariants were derived by Professor Aroian and applied to the following types of distributions: normal, binomial, Poisson, and Pearson type III. The results were further applied to the Gram-Charlier distribution, and to the problem of the distribution of means in samples of N from an infinite population. The exposition was based on class notes of Professor C. C. Craig.

2. In this paper Professor Lewis shows a method of separating the roots of

a trinomial equation by dividing concentric rings, having the origin of the complex plane as center, into equal sections each containing just one root of the given equation.

3. Mr. Rainville considered what may be a new solution for the problem of the conduction of heat in an infinite slab initially at a constant temperature and with its surfaces held thereafter at a different constant temperature. The newer solution presents marked advantages for purposes of extensive computations as compared to the classical solution of Fourier.

4. A very simple mechanical rule was established by Professor Kempner for the determination of the coefficients of the irreducible Kreisteilungs-Gleichung.

5. In 1933 Miss Achenbach presented a method for finding the unique solution of each of the sixteen systems of linear matrix equations which may be obtained from the system $A_iX+B_iY=C_i(i=1, 2)$, by permitting the coefficients A_i, $B_i(i=1, 2)$ to be transferred from left to right-hand multipliers of their respective unknowns in all possible combinations one, two, three, and four at a time. In solving the simple system in which all the coefficients are on the left, she included among the restrictions on the system that the matrices A_i, B_i $(i=1, 2)$ be non-singular. Professor Rechard presented four notes concerning this system of equations.

Note I shows that a solution of the system can be found even if one of the four coefficients is singular. Note II establishes in general a solution, other than the trivial one in case C_1 and C_2 are both zero. Note III points out the fact that the sixteen systems treated are all special cases of the general one, $A_iXB_i+C_iYD_i=E_i(i=1, 2)$. This inclusive system yields to the method employed to solve the fourteen systems in which all the coefficients are not on the same side of the unknowns. In Note IV the solution for a system in three unknowns, in which the coefficients are all on the same side of their respective unknowns, is developed under suitable restrictions on the singularity of the matrices involved.

6. To compare the mathematical development of early China with that of other nations, the ancient "Arithmetic in Nine Sections" was reviewed by Professor Hebel. He gave the mensuration rules of the early Chinese and their processes for the manipulation of fractions and the solution of certain systems of equations in more detail than is given in the usual mathematical histories, thus establishing the Chinese among the pioneers in mathematical science.

7. Pohlke's theorem, in four dimensions, was discussed by Professor Sisam. This theorem can be stated as follows: Let OP_i $(i=1, 2, 3, 4)$ be any four given line segments originating at 0 and lying in a space π of three dimensions. There exist four equal, mutually orthogonal segments $O^*P_i^*$, lying in a four dimensional space that contains π, from which the given segments may be obtained by a sequence of (at most) two parallel projections of which the first projects the four orthogonal segments on a three-space π' and the second is orthogonal to π'.

8. Mr. Stewart discussed the following locus problem: Let a general triangle with the vertices A, B and C, and the opposite sides a, b and c respectively, be considered. Let any straight line l intersect the sides, or the sides extended, in the finite points a', b' and c' respectively. Then let three circles be drawn; the first with center at A and radius Aa'; the second with center at B and radius Bb'; and the third with center at C and radius Cc'. Determine the locus of the radical center of these three circles, as the line l rotates about a finite fixed point (x_j, y_j), and the asymptotes of this locus and find the locus of this radical center when the point (x_j, y_j) moves to infinity in various directions. Results were obtained by elementary methods for the general situation and certain limiting cases.

9. In this expository paper Professor Kendall discussed in some detail the projective generation of point-rows of the second order from two non-concentric, non-perspective pencils of lines lying in the same plane, and the dual problem concerning sheaves or pencils of rays of the second class, these loci being identified with our ordinary curves of the second order. Following this she discussed the generation of curves of the third order formed from the points of intersection of corresponding rays of a sheaf of rays of the first order projective to a sheaf of rays of the second order. Brief mention was made of the extension of these methods of generating curves of higher order and of generating surfaces.

A. J. Lewis, *Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 9, 1936. The Chairman, Professor G. T. Whyburn, of the University of Virginia, presided over both sessions, morning and afternoon. Commander J. A. Logan, executive officer and acting head of the Post-graduate School of the U. S. Naval Academy, officially welcomed the members and their guests. Five papers were presented at the morning session, while in the afternoon, at the invitation of the Section, Professor Tobias Dantzig of the University of Maryland delivered a lecture entitled "Some curious aspects of mathematical history."

The attendance was sixty-nine including the following forty-three members of the Association: O. S. Adams, N. H. Ball, C. C. Bramble, Paul Capron, Randolph Church, G. R. Clements, Abraham Cohen, A. E. Currier, J. H. Curtiss, Alexander Dillingham, J. A. Duerksen, P. J. Federico, Michael Goldberg, Harry Gwinner, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, C. L. Leiper, S. B. Littauer, G. A. Lyle, Carol V. McCamman, E. J. McShane, Ruth G. Mason, Florence M. Mears, T. W. Moore, F. D. Murnaghan, J.L. Nagle, Walter Penney, C.H. Rawlins, A.W. Richeson, R.E. Root,

F. W. Sohon, J. L. Stearn, J. M. Stetson, John Tyler, F. M. Weida, C. H. Wheeler, G. T. Whyburn, John Williamson, E. W. Woolard, Oscar Zariski, R. T. Zoch.

The following officers were elected for the year 1936–37: Chairman, John Williamson, The Johns Hopkins University; Secretary, Michael Goldberg, Navy Department; Members of the Executive Committee, J. B. Scarborough, U. S. Naval Academy, and C. H. Wheeler, University of Richmond. The fall meeting will be held at the Bureau of Standards, Washington, D. C. on Saturday, December 5, 1936.

The following six papers were read:

1. "An examination for instructors in mathematics" by Professor G. R. Clements, U. S. Naval Academy.

2. "On the abstract definition of a group" by Dr. Abraham Sinkov, Washington, D. C., introduced by Mr. Goldberg.

3. "An algebraic approach to weather forecasting" by Professor R. E. Root, U. S. Naval Academy.

4. "Newton's method and the method of iteration" by Professor J. M. Stetson, College of William and Mary.

5. "A method for finding equations of composite surfaces" by Professor G. C. Vedova, St. John's College, introduced by Professor Rawlins.

6. "Some curious aspects of mathematical history" by Professor Tobias Dantzig, University of Maryland, by invitation of the Program Committee.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Clements gave an account of a recent examination for instructors in mathematics at the United States Naval Academy. A substantiating examination in college mathematics had been given to an outstanding group of candidates trained at a widely distributed group of colleges and universities east of the Mississippi River. While the results were on the whole satisfactory, instances were cited which seriously question the thoroughness of undergraduate teaching of mathematics, and current methods of judging the ability of a teacher.

2. The problem frequently arises in abstract group theory to determine the group which is generated by a set of operators satisfying certain given conditions. Dr. Sinkov discussed a solution of this problem that has been obtained by H. S. M. Coxeter and J. A. Todd, both of Cambridge. He illustrated their method, which is purely mechanical, by means of a particular example.

3. Professor Root discussed numerical methods for predicting changes in the field of pressure pertinent to the problem of weather forecasting. Referring to the work of Sverre Petterssen of Norway, and of I. M. Angervo of Finland, he emphasized the simplicity of direct algebraic processes which yield the same results. He indicated the derivation of formulas for the motion of isobars, of isallobars, of trough or wedge lines, of pressure centers, and of warm or cold fronts. He suggested possible advantages to be gained by considering the pres-

sure field as given by an algebraic polynomial, and by treating this as an interpolation formula from the point of view of statistical analysis.

4. Writers on the numerical solution of equations by the method of iteration usually give a rough criterion for the convergence and rapidity of convergence for any particular formula of iteration, but pay less attention to deriving such formulas. Using the usual criterion as a guide, Professor Stetson showed how to find as many such formulas as one pleases. Of these, the "simplest" is a common modification of Newton's Method, and another "almost as simple" is Newton's Method itself.

5. The method used by Professor Vedova for finding the equations of composite surfaces was based mainly on a function $S(a, b, x)$ defined by means of a limiting process, and its employment in the way suggested by the following

THEOREM: *If $f_i(x)$ is continuous in $a_i \leqq x \leqq a_{i+1}$, and $f_{i+1}(x)$ in $a_{i+1} \leqq x \leqq a_{i+2}$, where $a_i < a_{i+1} < a_{i+2}$, and if $f_i(a_{i+1}) = f_{i+1}(a_{i+1})$, then the function*

$$y = \sum_{i=1}^{n} S(a_i, a_{i+1}, x) f_i(x)$$

is continuous and takes on the value $f_i(x)$ in $a_i \leqq x < a_{i+1}$, for $i = 1, 2, \cdots, n$, and the value zero when $a_1 > x$ or $x \geqq a_{n+1}$.

By easy extensions of the method to two-dimensional space, in which the intervals on the x-axis were replaced by regions in the xy-plane, and the curves $y = f_i(x)$ by surfaces $z = F_i(x, y)$, equations of the analogous "composite surfaces" were obtained. Professor Vedova exhibited drawings of the composite surfaces—chosen from among the commoner ones in architecture—whose equations were developed as particular illustrations.

6. There is probably no other field of scientific endeavor about which the lay public entertains as many distorted notions as about mathematics. One of these notions could be called impeccability of mathematical progress. The idea prevails that, unlike any other body of human knowledge, which admittedly advanced by trial and error, the edifice of mathematics was reared on pure reason and according to a preconceived plan which was executed without blunder or hesitancy, as though not by the erring hand of man, but by the infallible spirit of God. These and similar contentions were discussed by Professor Dantzig in the light of history. He showed by illustrations, drawn from many fields of mathematics both pure and applied, that, far from following the systematic exposition of our formal treatises, the historical course of mathematics has been most erratic; that, while some achievements have been the result of prolonged reflections by mature minds, many others just as far-reaching in their effects sprang from the imaginations of mere adolescents; that some of our most fundamental concepts are empirical in origin, while many others remained inarticulate for thousands of years; that many a mathematical discovery was purely accidental, while others remained unutilized, only to be rediscovered

independently many decades later; that there is hardly one outstanding mathematician who at one time or another did not come out with erroneous statements, and that so great is our reliance on authority, that many such errors remained undetected or at least unchallenged for decades at a time; that the reasoning of many a mathematical genius has been steeped in occultism, and that to this day the occult in subtler guises still permeates such conceptions as space, time, matter and the infinite.

JOHN WILLIAMSON, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Gustavus Adolphus College, St. Peter, Minnesota, on May 16, 1936. Sessions were held at 11:00 A.M. and at 2:15 P.M. Luncheon was served at 12:45 P.M.

At the morning session the chairman of the section, Professor Inez Rundstrom, presided, and at the afternoon session Professor L. E. Bush acted as chairman. Eighty-five persons attended the meeting included the following twenty-seven members of the Association: W. E. Brooke, L. E. Bush, W. H. Bussey, Elizabeth Carlson, C. S. Carlson, Sister M. Claudette, R. W. Cowan, H. H. Dalaker, Margaret C. Eide, Gladys Gibbens, W. L. Hart, H. E. Hartig, Dunham Jackson, C. M. Jensen, W. H. Kirchner, Juliette E. Marshall, J. M. H. Olmsted, A. L. O'Toole, G. C. Priester, L. J. Quaid, Inez Rundstrom, M. G. Scherberg, F. J. Taylor, Ella Thorp, A. L. Underhill, K. W. Wegner, G. L. Winkelmann; and Sister Thomas à Kempis, institutional member representative.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of Gustavus Adolphus College, and the efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman, A. L. O'Toole, College of St. Catherine; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: E. N. Oberg, Rochester Junior College, Marion B. White, Carleton College, Inez Rundstrom, Gustavus Adolphus College.

The following fifteen papers were presented:

1. "Certain trigonometric integrals with infinite limits" by Professor H. P. Thielman, The College of St. Thomas.

2. "Experimental testing in mathematics in the General College" by Mary Elvebach, University of Minnesota, introduced by the Secretary.

3. "Properties of symmetry of orthogonal polynomials in two variables" by Andrew Sobczyk, University of Minnesota, introduced by the Secretary.

4. "Note on the convergence of improper integrals" by Earl Nelson, St. Olaf College, introduced by Professor C. S. Carlson.

5. "A course in synthetic metric geometry" by Professor Elizabeth Carlson, University of Minnesota.

6. "Enumeration of the terms in the product of two determinants" by Professor W. E. Brooke, University of Minnesota.

7. "A method of finding the solution of certain types of linear functional equations" by Dr. E. N. Oberg, Rochester Junior College.

8. (a) "A note on the construction of the circle of Apollonius" and (b) "A triangle-circle construction" by Professor F. J. Taylor, College of St. Thomas.

9. "The fundamental theorem of summation" by Professor A. L. O'Toole, College of St. Catherine.

10. "A certain cube root" by Professor C. S. Carlson, St. Olaf College.

11. (a) "A finger rule in trigonometry," (b) "A Peano curve" and (c) "Logarithms" by Dr. M. G. Scherberg, University of Minnesota.

12. "Some formulae of spherical trigonometry" by Professor Dunham Jackson, University of Minnesota.

13. "Solution of the linear homogeneous difference equation with quadratic coefficients" by Professor R. W. Cowan, College of St. Scholastica.

14. "On integration of the general diffusion equation" by Kyozo Aryiama, University of Minnesota, introduced by the Secretary.

15. "The variation of the roots of a quadratic equation near double points" by Dr. K. W. Wegner, University of Minnesota.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Thielman called attention to the fact that a certain method for evaluating definite integrals which is used in some books on the advanced calculus, is a particular case of the following theorem: If $f(p)$ is the Laplace transform of $F(x)$, i.e., if

$$f(p) = \int_0^\infty e^{-px}F(x)dx,$$

then

$$\int_0^\infty \frac{F(x)}{x}dx = \int_0^\infty f(p)dp$$

provided the integrals exist.

Use was then made of this theorem to evaluate integrals of the form

$$\int_0^\infty \frac{\sin^m x \cos^m x}{x^\alpha}dx.$$

He pointed out that by expanding $\sin^m x$ and $\cos^n x$ by means of the multiple angle formulae, integrating these expansions term by term, and then equating them to the values of the integrals obtained in some other way, compact expressions for various sums of the binomial coefficients can be obtained.

2. A placement test on elementary algebra was given to the students in mathematics in the General College at the first meeting of the class in the fall

of 1935, and again at the first meeting of the second quarter. Miss Elvebach discussed the results of these tests, and explained their significance.

3. In a recent paper presented to the American Mathematical Society, Professor Dunham Jackson has given a theory of systems of orthogonal polynomials in two variables (see Abstract No. 42-1-18, Bull. Amer. Math. Soc., Jan. 1936, p. 26). On the basis of that paper, Mr. Sobczyk showed that corresponding to a given region of definition and weight function, there is a particular set of orthogonal polynomials which are either invariant or else merely change sign, under any self-inverse linear homogeneous transformation of the variables which leaves the region of definition and weight function invariant.

4. Mr. Nelson showed that certain improper integrals can be tested for convergence by constructing an infinite series from the integrand, and by testing this series for convergence.

5. In this paper Professor Carlson gave a statement of the aims and content of a course in college geometry taught at the University of Minnesota.

6. If one expands a determinant of third order, one obtains six terms. The product of two such expansions involves 36 terms. If one multiplied the two determinants together first, and expanded the product without simplifying, 162 terms are obtained of which 126 terms cancel. Professor Brooke considered the analogous situation for n-th order determinants, and noted that $n!(n^n-n!)$ terms cancel, which if $n \geqq 7$ is more than 99% of the terms obtained.

7. Dr. Oberg gave a brief review of Volterra's method for solving linear integral equations followed by the application of this method to solving equations of the type $f(x)=u(x)+L(u)$ in which $L(u)$ is a linear functional of the unknown $u(x)$. A set of sufficient conditions was obtained for the uniform convergence of the infinite series which is derived as the solution for the unknown $u(x)$.

8(a). A method of constructing the circle of Apollonius, which is the locus of a point whose distances from two given points are in a given ratio, was described by Professor Taylor. Two points are found whose distances from the given points are in the given ratio, and a perpendicular bisector is constructed which meets the line between the given points in the center required. As the ratio of the distances approaches unity the circle approaches a straight line perpendicular to the line between the given points and as one term of the ratio approaches zero the circle approaches one or other of the given points.

8(b). Professor Taylor also discussed the following construction problem: To construct a triangle with a given length of base in one circle, in such a way that the base lies in a tangent line to a second circle, and one of the other sides lies in a tangent line to a third circle. Thirty-two constructions are possible, depending upon the arrangement of the given conditions.

9. Certain sections of elementary mathematics courses are often presented in a way which discourages students by giving them the impression that excellence in mathematics is a matter of trick methods, and even of legerdemain. It may be that the usual treatment in college algebra of the summation of series

is an example of this. By special devices, formulas are derived for the sums of arithmetic and geometric progressions. The suggestion of this to the student is that special methods must be devised for every series encountered. Professor O'Toole believes that general methods should be used whenever possible, and that good students will be encouraged more by one general method than by many special devices. Having given the student the stimulation that he will get from such a general mode of attack, it may then be quite proper to consider special devices which will work for a particular series.

10. Professor Carlson gave a method of simplifying certain of the cube roots of complex numbers occurring in Cardan's solution of the cubic equation.

11. Two formal rules and one construction were presented by Dr. Scherberg. The first of these was the finger rule which tells us that if we label the thumb and fingers in the order in which they occur on a hand as 0, 1, 2, 3, 4 to correspond to the angles 0, 30, 45, 60, 90 then the sine of the angle is one-half the square root of the number of the finger. The fraction rule in logarithms tells us that certain identities, involving products of logarithms, can best be remembered by comparison with algebraic identities made up of fractions involving the bases and arguments of the logarithms. Thus, compare $\log_a b \cdot \log_b c = \log_a c$ with $(b/a) \cdot (c/b) = c/a$. A simple construction of a continuous Peano curve $x = x(t)$, $y = y(t)$ filling the space of a unit square and passing through each point only once was also presented.

12. In this paper Professor Jackson pointed out that the principal formulas of spherical trigonometry can be deduced by methods closely analogous to those which are familiar in plane trigonometry, on the basis of three formulas, taken as fundamental, which correspond to the ratio definition of the sine, the ratio definition of the cosine, and the Pythagorean theorem.

13. The first part of Professor Cowan's paper was concerned with finding a particular solution of the linear homogeneous difference equation of the second order with quadratic coefficients by a consideration of the solution of the differential equation whose recursion relation yields the original difference equation. The aim of the second part was to find two solutions of the difference equation; one valid when certain quantities were positive, the other when they were negative.

14. The diffusion equations in cylindrical and spherical coordinates with some boundary conditions for cases in which the diffusion constant is a general function of radial coordinates alone were solved by Mr. Aryiama. A special example was considered, the method of solution being a generalization of that which is due to Stokes.

15. Dr. Wegner discussed the variation of the roots of a quadratic equation when the coefficients of the quadratic are allowed to vary along particular curves, such as radial lines. The corresponding paths of the two roots were found. Special attention was given to the direction in which the two roots enter and leave double points.

A. L. UNDERHILL, *Secretary*

THE EVOLUTION OF PROBLEMS OF THE CALCULUS OF VARIATIONS*

By G. A. BLISS, University of Chicago

The problems of the calculus of variations whose evolution through the literature is to be traced in this paper have reached their latest development in one to which the name of Bolza has been frequently attached in recent years. Geometrically his problem is set in the $(n+1)$-dimensional space of points $(x, y_1, \cdots, y_n)$. For such a point the notation (x, y) will be used freely in the following pages. Similarly, the notations $y(x)$, $y'(x)$ will designate the set of functions $[y_1(x), \cdots, y_n(x)]$ and the set of their derivatives, respectively. The problem of Bolza is then that of finding in a class of arcs $y_i = y_i(x)$ defined in (x, y)-space by functions of the form

$$y_i(x) \qquad (x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n) \tag{1}$$

and satisfying differential equations and end-conditions

$$\phi_\beta(x, y, y') = 0 \qquad (\beta = 1, \cdots, m < n), \tag{2}$$

$$\psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0 \qquad (\mu = 1, \cdots, p \leqq 2n + 2), \tag{3}$$

one which minimizes a sum

$$J = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f[x, y(x), y'(x)]dx. \tag{4}$$

The number m of the equations (2) is restricted to be less than n since otherwise these differential equations might have no arc satisfying them, or only a family of arcs depending upon arbitrary constants, in which cases the problem would be a trivial one. For similar reasons the number p of end-conditions (3) does not exceed the number $2n+2$ of end-values of the variables x and y at x_1 and x_2. When $p = 2n+2$ the numbers of equations and end-values of x and y are the same and the problem is in general one for which the end-points are the same for all of the arcs of the class in which a minimizing arc is sought.

The problem of Bolza is a very general one which reduces to the famous old problem of Lagrange when the function g is identically zero, and to the problem of Mayer in the case when f is identically zero. It is true, conversely, as I have recently indicated [41, §2],† that the problem of Bolza is transformable into one of Lagrange, or into a problem of Mayer with variable end-points, by tranformations which are considerably less convenient than the specializations indicated above which take the problem of Bolza into the others. All of the problems of the wide variety of types now usually classified as belonging to the calculus of variations seem to be expressible as problems of Bolza by means of

* Presented by invitation at the meeting of the Mathematical Association of America at Cambridge, Mass., August 31, 1936.

† The numbers in square brackets here and elsewhere refer to the bibliography at the end of this paper.

simple devices, though this would doubtless not be true of some of the recently studied questions concerning maxima and minima of functionals of more general kinds than those usually regarded as belonging to the calculus of variations.

It would be relatively unimportant to have a general formulation for problems of the calculus of variations in terms of which all special cases might be expressed, if the theory of the general problem were still undeveloped. Until quite recently the theories of the problems of Lagrange and Mayer with variable end-points, and of the problem of Bolza, have been incomplete. Bolza himself formulated in 1913 [27] a problem equivalent to the one described in the first paragraph above, deduced for it the multiplier rule which must be satisfied by its minimizing arcs, and made important remarks concerning what is called its normality or abnormality. In recent years papers by Cope, Graves, Hestenes, Morse, Reid, and myself have advanced the theory greatly, so that its present stage of development is comparable in many ways with that of simpler problems of the calculus of variations.* Two papers of Graves and Hestenes, in particular, for the first time freed the theory from assumptions of normality on sub-intervals which had hitherto prevented it from including other problems as special cases, especially the very important and well-known problem of Mayer which is abnormal on every sub-interval. Hestenes made three complete sufficiency proofs, and a fourth still dependent upon weak assumptions of normality on sub-intervals. Methods have been given independently by Reid and Morse for the removal of these normality assumptions in this fourth case also.

In the early literature of the calculus of variations, with which the present paper is most concerned, the results attained for general problems with variable end-points were all related to the so-called multiplier rule. For the problem of Bolza this rule says [41, p. 26] that for each minimizing arc there must be constants λ_0, $e_\mu(\mu=1, \cdots, p)$ and a function

$$F = \lambda_0 f + \lambda_1(x)\phi_1 + \cdots + \lambda_m(x)\phi_m$$

such that along the minimizing arc the differential equations

$$\frac{d}{dx}F_{y_i'} - F_{y_i} = 0, \qquad \phi_\beta = 0 \tag{5}$$

are satisfied, and furthermore such that the end-values of x and y on the minimizing arc satisfy the conditions

$$\Big[(F - y_i'F_{y_i'})dx + F_{y_i'}dy_i\Big]_{x_1}^{x_2} + \lambda_0 dg + e_\mu d\psi_\mu = 0, \qquad \psi_\mu = 0, \tag{6}$$

the former of which is to be an identity in the $2n+2$ differentials dx_1, dx_2, dy_{i1}, dy_{i2} of the end-values of x and y on the arc $y(x)$. In these equations the literal subscripts y_i and y_i' attached to F indicate partial derivatives, and it is understood that terms with repeated indices i or μ represent sums of such terms over

* An account of these results and a bibliography is given in the lecture notes of Bliss [41].

the ranges $i=1, \cdots, n$ and $\mu=1, \cdots, p$, as is customary in tensor analysis. The number of constants of integration in the solutions of the differential equations (5) is found upon investigation to be $2n$, and these with the $p+2$ constants x_1, x_2, e_μ are equal in number to the $2n+p+2$ equations which one deduces from (6) when the $2n+2$ equations obtained by equating to zero the coefficients of the differentials dx_1, dx_2, dy_{i1}, dy_{i2} are adjoined to the p end-conditions $\psi_\mu=0$.

By far the simplest case of Bolza's general problem is that in which the differential equations (2) are not present and the end-points are fixed. It is interesting to note, however, that problems of the more difficult sort, having differential equations $\phi_\beta=0$ as side conditions, appeared from the very first in the literature of the calculus of variations. Results of interest were obtained for them long before the theory of the simpler problems was in any sense complete as we know it today. The emphasis of this paper is on the evolution of these more complicated types which may be thought of as precursors of the problem of Bolza.

The earliest problem of the calculus of variations which has been noted is the classical isoperimetric problem of the ancient Greeks, that of finding in a class of simply closed arcs of given length l in the plane one which encloses the maximum area. It is easy to phrase this problem as one of Bolza's type, namely, that of finding in a class of arcs

$$x(s), \qquad y(s) \qquad\qquad (s_1 \leqq s \leqq s_2)$$

satisfying conditions of the form

$$x'^2 + y'^2 - 1 = 0,$$
$$s_1 = s_2 - l = x(s_1) - x(s_2) = y(s_1) - y(s_2) = 0$$

one which minimizes the area integral

$$J = \frac{1}{2}\int_{s_1}^{s_2}(xy' - x'y)ds.$$

Thus the earliest known problem of the calculus of variations, formulated in this way, already involves a differential equation as a side condition and has variable end-points.

An interesting history of this isoperimetric problem has recently been given by Porter in his master's dissertation at the University of Chicago [39]. The origin of the problem is lost in antiquity, but an early attempt to prove that the solution is a circle was made by Zenodorus who lived probably between 200 B.C. and 90 A.D. His theorems have been preserved for us by Theon and Pappus who are supposed to have lived in the fourth century A.D. One of the old writers said that it is a good thing to know about this problem because the real estate men of his time often assumed for purposes of sale that a plot of ground with a larger perimeter had a larger area. That the problem is a really difficult one is evidenced by the fact that it was not until the latter half of the

nineteenth century that a completely satisfactory proof of the isoperimetric property of the circle was given, and most of the now extensive list of such proofs are dated later than 1900. Proofs based upon the calculus of variations have been given by Weierstrass, Schwarz, Kneser, and Bolza; one using Fourier series is due to Hurwitz; and several based upon more geometric considerations have been given by Carathéodory, Study, Minkowski, and Blaschke. A bibliography of these papers and a discussion of their results are given in Porter's dissertation.

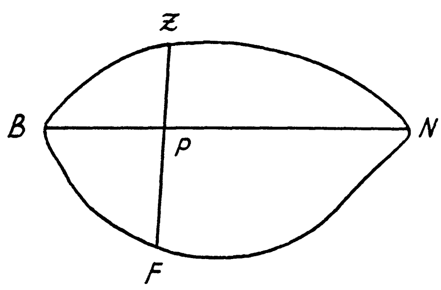
No report on the history of the calculus of variations can ignore the famous problem of the brachistochrone which was conceived but incorrectly solved by Galileo in 1630 [1, pp. 471–2; 2, p. 239], and which was re-proposed by John Bernoulli in 1696 [3] as a challenge to contemporary mathematicians. The problem is that of finding an arc in a vertical plane down which a particle starting with a given initial velocity will fall under the action of gravity from one given point to another in the shortest time. Analytically the problem is that of finding in a class of plane arcs $y(x)$ $(x_1 \leqq x \leqq x_2)$ satisfying end-conditions of the form

$$x_1 - a_1 = x_2 - a_2 = y(x_1) - b_1 = y(x_2) - b_2 = 0$$

one which minimizes an integral of the form

$$J = \int_{x_1}^{x_2} \left[\frac{1 + y'^2}{y - \alpha}\right]^{1/2} dx.$$

The time of descent is expressible as the product of such an integral by an unessential constant factor. In response to John Bernoulli's challenge the solution was shown to be a cycloid generated by a circle rolling on the under side of the line $y = \alpha$. This result was attained by John himself, by his older brother James, and by Newton, Leibniz, and l'Hospital. Only the solutions of the Bernoulli brothers were published [3].



In the paper in which James Bernoulli explained his solution of the brachistochrone problem he suggested the determination of the arc of quickest descent from a fixed point to a fixed vertical line, thus explicitly formulating for the first time a problem with variable end-points. He also proposed two much

more difficult problems, those of finding among all arcs BFN of given length l, as shown in the accompanying figure, two such that the area $BPNZ$ is a maximum when $PZ=PF^n$, or when $PZ=s^n$, where s is the length of the arc BF. Analytically these are the problems of finding in a class of arcs

$$y(x), \qquad s(x) \qquad\qquad (x_1 \leqq x \leqq x_2)$$

in (x, y, s)-space, satisfying conditions of the form

$$s' - [1 + y'^2]^{1/2} = 0,$$
$$x_1 - a_1 = x_2 - a_2 = y(x_1) = y(x_2) = s(x_1) = s(x_2) - l = 0,$$

two which maximize the integrals

$$J_1 = \int_{x_1}^{x_2} y^n dx, \qquad J_2 = \int_{x_1}^{x_2} s^n dx. \tag{7}$$

The solution of these problems caused much embarrassment to John Bernoulli whose first attempts were all incorrect. James published [4] his own correct but rather involved solution in 1701, four years before his death, and in 1718 John revised and greatly simplified his brother's treatment [5].

This last paper is an important one historically because it undoubtedly had great influence upon the illustrious Euler who was born in 1707 and who had his early training in mathematics under John Bernoulli. Euler adopted and systematized skillfully the method of the Bernoullis. For James Bernoulli's problems this method consists roughly in

(a) replacing the integrals of the problem by finite sums thus making them functions of a finite number of ordinates of the arc $y(x)$, derivatives being replaced by difference quotients;

(b) varying one or more arbitrarily selected ordinates, and calculating the resulting variations of the integrals;

(c) equating to zero the variation of the integral to be minimized, as well as that of the length integral which is to be kept constant, and interpreting the resulting difference equations as differential conditions on the minimizing arc by a somewhat crude limiting process.

Applied to the brachistochrone problem this method was relatively simple because when no differential or isoperimetric condition was present, as a restriction on the class of arcs in which a minimizing arc was sought, only one of the ordinates of the arc $y(x)$ needed to be varied. For an isoperimetric problem, however, more than one had to be varied and the calculation of the variations of the integrals and the interpretation of the resulting equations became much more complicated. It was the necessity of varying more than one ordinate which John Bernoulli at first failed to understand, so that his earlier efforts were unsuccessful.

By these methods Euler succeeded in finding in 1736 [7] a formula for the

differential equation, the famous Euler equation, of the minimizing arcs for an integral of the form

$$J = \int_{x_1}^{x_2} f(x, y, y', y'', \cdots)dx.$$

Inspired by the isoperimetric problem of James Bernoulli he also formulated much more general problems in which the integrand function f may involve not only the variables $x, y, y', \cdots$ indicated in the last integral above, but also more general integrals than the length integral s in the formula (7) for J_2. Three of these more general problems are here formulated in notations similar to those used above for the problem of Bolza:

$$\text{I}\quad \begin{aligned} &y(x), \qquad y_1(x) \qquad\qquad (x_1 \leqq x \leqq x_2),\\ &y_1' - \phi(y_1, x, y, y', \cdots) = 0,\\ &J = \int_{x_1}^{x_2} f(y_1, x, y, y', \cdots)dx = \min. \end{aligned}$$

$$\text{II}\quad \begin{aligned} &y(x), \qquad y_1(x), \qquad y_2(x) \qquad\qquad (x_1 \leqq x \leqq x_2),\\ &y_1' - \phi_1(x, y, y', \cdots) = 0, \qquad y_2' - \phi_2(y_1, x, y, y', \cdots) = 0,\\ &J = \int_{x_1}^{x_2} f(y_2, x, y, y', \cdots)dx = \min. \end{aligned}$$

$$\text{III}\quad \begin{aligned} &y(x), \qquad y_1(x) \qquad\qquad (x_1 \leqq x \leqq x_2),\\ &y_1' - \phi(y_1, x, y, y', \cdots) = 0,\\ &J = y_1(x_2) = \min. \end{aligned}$$

The end-conditions are omitted because Euler himself usually did not give any. He apparently sought first to find a differential equation for his minimizing arc, after which the number of constants available in his solution suggested to him the number and type of the end-conditions which he could impose. His first attempts to find the differential equations of solutions of these very general problems in his papers of 1734 [6] and 1736 [7] were apparently unsuccessful. But in his famous treatise on the calculus of variations of 1744 he systematized and improved his methods [8, pp. 106, 107, 132, 133] and listed differential equations for a large number of types of problems, many of which have been checked and found correct provided that one supplies suitable end-conditions. He does not in general secure end-conditions analogous to the conditions(6) of the multiplier rule. The three problems listed above, treated in Chapters III and IV of the *Methodus Inveniendi*, seem to have received much less attention than the results of Chapters I, II, a part of V, and VI, which were translated from the original Latin into German and published in Ostwald's Klassiker der exakten Wissenschaften [3].

Euler also considered [8, p. 161 and elsewhere] many special cases and also the general case of the problem of finding an arc $y(x)$ $(x_1 \leqq x \leqq x_2)$ in the plane which minimizes a function $\psi(A_1, \cdots, A_n)$ involving n integrals

$$A_i = \int_{x_1}^{x_2} \phi_i(x, y, y', \cdots)dx \qquad (i = 1, \cdots, n).$$

Condensed and in the notation of the problem of Bolza this problem may be suggested by the formulas

$$\begin{array}{lll} \text{IV} & y(x), \qquad y_i(x) & (x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n), \\ & y_i' - \phi_i(x, y, y', \cdots) = 0 & (i = 1, \cdots, n), \\ & y_i(x_1) = 0 & (i = 1, \cdots, n), \\ & J = \psi[y_1(x_2), \cdots, y_n(x_2)] = \min. & \end{array}$$

The end-conditions are again incompletely prescribed. Euler found correct differential equations for the solutions in special cases, but Bolza's discussion of two of them [28] shows that much remains to be said about problems of this type.

The theory of problems of the calculus of variations was greatly improved as a result of the studies of Lagrange (1736–1813). He discarded the geometric-analytic arguments of Euler and the Bernoullis and by the introduction of his so-called variations gave the theory an approximately modern outward form. Instead of varying individual ordinates of the minimizing arc $y(x)$, as his predecessors had done, he considered neighboring arcs defined by functions of the form $y(x)+\delta y(x)$, where $\delta y(x)$ is a function of x called a variation of $y(x)$. He calculated the variations of the integrals involved in the problem, due to such variations of $y(x)$, and by means of the integrations by parts now so commonly used secured conditions on the minimizing arc $y(x)$.

In his paper [9] of 1762 Lagrange applied his methods to re-establish Euler's results, and he furthermore deduced for the first time end-conditions which must be satisfied by a minimizing arc for problems of the calculus of variations of non-special types having variable end-points. In a later paper [11] of 1770 he proposed the following very general problem, summarized here with notations again chosen to exhibit its relationships to the problem of Bolza stated above:

$$\begin{array}{ll} y_0(x), \qquad y_i(x) & (x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n), \\ \phi(y_0, y_0', \cdots, y_0^{(r)}; x, y, y', \cdots) = 0, \quad \phi_\beta(x, y, y', \cdots) = 0, & \\ \psi_\mu[y_0(x_1), \cdots, y_0^{(r-1)}(x_1); y(x_1), y'(x_1), \cdots, y(x_2), y'(x_2), \cdots] = 0, & \\ J = y_0(x_2) = \min. & \end{array}$$

The formulation of the end-conditions $\psi_\mu = 0$ given here is more explicit than that of Lagrange. He mentions various possibilities in his paper [11, pp. 38, 40, 41] and explains how the end-conditions on the minimizing arc may be deter-

mined. The conditions $\psi_\mu = 0$ set down above seem to correspond to the most general case which he considered.

Lagrange regarded the equation $\phi = 0$ as determining the end-value $y_0(x_2)$ when the set of functions $y(x)$ and the initial values at x_1 of $y_0(x)$ and its first $r-1$ derivatives are given. However if one expresses the value $y_0(x_2)$ to be minimized in the form

$$J = y_0(x_1) + \int_{x_1}^{x_2} y_0' \, dx$$

and thinks of $\phi = 0$ as a differential side-condition like the equations $\phi_\beta = 0$, the problem has the form of the problem of Bolza described above except that it involves derivatives of the functions $y_0(x)$, $y_i(x)$ $(i = 1, \cdots, n)$ of orders higher than the first. In particular if $y_0(x_1)$ is zero or a constant the problem is equivalent to a problem of the type obtained from the problem of Bolza when the function g in (4) is identically zero, provided at least that one again permits the functions ϕ_β and the integrand f to involve derivatives of the functions $y(x)$ of orders higher than the first. It was this problem, discussed by Lagrange in his "Leçons sur les calculs des fonctions" of 1806 [13, pp. 420, 422], which has since been called the problem of Lagrange in the calculus of variations.

There is actually no gain in generality by permitting derivatives of the functions $y_i(x)$ $(i = 1, \cdots, n)$ of orders higher than the first in the functions ϕ_β and f of (2) and (4). If the highest derivative of a particular function $y(x)$ involved were of order s one could replace the derivatives $y^{(\sigma)}$ $(\sigma = 1, \cdots, s-1)$ by new variables z_σ and $y^{(s)}$ by z_{s-1}', and adjoin to the side conditions the differential equations

$$y' - z_1 = z_1' - z_2 = \cdots = z_{s-2}' - z_{s-1} = 0 .$$

If this were done for each of the functions $y_i(x)$ $(i = 1, \cdots, n)$ the problem would reduce to one of precisely the type described in (1)–(4) involving only first derivatives.

Prompted doubtless by a criticism by Borda of his treatment of the brachistochrone problem with first end-point variable on a curve [10], Lagrange further considered the case when end-values of the functions $y(x)$ occur in the functions ϕ_β and f as well as in the end-conditions $\psi_\mu = 0$. This is also not an essential generalization. For if $y_1(x_1)$, for example, were such an end-value one could replace it in ϕ_β and f by $z(x)$ and adjoin the differential equation and end-condition

$$dz/dx = 0, \qquad z(x_1) - y_1(x_1) = 0.$$

After similar modifications for the other end-values occurring in ϕ_β and f the problem would again be of the Bolza type described in (1)–(4).

For his general problems Lagrange obtained results equivalent to those of the multiplier rule for the problem of Bolza. His proof was quite imperfect, little more, in fact, than the recitation of an analogy with the proof of the

multiplier rule for the minimum of a function of a finite number of variables subjected to side conditions [13, pp. 421–2]. One of its defects was removed by A. Mayer in 1886 [18], and the proof was completed by Hilbert in 1905 [21].

In 1878 A. Mayer studied a problem of the calculus of variations which has since that time been designated by his name. By a sufficiently liberal interpretation it can be made to appear as a special case of the problem considered by Lagrange in 1770, a type which seems to have been later abandoned in the literature in favor of the problem of Lagrange of 1806. In terms of the notations used above the problem of Mayer has the following summary:

$$y_i(x) \qquad (x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n),$$

$$\phi_\beta(x, y, y') = 0 \qquad (\beta = 1, \cdots, m < n),$$

$$x_1 - a_1 = x_2 - a_2 = y_i(x_1) - b_{i1} = y_\sigma(x_2) - b_{\sigma 2} = 0 \qquad (i = 1, \cdots, n;\ \sigma = 2, \cdots, n),$$

$$J = y_1(x_2) = \min.$$

A large part of the literature of the calculus of variations of the nineteenth century was devoted to the study of transformations of the so-called second variations of a great variety of problems having fixed end-points, and to the applications of these transformations to the determination of necessary conditions for a minimum analogous to those deduced by Legendre in 1786 and Jacobi in 1837 for simple cases [12, 14]. Mayer himself made important contributions [17]. He noticed, however, that the problem last summarized above is essentially a problem with variable end-points when transformed into a problem of Lagrange by the customary substitution of the expression

$$J = \int_{x_1}^{x_2} y_1' \, dx$$

for the function $J = y_1(x_2)$, a fact which was later emphasized also by Hadamard [26, p. 224] and Bolza [24, pp. 573–4]. Clebsch seems to have failed to notice this when in 1858 he stated the problem [15, p. 337] without clearly specifying end-conditions, and dismissed it with the assertion that it was equivalent to a problem of Lagrange which he then proceeded to study. Mayer's more careful formulation showed the necessity at that time of an independent study of his problem. He emphasized clearly the role of the end-conditions, proved the multiplier rule, and discussed the analogues of the necessary conditions of Legendre and Jacobi by means of his theory of the second variation. It should be noted here, however, as was indicated for more general problems in an earlier paragraph, that in view of recently completed theories the problem of Mayer can now be treated as a problem of Lagrange with one variable end-point.

In view of the variety of problems which have been described above it is clear that Bolza was well justified in his formulation [27] in 1913 of a more general problem which included all earlier ones by easy specializations or simple

transformations. The emphasis of his paper is upon normality considerations, and upon the multiplier rule which he proves on the basis of an unsymmetric assumption often used before. In 1918 the author of the present paper removed this lack of symmetry for the problem of Mayer with variable end-points which he showed to be equivalent to the problem of Bolza [31]. The problem of Bolza remains the most convenient, however, not only because other general problems are simple sub-cases of it, but also for the very important reason that the so-called accessory minimum problem of the second variation, by means of which the analogue of the Jacobi condition may be deduced, has again the same form. The accessory minimum problems of the problems of Lagrange and Mayer with variable end-points are also problems of Bolza, and hence require a theory different from that of the original problems of Lagrange or Mayer with which they are associated.

The recent advancement of the theory of the problem of Bolza, noted in a preceding paragraph, has for the first time established results for problems of the calculus of variations with variable end-points comparable with those which have been known for some time past for problems with fixed end-points. Further, Denbow has shown quite recently, in a thesis [42] to be presented for the doctor's degree at the University of Chicago, that a problem (1)–(4) involving in the function g of (4) the coordinates of a finite number of points on the arc (1) besides the end-points, can be reduced to a problem of Bolza by a suitable transformation. The problem (1)–(4) with such arguments for the function g is a generalization of a problem cited by Hadamard [26, pp. 175–7] and studied in detail for a special case by Sinclair [25] and for a more general case by Clarke [33]. A transformation similar to that of Denbow can be used to reduce the problem with discontinuous integrand studied by Mason and Bliss [22] to a problem of Bolza, and probably also to effect the same result for discontinuous solutions of problems of the calculus of variations such as have been studied by Carathéodory [19], Bolza [23], Graves [37, 38], Reid [40], and others [see 38]. In a thesis [43], also to be presented for the doctor's degree at the University of Chicago, Valentine has transformed into problems of Bolza problems involving differential inequalities as well as differential equations in (2), such as were studied by Bolza [29]. His transformations are applicable further to problems involving minima relative to one-sided variations such as were studied by Bliss [20], and Bliss and Underhill [30]. The rather unusual problems encountered by Roos [35, 36] in his applications of the calculus of variations to economic questions are easily reducible to problems of Bolza.

From these remarks it is clear that the theory of the problem of Bolza is likely to be an effective one for the calculus of variations for some time to come. Interesting studies of the determination of maxima or minima for functionals of less restricted types than those of the calculus of variations, have been made by LeStourgeon [32], Hahn [34], and within the past year by Goldstine [44]. What has been gained in generality seems to have been counter-balanced in each of these cases, however, by a loss of some of the properties of the func-

tionals of the calculus of variations upon which most interesting parts of the theory depend. For the present the problem of Bolza is one of the most satisfactory of the general ones in which these properties are preserved. Its location as a chapter in a more inclusive theory of maxima and minima of functionals is one of the interesting problems of the future.

List of References

1. Galileo, *Dialog über die beiden hauptsächlichsten Weltsysteme* (1630), translation by Strauss (1891).

2. Galileo, *Dialogues concerning two new sciences* (1638), translation by Crew and De Slavio (1914).

3. See Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 46, for translations of three papers by John Bernoulli in the Acta Eruditorum of 1696 and 1697, of one by James Bernoulli in the same journal for 1697, and of Chapters I, II, part of V, and VI of Euler's *Methodus Inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne et Genève (1744).

4. James Bernoulli, *Analysis magni problematis isoperimetrici*, Acta Eruditorum (1701). See also his *Oeuvres*, vol. 2 (1744), pp. 897–920.

5. John Bernoulli, *Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes sur les isoperimètres*, Mémoires de l'Académie Royale des Sciences de Paris (1718), p. 100. See also his *Oeuvres*, vol. 2 (1742), pp. 235–69.

6. Euler, *De linea celerrimi discensus in medio quocumque resistente*, Commentarii Academiae Scientiarum Imperialis Petropolitanae, vol. 7 (1734–1735), pp. 135–49.

7. Euler, *Curvarum maximi minimive proprietate gaudentium inventio nova et faciles*, Commentarii Academiae Scientiarum Imperialis Petropolitanae, vol. 8 (1736), pp. 159–190.

8. Euler, *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne et Genève (1744). See also Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 46, for a translation into German of Chapters I, II, part of V, and VI.

9. Lagrange, *Nouvelles méthodes pour déterminer les maxima et les minima des formules intégrales indéfinies*, Miscellanea Taurinensia (1760–1761), see also his *Oeuvres*, vol. 1 (1867), pp. 335–362; and for a translation into German, Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 47. The date of Lagrange's paper actually was 1762.

10. Borda, *Éclaircissements sur les méthodes de trouver les courbes qui jouissent de quelque propriété du maximum ou du minimum*, Histoire de l'Académie Royale des Sciences (Paris) 1767, éd. 1770, Hist. p. 90, Mém. pp. 551–63.

11. Lagrange, *Sur la méthode des variations*, Miscellanea Taurinensia, vol. 4 (1766–1769). See also his *Oeuvres*, vol. 2 (1868), pp. 37–63; and for a translation into German, Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 47. The date of this paper of Lagrange was actually 1770.

12. Legendre, *Mémoire sur la manière de distinguer des maxima et minima dans le calcul des variations*, Histoire de l'Académie Royale des Sciences (Paris) 1786, pp. 7–37; Addition, *ibid.*, 1787, pp. 348–51. See also Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 47.

13. Lagrange, *Leçons sur le calcul des fonctions*, Paris, 1896. See also his *Oeuvres*, vol. 10 (1884), pp. 364–451.

14. Jacobi, *Zur Theorie der Variations-Rechnung und der Differential-Gleichungen*, Journal für die reine und angewandte Mathematik, vol. 17 (1837), pp. 68–82. See also his *Werke*, vol. 4 (1886), pp. 50–52; and Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 47.

15. Clebsch, *Über diejenigen Probleme der Variationsrechnung welche nur eine unabhängige Variable enthalten*, Journal für die reine und angewandte Mathematik, vol. 55 (1858), pp. 335–55.

16. Mayer, *Über die Kriterien des Maximums und Minimums der einfachen Integrale*, Journal für die reine und angewandte Mathematik, vol. 69 (1868), pp. 238–63.

17. Mayer, *Über das allgemeinste Problem der Variationsrechnung bei einer einzigen Variabeln*, Berichte der Königlichen Sächischen Gesellschaft der Wissenschaft, vol. 30 (1878), pp. 16–32.

18. Mayer, *Begründung der Lagrange'schen Multiplicatoren-Methode in der Variationsrechnung*, Mathematische Annalen, vol. 26 (1880), pp. 74–82.

19. Carathéodory, *Über die diskontinuierlichen Lösungen in der Variationsrechnung*, Dissertation, Göttingen, 1904.

20. Bliss, *Sufficient conditions for a minimum with respect to one-sided variations*, Transactions of the American Mathematical Society, vol. 5 (1904), pp. 447–92.

21. Hilbert, *Zur Variationsrechnung*, Göttinger Nachrichten (1905), pp. 159–80. See also Mathematische Annalen, vol. 62 (1906), pp. 351–68.

22. Bliss and Mason, *A problem of the calculus of variations in which the integrand is discontinuous*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 325–36.

23. Bolza, *The determination of the conjugate points for discontinuous solutions in the calculus of variations*, American Journal of Mathematics, vol. 30 (1908), pp. 209–21.

24. Bolza, *Vorlesungen über Variationsrechnung*, 1909; 2nd ed., 1933.

25. Sinclair, *Concerning a compound discontinuous solution in the problem of the surface of revolution of minimum area*, Annals of Mathematics, Series 2, vol. 10 (1909), pp. 55–71.

26. Hadamard, *Leçons sur le calcul des variations*, 1910.

27. Bolza, *Über den anormalen Fall beim Lagrangeschen und Mayerschen Problem mit gemischten Bedingungen und variabeln Endpunkten*, Mathematische Annalen, vol. 74 (1913), pp. 430–6.

28. Bolza, *Über zwei Eulersche Aufgaben aus der Variationsrechnung*, Annali di Matematica, Series 3, vol. 20 (1913), pp. 245–50.

29. Bolza, *Über Variationsprobleme mit Ungleichungen als Nebenbedingungen*, Mathematische Abhandlungen Hermann Amandus Schwarz zu seinem Fünfzigjährigen Doktorjubiläum (1914), pp. 1–18.

30. Bliss and Underhill, *The minimum of a definite integral for unilateral variations in space*, Transactions of the American Mathematical Society, vol. 15 (1914), pp. 291–310.

31. Bliss, *The problem of Mayer with variable end-points*, Transactions of the American Mathematical Society, vol. 19 (1918), pp. 305–14.

32. Le Stourgeon, *Minima of functions of lines*, Transactions of the American Mathematical Society, vol. 21 (1920), pp. 357–83.

33. Clarke, *On the minimum of the sum of a definite integral and a function of a point*, Dissertation, University of Chicago, 1922. For an abstract see *Abstracts of Theses*, The University of Chicago Science Series, vol. 1 (1922–1923), pp. 9–15.

34. Hahn, *Über die Lagrangesche Multiplikatorenmethode*, Sitzungsberichte der Akademie der Wissenschaften zu Wien, vol. 131, Abteilung 2a (1922), pp. 531–50.

35. Roos, *A mathematical theory of depreciation and replacement*, American Journal of Mathematics, vol. 50 (1928), pp. 147–157.

36. Roos, *A problem of minimizing an integral with discontinuous integrand*, Transactions of the American Mathematical Society, vol. 31 (1929), pp. 58–70.

37. Graves, *Discontinuous solutions in space problems of the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 1–28.

38. Graves, *Discontinuous solutions in the calculus of variations*, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 831–46.

39. Porter, *A history of the classical isoperimetric problem*, Contributions to the Calculus of Variations 1931–1932, The University of Chicago Press.

40. Reid, *Discontinuous solutions in the non-parametric problem of Mayer in the calculus of variations*, American Journal of Mathematics, vol. 57 (1935), pp. 69–93.

41. Bliss, *The problem of Bolza in the calculus of variations*, mimeographed lecture notes, The University of Chicago, 1935.

42. Denbow, *A generalized form of the problem of Bolza*, Dissertation, The University of Chicago, 1936.

43. Valentine, *The problem of Lagrange with differential inequalities as side conditions*, Dissertation, The University of Chicago, 1936.

44. Goldstine, *Conditions for a minimum of a functional*, Dissertation, The University of Chicago, 1936.

ANORMAL SYSTEMS OF NUMERATION*

By A. J. KEMPNER, University of Colorado

I. We admit as base of a number system any real number $a>1$, not an integer.

By following exactly the method of procedure for the representation of numbers when the base a is an integer >1, we are led to the following

ALGORITHM: *For a base $a>1$, not integral, we represent a number $c>0$ by the following chain of operations*:

$$\begin{array}{lll}
a^{n+1} > c \geqq a^n; & & \\
c = \alpha_n a^n + \beta_n, & 0 < \alpha_n \leqq [a], & 0 \leqq \beta_n < a^n, \\
\beta_n = \alpha_{n-1}a^{n-1} + \beta_{n-1}, & 0 \leqq \alpha_{n-1} \leqq [a], & 0 \leqq \beta_{n-1} < a^{n-1}, \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
\beta_1 = \alpha_0 a^0 + \beta_0, & 0 \leqq \alpha_0 \leqq [a], & 0 \leqq \beta_0 < a^0, \\
\beta_0 = \alpha_{-1}a^{-1} + \beta_{-1}, & 0 \leqq \alpha_{-1} \leqq [a], & 0 \leqq \beta_{-1} < a^{-1}, \\
\beta_{-1} = \alpha_{-2}a^{-2} + \beta_{-2}, & 0 \leqq \alpha_{-2} \leqq [a], & 0 \leqq \beta_{-2} < a^{-2}, \\
\cdots\cdots & \cdots\cdots & \cdots\cdots
\end{array}$$

This process may terminate (if $\beta_i=0$, all subsequent β's and α's are 0), or it may continue indefinitely. The representation is called canonical, and is designated by

$$c = \alpha_n \cdots \alpha_0 \cdot \alpha_{-1}\alpha_{-2} \cdots {}_{(a)}(\text{can}),$$

with or without the subscript (a).

Thus $2=1\cdot(3/2)+1/2$, $1/2=0\cdot(3/2)^0+1/2$, $1/2=0\cdot(3/2)^{-1}+1/2$, $1/2=1\cdot(3/2)^{-2}+1/18$, $1/18=0\cdot(3/2)^{-3}+1/18$, $\cdots$, $1/18=0\cdot(3/2)^{-7}+1/18$, $1/18=1\cdot(3/2)^{-8}+217/(2\cdot 3^8)$, etc.; and hence

$$2 = 10.01000001 \cdots {}_{(3/2)}(\text{can}).$$

A negative number is characterized by prefixing a minus sign.†

THEOREM: *Every real number can be represented canonically to any base $a>1$, and this canonical representation is unique.*

The proof of this theorem is obvious.

For an integral base $a>1$ the canonical representation is the only one possible if the digits are restricted to $0\leqq\alpha_i<a$, except for the ambiguity represented

* Presented to the American Mathematical Society, Ann Arbor, September, 1935. This paper is the answer to a question raised by one of my students, Joy Gilder, who wanted to know whether the number e could be used as the base of a number system.

† The numbers 0 and 1 are clearly excluded as bases. Positive numbers less than 1 and negative numbers may be used as bases with slight modifications of the process and suitable restrictions on the set of digits employed.

by $1=.99\bar{9}\cdots$. We shall now show that for a non-integral base $a>1$ the restriction of the digits to this range is in no sense sufficient to secure uniqueness. This is illustrated by

$$2 = (3/2)^{-1} + (3/2)^{-2} + \cdots = .111\cdots_{(3/2)}$$

which differs from the canonical representation given above; or by

$$\begin{aligned} 1 &= 1_{(3/2)}(can), \\ &= 1\cdot(3/2)^{-1} + 1\cdot(3/2)^{-3} + 1\cdot(3/2)^{-9} + \cdots = .101000001\cdots_{(3/2)}. \end{aligned}$$

In order to have a representation of c to base $a>1$, non-integral, it is clear that the monotonically decreasing sequence $\beta_n, \beta_{n-1}, \cdots$ must approach 0, i.e.,

$$\lim_{k\to\infty} \beta_{n-k} = 0;$$

(this may include the case where all but a finite number of β are zero). In the divisions involved in $2=10.010\cdots_{(3/2)}$(can), $2=.11\bar{1}\cdots_{(3/2)}$, $1=.1010\cdots_{(3/2)}$, this condition is satisfied. But if we attempt to represent 2 by starting $.01\cdots_{(3/2)}$, the divisions can be carried out, and the β form a decreasing sequence, yet they do not approach 0. This is obvious, since the largest number which can be represented to base 3/2 starting $.01\cdots$ is

$$(3/2)^{-2} + (3/2)^{-3} + \cdots = .011\bar{1}\cdots = 4/3 < 2.$$

It is therefore not permissible to say that we obtain non-canonical representations of c to the base a by omitting in our algorithm the third column. Instead, remembering that

$$[a](a^{r-1} + a^{r-2} + \cdots) = [a]\frac{a^r}{a-1},$$

we have the

THEOREM: *Necessary and sufficient for the representation of $c>0$ to the base $a>1$, non-integral, is*

$$\begin{array}{lll} c = \alpha_n a^n + \beta_n, & 0 < \alpha_n \leqq [a], & 0 \leqq \beta_n \leqq [a]a^n/(a-1), \\ \beta_n = \alpha_{n-1}a^{n-1} + \beta_{n-1}, & 0 \leqq \alpha_{n-1} \leqq [a], & 0 \leqq \beta_{n-1} \leqq [a]a^{n-1}/(a-1), \\ \cdots & \cdots & \cdots \\ \beta_0 = \alpha_{-1}a^{-1} + \beta_{-1}, & 0 \leqq \alpha_{-1} \leqq [a], & 0 \leqq \beta_{-1} \leqq [a]a^{-1}/(a-1) \\ \cdots & \cdots & \cdots \end{array}$$

If in addition the β satisfy the stronger inequalities in the third column of the algorithm, the representation is canonical.

The notation

$$c = \alpha_n\alpha_{n-1} \cdots \alpha_0 \cdot \alpha_{-1}\alpha_{-2} \cdots {}_{(a)}$$

indicates that the representation may be either canonical or non-canonical. If $\alpha_{-1} = \alpha_{-2} = \cdots = 0$, we say that c is represented as a "whole number" to the base a (in distinction to ordinary "integers"). Finite decimals, infinite decimals, periodic decimals, pure and mixed, are defined in the obvious manner. The numerical illustrations already given show: a whole number to the base a is not necessarily an integer; an integer is not necessarily a whole number; a whole number may equal an infinite decimal ($1_{(3/2)} = .10100 \cdots {}_{(3/2)}$); the sum of two whole numbers is not necessarily a whole number ($1_{(3/2)} + 1_{(3/2)} = 10.010 \cdots {}_{(3/2)}$); the product of two whole numbers is not necessarily a whole number $[(5/2)\cdot(5/2) = 11_{(3/2)} \cdot 11_{(3/2)} = 10001.00001 \cdots {}_{(3/2)}]$. On account of these properties, the notion of a "whole number" cannot be fundamental in such number systems, since it possesses no closure properties. However, it was obvious from the start that only the representation of the individual number in the system, and the abstract structure of the system, can interest us, not its use for arithmetic.

From the definition of canonical representation it follows immediately that:

(a) If $\alpha_n\alpha_{n-1} \cdots \alpha_0 \cdot \alpha_{-1} \cdots$ is canonical, any displacement of the decimal point gives a canonical representation of the new number; and

(b) If $\alpha_n\alpha_{n-1} \cdots \alpha_0 \cdot \alpha_{-1} \cdots$ is canonical, all numbers obtained by omitting the first λ digits from the left, $\lambda = 1, 2, \cdots$, are canonical.

The canonical form

$$c = \alpha_n\alpha_{n-1} \cdots \alpha_0 \cdot \alpha_{-1}\alpha_{-2} \cdots {}_{(a)}(\text{can})$$

is characterized as follows:

For finite decimals or "whole numbers,"

$$\begin{aligned} \alpha_{n-1}a^{n-1} + \alpha_{n-2}a^{n-2} + \cdots + \alpha_r a^r &< a^n \\ \alpha_{n-2}a^{n-2} + \cdots + \alpha_r a^r &< a^{n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \alpha_r a^r &< a^{r+1}, \end{aligned}$$

where α_r is the last non-vanishing digit (left to right). This finite chain of inequalities, each involving a finite number of terms, is replaced in the case of an infinite decimal by the infinite set of inequalities, each involving an infinite number of terms:

$$\begin{aligned} \alpha_{n-1}a^{n-1} + \alpha_{n-2}a^{n-2} + \cdots &< \alpha^n \\ \alpha_{n-2}a^{n-2} + \cdots &< \alpha^{n-1} \\ \cdots\cdots\cdots\cdots\cdots & . \end{aligned}$$

Therefore, for a whole number or a finite decimal to the base a, a finite number of steps suffices to decide whether a given representation is canonical or not. For an infinite decimal the decision generally offers a difficult problem.

However, some special results for $a>1$ are very simple.

THEOREM: *Given a number to the base a in canonical form,*

$$c = \alpha_n\alpha_{n-1}\cdots\alpha_0\cdot\alpha_{-1}\alpha_{-2}\cdots \text{(can)},$$

and the same number in some non-canonical form,

$$c = \beta_m\beta_{m-1}\cdots\beta_0\cdot\beta_{-1}\beta_{-2}\cdots,$$

then $n \geqq m$. *If* $n=m$, *then* $\alpha_n \geqq \beta_n$; *if* $n=m$, $\alpha_n=\beta_n$, *then* $\alpha_{n-1}\geqq\beta_{n-1}$; *etc.*

THEOREM: *If* $c_1>c_2$, *and*

$$c_1 = \alpha_n\alpha_{n-1}\cdots\alpha_0\cdot\alpha_{-1}\alpha_{-2}\cdots \text{(can)}$$

and

$$c_2 = \beta_m\beta_{m-1}\cdots\beta_0\cdot\beta_{-1}\cdot\beta_{-2}\cdots \text{(can)},$$

then the same relations as in the preceding theorem hold between the α *and* β. *In other words, canonical numbers in increasing order of magnitude are lexicographically arranged to the base a.*

The root of the ambiguities in our representations lies in the following

LEMMA: *For* $a>1$, *not integral, there exist a positive integer* k *and* $k+1$ *integral coefficients* α_i, $0\leqq\alpha_i\leqq[a]$, *such that*

$$\alpha_n a^n + \alpha_{n-1}a^{n-1} + \cdots + \alpha_{n-k}a^{n-k} > a^{n+1}.$$

Proof: We will show that if we take for each α its greatest possible value $[a]$, the required inequality will hold for a sufficiently large k.

Since $[a]>a-1$, we have

$$[a]\frac{a^{n+1}}{a-1} = [a](a^n + a^{n-1} + \cdots + 1 + a^{-1} + a^{-2} + \cdots) > a^{n+1}.$$

Hence, for k sufficiently large,

$$[a]a^n + [a]a^{n-1} + \cdots + [a]a^{n-k} > a^{n+1}.$$

For an integral base a, the greatest permissible digit is not greater than $a-1$ but equal to $a-1$, leading to the (trivial) ambiguities in the representation of numbers in such systems ($.99\overline{9}\cdots=1$, etc.). For $a>1$, not integral, an unending repetition of the greatest digit $[a]$ cannot occur in a number in canonical form. For example,

$$.11\overline{1}\cdots{}_{(3/2)}\text{(non-can)} = 10.10100_{(3/2)}\text{(can)} = 2.$$

The following two properties (to which Professor W. B. Carver calls my attention) are easily verified:

(1) Any representation $\alpha_n\cdots\alpha_0\cdot\alpha_{-1}\alpha_{-2}\cdots$ is canonical if it contains no digit $>a-1$, i.e., if it does not contain the greatest digit $[a]$.

(2) Any representation containing the digit $[a]$ is canonical if it is made up of fewer than

$$\frac{\log [a] - \log \{[a] - a + 1\}}{\log a}$$

digits.

II. $a = 3/2$. As representative of the case a rational, we carry out in some detail the examination of the special case $a = 3/2$. All types of anormality occur already in this illustration, which is particularly simple to handle, because of the small numbers in numerator and denominator of a, and because the digits are restricted to 0, 1.

For simplicity of notation, but as is immediately clear, without real loss of generality, we consider fractions (to base 3/2) starting $1. \cdots$. A shift of decimal point corresponds to multiplication throughout the whole process by some positive or negative integral power of the base, and causes no difficulty.

We see that a canonical representation, base 3/2, cannot start with 1.1 (and therefore also not with $.11 \cdots, 11. \cdots$, etc.) This follows immediately from

$$1 + 1\cdot(3/2)^{-1} = 5/3 > 3/2.$$

Therefore a combination $\cdots 11 \cdots$ may not occur *anywhere* in a canonical representation to base 3/2.

Consider next 1.0 as first two digits of a canonical representation. Is it possible to continue 1.01? This is satisfactory $(1+0\cdot(3/2)^{-1}+1\cdot(3/2)^{-2} = 13/9 < 3/2)$, so that we have no conflict with the canonical character. Assume therefore as first three digits $101 \cdots$. Then 1011 impossible, on account of the sequence 11. Try $10101 \cdots$; now

$$1 + 0\cdot(3/2)^{-1} + 1\cdot(3/2)^{-2} + 0\cdot(3/2)^{-3} + 1\cdot(3/2)^{-4} > 3/2,$$

so that this sequence may not occur, either at the beginning or anywhere else in the decimal representation. In the same way 101001, 1010001, 10100001 are all excluded, and 101000001 is the first permissible sequence starting with 101.

We start over again with 1001, which is permissible $(1+8/27 < 3/2)$; 10011 is impossible; 100101 possible.

These considerations may be continued, but they can lead only to necessary, not sufficient, conditions for a representation to be canonical. Hence:

LEMMA: *Necessary but not sufficient conditions for a number to be in canonical form to the base* 3/2 *are, reading digits from left to right*:

1. *there must be at least one* 0 *between two consecutive* 1's,
2. *the sequence* 101 *must be followed at the right by at least five* 0's: $10100000 \cdots 1 \cdots$;
3. *the combination* 1001 *must be followed at the right by at least one* 0: $10010 \cdots 1 \cdots$.

According to this lemma, $1001001 \cdots$ may or may not occur in a canonical representation: it actually may, since

$$1 + (3/2)^{-3} + (3/2)^{-6} + \cdots = \frac{1}{1 - (3/2)^{-3}} = \frac{27}{19} < 3/2.$$

On the other hand, 1001010000100 is not canonical.

If we ask what relation to the base 3/2 corresponds to $1 = .9\overline{9} \cdots$ to the base 10, we find

$$1 = (3/2)^{-1} + (3/2)^{-3} + (3/2)^{-8} + \cdots = .101000010 \cdots_{(3/2)},$$

infinite and non-canonical.

It seems likely that every real number $\neq 0$ admits, to base $a > 1$, non-integral, an infinite number of distinct non-canonical representations.

III. $a = u/v$, u prime to v.

THEOREM: *A necessary condition for a fraction $c = r/s$, r prime to s, to admit a periodic representation to the base u/v (canonical or otherwise) is that all prime factors of s must be factors of $uv(u^\lambda - v^\lambda)$, where λ is the number of digits in the period.*

Proof: Let

$$\begin{aligned} c &= \alpha_n \cdots \alpha_0 \cdot \alpha_{-1} \cdots \alpha_{-\rho+1} \overline{\alpha_{-\rho} \cdots \alpha_{-\rho-\lambda+1}} \cdots_{(u/v)} \\ &= \alpha_n a^n + \cdots + \alpha_{-\rho+1} a^{-\rho+1} + (\alpha_{-\rho} a^{-\rho} + \cdots + \alpha_{-\rho-\lambda+1} a^{-\rho-\lambda+1}) \cdot \frac{a^\lambda}{a^\lambda - 1} \end{aligned}$$

or

$$\frac{r}{s} = \sum \alpha_\sigma \frac{u^\sigma}{v^\sigma} + \frac{u^\lambda}{u^\lambda - v^\lambda} \sum \alpha_{-\tau} \frac{v^\tau}{u^\tau},$$

with the summations taken over appropriate finite ranges for σ and τ.

Of course no irrational number can be periodic to a rational base a, canonical or otherwise; and the last formula gives the class of rational numbers representable in at least one way (canonical or otherwise) in periodic form. To actually determine the numbers periodic in canonical form is again a difficult problem. (On the other hand, we know that an integer or a rational number may lead to an infinite non-periodic canonical representation.) In terms of the algorithm, a necessary and sufficient condition for periodicity of a canonical representation (period of λ digits starting with $\alpha_{-\rho}$) is

$$\beta_{-\rho} = \beta_{-\rho-\lambda} a^\lambda.$$

This follows immediately from the algorithm.

IV. *a irrational.* Let $a = s^{1/k}$ (irrational), $s > 1$.

$$c = \alpha_n \cdots \alpha_0 \cdot \alpha_{-1} \cdots$$
$$= \alpha_n s^{n/k} + \alpha_{n-1} s^{(n-1)/k} + \cdots + \alpha_0 + \alpha_{-1} s^{-1/k} + \cdots,$$
$$0 \leqq \alpha_i < [s^{1/k}].$$

Multiplication by $s^{\gamma/k}$ is of course accomplished by shifting the decimal point γ units to the right; but addition or multiplication of two numbers to base $s^{1/k}$ is generally not obvious.

Making use of $a^k = s$, the representation to base $s^{1/k}$ (canonical or otherwise) corresponds to the (always possible) representation of $c > 0$ in the form

$$c = f_0(s) + f_1(s) \cdot s^{1/k} + \cdots + f_{k-1}(s) \cdot s^{(k-1)/k},$$

where $f_i(s)$ is a polynomial, or power series with a finite number of positive and a finite or infinite number of negative exponents, all with non-negative coefficients $\leqq [s^{1/k}]$. The canonical representation of a number to base a, a a quadratic irrationality, proceeds in obvious fashion. With $a = \sqrt{5}$, for example, we have

$$\begin{aligned}
1/2 &= 1 \cdot 5^{-1/2} + (1/2 - 5^{-1/2}) \\
&= 1 \cdot 5^{-1/2} + 0 \cdot 5^{-1} + (1/2 - 5^{-1/2}) \\
&= 1 \cdot 5^{-1/2} + 0 \cdot 5^{-1} + 0 \cdot 5^{-3/2} + (1/2 - 5^{-1/2}) \\
&= 1 \cdot 5^{-1/2} + 0 \cdot 5^{-1} + 0 \cdot 5^{-3/2} + 1 \cdot 5^{-2} + (23/50 - 5^{-1/2}) \\
&= 1 \cdot 5^{-1/2} + 0 \cdot 5^{-1} + 0 \cdot 5^{-3/2} + 1 \cdot 5^{-2} + 0 \cdot 5^{-5/2} + 1 \cdot 5^{-3} + \cdots \\
&= .100101 \cdots_{(\sqrt{5})}(\text{can}).
\end{aligned}$$

For a given irrational base necessary conditions for a representation to be canonical may be derived as for $a = 3/2$.

For example, for the base $a = \sqrt{5}$, consider the representation

$$c = \cdot \alpha_{-1} \alpha_{-2} \cdots.$$

Try $\alpha_{-1} = 2$, $\alpha_{-2} = 2$, i.e., $c = .22 \cdots$. Since $2 \cdot 5^{-1/2} + 2 \cdot 5^{-1} > 1$, the sequence 22 cannot occur in a canonical representation. Similarly 21 impossible; 20 of course possible; 10, 11, 12, 00, 01, 02 all possible. The following sequences of three digits cannot occur: 222, 221, 220, 212, 211, 210, 202, 122. This can of course be continued to take care of sequences of any assigned length, but can clearly lead only to necessary conditions for the representation to be canonical to base $\sqrt{5}$.

Consider the case where the base a is a root of the quadratic equation

$$a_0 a^2 + a_1 a + a_2 = 0,$$

a_i integers and $D = a_1^2 - 4a_0 a_2$ a non-square; so that

$$a = h + k\sqrt{D}, \qquad h, k \text{ rational}.$$

Numbers c which can in at least one way be represented as whole numbers (canonical or otherwise) are

$$c = \alpha_n(h + k\sqrt{D})^n + \alpha_{n-1}(h + k\sqrt{D})^{n-1} + \cdots \alpha_1(h + k\sqrt{D}) + \alpha_0,$$
$$0 \leqq \alpha_i \leqq [a].$$

Since the rational part of $(h+k\sqrt{D})^i$ and the part involving $\sqrt{D}$ are respectively

$$\frac{(h + k\sqrt{D})^i + (h - k\sqrt{D})^i}{2} \quad \text{and} \quad \frac{(h + k\sqrt{D})^i - (h - k\sqrt{D})^i}{2},$$

the numbers are $c = A + B\sqrt{D}$,

$$A, B = \frac{1}{2}\sum_{i=0}^{n} \alpha_i[(h + k\sqrt{D})^i \pm (h - k\sqrt{D})^i], \qquad 0 \leqq \alpha_i \leqq [a].$$

Numbers permitting at least one representation as finite or periodic decimals are similarly characterized. For an algebraic number of order $n > 2$, we may make use of the reduction

$$a^n = -(a_1 a^{n-1} + a_2 a^{n-2} + \cdots + a_n)/a_0,$$

without however obtaining results of particular interest.

V. *a transcendental.* The same methods lead to the same general type of results as in the other cases. Certain (denumerable) classes of transcendental numbers connected rationally with the given base a are obtained for the totalities of numbers representable as periodic decimals, in at least one way:

$$c = \sum \alpha_i a^i + \frac{a^\lambda}{a^\lambda - 1}\sum \alpha_i a^i, \qquad \text{periodic decimals},$$

with summations taken over appropriate ranges. Thus, to the base e, the periodic decimal

$$\overline{\cdot \alpha_{-1} \cdots \alpha_{-\lambda}} \cdots {}_{(e)} = \frac{e^\lambda}{e^\lambda - 1}\sum_{i=-\lambda}^{-1} \alpha_i e^i, \qquad 0 \leqq \alpha_i \leqq 2.$$

Nothing is known concerning the class of numbers permitting canonical periodic representation except the trivial fact that it must be a sub-class of the less restricted set.

VI. *Occurrence of digits in canonical representations.*

For $a = 3/2$, with the two possible digits 0, 1, we found that the sequence 11 is excluded, that 101 must be followed by five zeros at least, that 1001 may be followed by 01. It is clear that we must have in every infinite canonical representation to base 3/2 a preponderance of 0's, and that a maximum percentage of 1's will occur in $.1001\overline{001} \cdots$ (can), leading to the maximum density of 1's as compared with 0's of 1:2.

For a base $a > 2$, the largest digit $[a]$ seems to play in infinite canonical representations a role similar to that of the digit 1, to the base 3/2.

ON THE FOCI OF PLANE ALGEBRAIC CURVES WITH APPLICATIONS TO SYMMETRIC CUBIC CURVES*

By ETHEL A. RICE, Hereford, Texas.

Introduction. The points called *foci* were first encountered in connection with conic sections. Apollonius of Perga, who lived in the third century B.C., incidentally discovered the foci of an ellipse and hyperbola. Five hundred years later Pappus found the focus of a parabola. The theory of the foci of conics was first worked out by Kepler, who introduced the name of focus, in the early part of the seventeenth century.† The general idea of a focus of a curve of degree higher than the second, however, was not established until 1832 by Plücker.‡

A general discussion of foci and their properties may be found in Salmon,§ Basset,||, Wieleitner,¶ Hilton,** Coolidge,†† and others. Hilton and Basset show graphically the location of the foci of a few cubics and quartics. In addition to a good many papers dealing with special theorems concerning foci, there are a few dealing with the foci of cubics. For instance, R. A. Roberts§§ has several papers on foci and confocal systems of plane curves where applications are made to confocal cubics; H. M. Jeffrey|||| in a number of articles considers cubics of the third class with specified types of foci and cubics with a double and single focus; A. Emch¶¶ in dealing with curves having a given system of foci finds the foci of a special cubic.

Part I of this paper is devoted to the general definition of foci and to a summary of the known results concerning the number of foci of algebraic curves, with applications to cubic curves. In part II the real foci of certain plane algebraic cubics are determined, although the fact that imaginary foci exist must not be overlooked. Only curves are considered whose equations, expressed in Cartesian coordinates, have real coefficients. Furthermore, the curves are symmetric with regard to an axis or a point. The writer particularly desires to portray graphically the location of the real foci of these special cubics.

I. GENERAL CONSIDERATIONS

1. *Definition.* It is well known that every circle intersects the line at in-

* This paper is an abstract of a thesis for the Master's degree, University of Colorado, 1932, under the supervision of Dr. Claribel Kendall. Presented at the meeting of the Rocky Mountain Section of the Mathematical Association of America, April 14, 1933.

† Gow, *History of Greek Mathematics*, 1884, p. 252.

‡ Plücker, *Gesammelte Mathematische Abhandlungen*, vol. 1 (1895) p. 291.

§ *Higher Plane Curves*, third edition, 1879.

|| *An Elementary Treatise on Cubic and Quartic Curves*, 1901.

¶ *Theorie der ebenen Algebraischen Kurven höherer Ordnung*, 1905.

** *Plane Algebraic Curves*, 1920.

†† *A Treatise on Algebraic Plane Curves*, 1931.

§§ Quarterly Journal of Mathematics, vol. 32 (1900), pp. 141–182; vol. 35 (1904), pp. 297–394.

|||| Ibid., vol. 14 (1877), pp. 127–139, 359–376. vol. 16 (1879), pp. 65–81, 348–369.

¶¶ Bulletin of the American Mathematical Society, vol. 25 (1918), pp. 157–161.

finity in two imaginary points, say I and J. These points are called the *circular points at infinity*, and their homogeneous Cartesian coordinates are $(1, i, 0)$ and $(1, -i, 0)$, respectively, where $i^2=-1$. In the case of a curve with real coefficients the two points I and J are related in a similar manner to the curve; for example, if I is a multiple point on a curve, then J is also a multiple point of the same multiplicity. Lines passing through the circular points, other than the line at infinity, are called *circular lines* or *isotropic lines*.

If from a focus of a conic we draw two tangents to the curve, these pass respectively through the two circular points at infinity. This fact led Plücker to a generalized conception of the foci of curves of higher degree. His definition, as stated by Cayley,* is as follows: "If from each of the circular points at infinity . . . tangents are drawn to the curve, the intersections of each tangent from the one point with each tangent from the other point are the foci of the curve." This definition not only gives real foci but also imaginary foci corresponding to the imaginary intersections of the tangents from the circular points to the curve.†

If the circular points I and J do not lie on the curve, the finite intersections of ordinary tangents (tangents having ordinary contact) from I and J to the curve are called *ordinary foci*. If I and J are on the curve, the finite intersections of the tangents to the curve at I and J are called *singular foci* of the curve. The intersections of the remaining tangents from I and J to the curve are ordinary foci.

2. *Number of Foci of a Plane Algebraic Curve*. Consider a curve of the mth class. Since m tangents can in general be drawn from a point to such a curve m tangents can be drawn to the curve from I and m from J. These isotropic tangents will intersect in m^2 points. If such a tangent from I is

$$x + iy = a + ib,$$

then

$$x - iy = a - ib$$

is a tangent from J. These tangents intersect in the real point (a, b), so that there are m real foci. There are no more than m real foci, for no tangent from I or J can contain more than one real point. Hence: "A curve of class m has in general m real and m^2-m unreal foci."‡

The number m of real foci will be decreased if the curve is tangent to the line joining the circular points (the line at infinity), or passes through the circular points. "If the line at infinity is a multiple tangent of order g, a curve of the mth class cannot have more than $m-g$ real foci."§ However, the line at in-

* Cayley, *Collected Papers*, vol. 6 (1893), p. 515.

† With this definition the ellipse, for instance, has four foci, two real and two imaginary.

‡ Hilton, *Plane Algebraic Curves*, 1920, p. 69.

§ Basset, *An Elementary Treatise on Cubic and Quartic Curves*, 1901, p. 47.

finity may be an inflexional tangent. Since such a tangent counts for two among the tangents at a given point,* the number of real ordinary foci is reduced by two for each inflexion on the line at infinity with that line as the inflexional tangent.

If the curve also passes through the circular points, the number of real ordinary foci is lowered still further. In this case there will be, however, singular foci. If I is an ordinary point, the tangent at this point counts as two among the tangents drawn from I to the curve. The same is true of the point J. There will then be $m-g-2$ real ordinary foci, and the tangents at I and J intersect in a real point giving a singular focus, commonly called a double focus. Similarly, if I and J are singular points of multiplicity k with distinct tangents, there are k double and $m-g-2k$ ordinary foci. If I and J are each an inflexion, or each a cusp, then the tangent at I or J counts three times among the I or J tangents; and there are from each point $m-g-3$ other tangents. Hence there are $m-g-3$ real foci, and the only other real focus is the intersection of the tangents at I and J, which is commonly called a triple focus as counting for three among the real foci.†

The points of contact of the m tangents which can be drawn from a given point to a curve of the mth class are the points of intersection of the curve with its first polar.‡ If a curve has a node, conjugate point, or cusp, the first polar passes through it and the point counts for two (or three in case of a cusp) among the intersections. (Since higher singularities reduce to the equivalent of double points, this statement takes care of singularities of order higher than two.) But the line joining the given point to the singular point is not a true tangent. Hence, the intersections of lines from I and J to the singular points must be excluded from among the foci.

The general algebraic curve of the third degree is of class six and has, therefore, six real foci. If, however, the cubic is tangent to the line at infinity the number of real foci is reduced as in the general case just discussed. A nondegenerate cubic can have at most one singularity which may be a node, conjugate point, or cusp. It is impossible, then, for the circular points of a cubic to be singular points, for if I were a singular point, then J also would be a singular point. If the cubic passes through the circular points, and they are ordinary points, the number of real ordinary foci is reduced by two. I and J may be inflexion points, however, and the number of real ordinary foci is reduced by three. Hence, for the cubic we will be concerned only with ordinary, double, and triple foci.

If a cubic is unicursal, the lines drawn from the circular points to the singular point are not, as has been stated above, true tangents to the curve. Therefore, their real intersections, which will be at the singular point, must be excluded from among the foci of the cubic.

* Coolidge, *A Treatise on Algebraic Plane Curves*, 1931, p. 21.

† Salmon, *Higher Plane Curves*, Third Edition, 1879, p. 121.

‡ Wieleitner, *Algebraische Kurven*, 1905, p. 18.

The degenerate cubics are composed of a conic and a straight line or three straight lines. The first type has two singularities and the number of real foci cannot exceed two. The real foci of the degenerate cubic are the foci of the conic. Since the number of real foci of a cubic cannot exceed six and since each singularity lowers the number of real foci by at least two, a degenerate cubic composed of three straight lines, and thus having three nodes, cannot have real foci. This statement accounts for the case of three concurrent lines; the point of intersection is a triple point which is equivalent to three double points.

II. FOCI OF SPECIAL PLANE ALGEBRAIC CUBICS

The following discussion deals with the actual determination of the real foci of special symmetric cubics.

1. *Cubics of the Type* $y^2=k(x-a)\ (x-b)\ (x-c)$, $k>0$. Curves represented by this equation divide into five principal classes. Newton has given the name *divergent parabolas* to these curves, and his theorem is that every cubic may be projected into one of these five divergent parabolas.* Cubics of this group have the line at infinity as an inflexional tangent. Hence the number of real ordinary foci is at least two less than the class.

When $a<b<c$ the curve is composed of an oval and an infinite branch and has no singularities. The class of this cubic is six, but since the line at infinity is an inflexional tangent there are only four real foci.

If a and b are conjugate imaginaries, we have again a non-singular curve of class six with four real foci.

If $a=b<c$ the oval becomes a point and we have an infinite branch with a conjugate point. This curve will have only two real foci, because the lines from I and J to the conjugate point do not count as real tangents and the class of the curve is reduced to four. Again the number of real foci is two less than the class.

When $a<b=c$ we have a node and the class of the curve is consequently four. Hence, a nodal cubic having the line at infinity as an inflexional tangent has two real foci.

Finally, if $a=b=c$ we have a cuspidal cubic of class three with only one real focus.

In all of the above cases except the last a special cubic may occur with a reduced number of real foci. This is the cubic which R. A. Roberts† discusses as follows: "The simplest non-singular cubic in relation to the circular points is the cubic with an inflexional focus and the line at infinity as an inflexional tangent. This curve has two single foci collinear with the inflexional focus. Taking rectangular coordinates the curve may be written

* Salmon, loc. cit., p. 164.

† R. A. Roberts, *On Foci and Confocal Systems of Plane Curves*, Quarterly Journal of Mathematics, vol. 35 (1904), p. 335. Although Roberts speaks of the non-singular case, this special cubic can occur with a node or a conjugate point.

$$(x-\alpha)^3-\mu\{y^2+(x-\beta)^2\}=0,$$

where $(\beta, 0)$ is the inflexional focus." The inflexional focus referred to is the intersection of two inflexional tangents which pass through the circular points.

Examples 1–7, with the correspondingly numbered figures, illustrate the various curves of the type

$$y^2=k(x-a)(x-b)(x-c), \qquad k>0.$$

EXAMPLE 1. *Oval and Infinite Branch.* Given the curve

$$y^2=4x^3/27-x^2+x/2+7/2. \tag{1}$$

To find the real ordinary foci, determine the value of c in

$$x+iy=c, \quad \text{where} \quad c=a+ib, \tag{2}$$

so that (2), an imaginary line through I, shall be tangent to (1). Eliminating y between (1) and (2) gives

$$4x^3/27+x(1/2-2c)+(c^2+7/2)=0. \tag{3}$$

The condition that (2) shall be tangent to (1) is that (3) have equal roots; that is, that the discriminant of (3) vanish. Hence,

$$c^4-8c^3+13c^2-1.500c+12.375=0. \tag{4}$$

The roots, correct to two decimal places, are

$$c=-0.18\pm 0.88i, \qquad 5.71, \qquad 2.65.$$

and therefore the real points of intersection of the tangents from I with the tangents from J, that is, the four real ordinary foci, are $(-0.18, 0.88)$, $(-0.18, -0.88)$, $(5.71, 0)$, and $(2.65, 0)$.

EXAMPLE 2. *Infinite Branch, Non-singular.* The four real ordinary foci* of the curve

$$y^2=4x^3/27-x^2+2x$$

are $(3.27, 1.79)$, $(3.27, -1.79)$, $(0.73, 0.21)$ and $(0.73, -0.21)$.

EXAMPLE 3. *Infinite Branch with Inflexional Focus.* Given the curve

$$(x+3)^3-(27/8)\{y^2+(x+4)^2\}=0. \tag{5}$$

To find the real ordinary foci, solve (5) with

$$x+iy=c, \tag{6}$$

and impose the condition that (6) shall be tangent to (5). This gives $c=\pm 3.46$ and $c=-4, -4$. Hence the real ordinary foci are $(3.46, 0)$ and $(-3.46, 0)$.

The focus $(-4, 0)$ is an inflexional focus, so called because it is the intersec-

* The method of obtaining the foci in this and following problems in which the solution is not shown is the same as in Example 1.

tion of two inflexional tangents which pass through I and J. The inflexions are not at I and J, however, and therefore each inflexional tangent counts as two among the tangents from the circular points. Hence, the factor $c+4=0$ must occur twice in the discriminant.

EXAMPLE 4. *Conjugate Point and Infinite Branch.* Given the curve

$$y^2 = 4x^3/27 - 7x^2/9.$$

When we solve with $x+iy=c$, the discriminant equated to zero is

$$c^2(27c^2 - 162c - 7) = 0. \tag{7}$$

For the factor $c^2=0$, we have $x+iy=0$, which is the equation of a line from I through the conjugate point. Therefore, (0, 0) is not considered among the foci. The second factor of (7) gives $c=7.19$ and -1.19. Therefore, the two real ordinary foci are (7.19, 0) and (-1.19, 0).

EXAMPLE 5. *Conjugate Point and Infinite Branch.* Given the curve

$$y^2 = x^3 - x^2.$$

This cubic is of the same type as the curve in Example 4 but is the special case discussed by Roberts, with $\alpha=\beta=0$. The real ordinary foci are (32/27, 0), and (0. 0), the conjugate point.

EXAMPLE 6. *Nodal Cubic.* The two real ordinary foci of the curve

$$y^2 = 4x^3/27 - 8x^2/9 + 4x/3$$

are (0.500, 0.096) and (0.500, -0.096).

EXAMPLE 7. *The Cuspidal Cubic.* The curve

$$y^2 = x^3/27$$

is of the third class and has one real ordinary focus at (-4, 0).

2. *Circular Cubics, Unicursal.* A circular cubic is a cubic which passes through the circular points at infinity. A unicursal circular cubic is a circular cubic with one double point which may be a node, conjugate point, or cusp.

Since nodal circular cubics are of the fourth class, the curve has one real singular or double focus and two ordinary foci. A nodal cubic not passing through the circular points would in general have four foci. The same is true for circular cubics with a conjugate point.

Cuspidal circular cubics are of the third class; therefore, they have one double focus and one ordinary focus.

It can be shown that the inverse of a conic with respect to a point on the curve is a circular cubic whose asymptote is parallel to the tangent to the conic at the center of inversion. The inverse points of the foci of the conic are the ordinary foci of the circular cubic.*

The following problems are examples of unicursal circular cubics.

* Basset, loc. cit., p. 90.

EXAMPLE 8. *Cissoid of Diocles (Cuspidal Cubic).* Given the curve*

$$x(x^2 + y^2) = ay^2. \tag{8}$$

This curve is the inverse of the parabola $y^2 = x/a$ whose vertex is the center of inversion; the focus of the parabola is $[1/(4a), 0]$, and the real ordinary focus of the cuspidal cubic is $(4a, 0)$.

Since the curve passes through I and J there is a singular focus. To find this focus substitute $y = mx + k$ in (8), and we have

$$x^3(1 + m^2) + x^2(2mk - am^2) + x(k^2 - 2amk) - ak^2 = 0. \tag{9}$$

Equating the coefficients of the two highest powers of x in (9) to zero gives $m = \pm i$ and $k = \pm ai/2$. Hence the isotropic asymptotes are

$$y = \pm ix \pm ai/2 \quad \text{or} \quad x \pm iy = -a/2.$$

Therefore, the real singular focus† is $(-a/2, 0)$.

EXAMPLE 9. *Circular Cubic with Conjugate Point.* The curve

$$x(x^2 + y^2) = x^2 + 2y^2$$

is the inverse of the ellipse $x^2 - x + 2y^2 = 0$ with a vertex as center of inversion; the foci of this ellipse are $[(2 \pm \sqrt{2})/4, 0]$, and the real ordinary foci of the circular cubic are $(6.82, 0)$ and $(1.22, 0)$. The real singular focus is $(1.50, 0)$.

EXAMPLE 10. *The Trisectrix of Maclaurin (Nodal Cubic).* The curve‡

$$2x(x^2 + y^2) = a(3x^2 - y^2)$$

is the inverse of the hyperbola $3ax^2 - ay^2 = 2x$ with a vertex as center of inversion; the foci of this hyperbola are $(1/a, 0)$ and $[-1/(3a), 0]$. This curve has points of inflexion at I and J, and the isotropic inflexion asymptotes intersect in a point which is called a triple focus. The class is three. Therefore, one of the foci of the hyperbola will be the inverse of the ordinary focus and the other one will be the inverse of the triple focus. The real ordinary focus is $(-3a, 0)$ and the real singular focus is $(a, 0)$.

EXAMPLE 11. *Logocyclic Curve (Nodal Cubic).* The curve

$$x(x^2 + y^2) + x^2 - y^2 = 0$$

is the inverse of the hyperbola $x^2 - y^2 + x = 0$ with a vertex as center of inversion; the foci of this hyperbola are $[(-1 \pm \sqrt{2})/2, 0]$, and the real ordinary foci of the cubic are $(4.82, 0)$ and $(-0.82, 0)$. The real singular focus is $(-1, 0)$.

3. *Circular Cubics, Non-singular.* A non-singular circular cubic is a cubic which passes through the circular points and which has no singularities. A proof of the following theorem may be found in Hilton's *Plane Algebraic Curves:*

---

* Hilton, loc. cit., p. 73.

† The real singular focus is indicated by a cross on the graphs.

‡ Hilton, loc. cit., p. 73.

"Any circular cubic is self-inverse with respect to each of four mutually orthogonal circles, and the sixteen foci lie by fours on these four circles."* If the circular cubic has two circuits—that is, an infinite branch and an oval—three of the circles with respect to which it is self-inverse are real. The remaining circle is unreal, but it has a real center on the side of the oval near to the infinite branch. But if the circular cubic has only one circuit, two of the circles are real, and the other two have imaginary centers. The centers of the circles are the points at which tangents are parallel to the real asymptote.

The inverse of a bicircular quartic with respect to a point on the curve is a circular cubic. Since foci invert into foci and a circle into a circle, the properties of the real ordinary foci of bicircular cubics and bicircular quartics are identical.†

Examples of one-circuited and two-circuited circular cubics may be found in Hilton's *Plane Algebraic Curves*.‡ A discussion of one special curve follows.

EXAMPLE 12. *Special One-Circuited Circular Cubic.* The foci of the curve§

$$x^3 + xy^2 + y = 0$$

were found by Emch by using line coordinates. The real ordinary foci are $(2-3\sqrt{2}/2,\ 2-3\sqrt{2}/2)$, $(-2+3\sqrt{2}/2,\ -2+3\sqrt{2}/2)$, $(-2-3\sqrt{2}/2,\ -2-3\sqrt{2}/2)$, and $(2+3\sqrt{2}/2,\ 2+3\sqrt{2}/2)$, and the real singular focus is $(0, 0)$. In this circular cubic the real foci are all collinear. The figure is omitted for lack of space.

4. *Cubics of the Type* $(x^2+y^2-r^2)x-ky^m=0$. We consider cubics of this type where $m=0, 1, 2$. They are all circular cubics, passing through the circular points I and J.

For $m=0$, except for the special values of k noted below the curve is non-singular and of class six, and there are four real ordinary foci and a singular focus. If $k=2r^3\sqrt{3}/9$, the curve has a conjugate point, and there are only two real ordinary foci and the singular focus.

When $m=1$, we get a one-circuited circular cubic with four real ordinary foci and a singular focus at the origin.

When $m=2$, the curve has an asymptote parallel to the y-axis and not the y-axis itself as in all the above cases. This cubic has the four real ordinary foci; the singular focus, however, is not at the origin but lies on the x-axis within the oval and near the center.

The following examples illustrate the cases discussed.

EXAMPLE 13. *Circular Cubic with Oval.* The four real ordinary foci of the curve

$$(x^2 + y^2 - 1)x - 0.10 = 0$$

---

* Hilton, loc. cit., p. 220.

† Hilton, loc. cit., p. 304.

‡ Ibid., pp. 217–223.

§ Emch, *On Plane Algebraic Curves with a Given System of Foci*, Bulletin of the American Mathematical Society, vol. 25 (1918), p. 159.

are (0.31, 1.32), (0.31, −1.31), (−0.31, −0.67), and (−0.31, 0.67), and the real singular focus is (0, 0).

EXAMPLE 14. *Circular Cubic with Conjugate Point.* The two real ordinary foci of the curve

$$(x^2 + y^2 - 9)x - 6\sqrt{3} = 0$$

are (1.73, 4.90) and (1.73, −4.90), and the real singular focus is (0, 0).

EXAMPLE 15. *Circular Cubic with Only Infinite Branch.* Given the curve

$$(x^2 + y^2 - 1)x - 1 = 0.$$

The real ordinary foci are (0.87, 2.02), (0.87, −2.02), (−0.13, 0), and (−1.61, 0). The real singular focus is again (0, 0).

EXAMPLE 16. *Circular Cubic with Oval.* The real ordinary foci of the curve

$$(x^2 + y^2 - 4)x - y^2 = 0$$

are (2.73, 2.73), (2.73, −2.73), (−0.73, 0.73) and (−0.73, −0.73), and the real singular focus is (−0.50, 0).

EXAMPLE 17. *One-circuited Cubic.* The four real ordinary foci of the curve

$$(x^2 + y^2 - 1)x - 0.50y = 0$$

are (0.95, 1.74), −0.95, (−1.74), (−0.46, 0.32), and (0.46, −0.32), and the real singular focus is (0, 0).

5. *Miscellaneous Examples.** The foci of five well-known symmetric cubics have been determined but they cannot be placed in the preceding groups.

EXAMPLE 18. *Folium of Descartes.* The real ordinary foci of the curve

$$x^3 + y^3 - 3xy = 0$$

are (−0.09, 0.73), (0.73, −0.09), (1.41, 1.41), and (−2.04. −2.04).

The Folium of Descartes is of class four and has four real ordinary foci, two lying on the axis of symmetry and the other two symmetric with respect to this axis. The curve does not pass through the circular points, and therefore has no singular focus.

EXAMPLE 19. The real ordinary foci of the curve

$$xy^2 + y^2 - x^2 = 0$$

are (−3.6, 0), (0.43, 0.96) and (0.43, −0.96).

This curve is of class four, but it is tangent to the line at infinity which reduces the number of real ordinary foci to three. There is no singular focus since the curve does not pass through the circular points.

The curve, instead of having a finite loop as does the Folium of Descartes, goes to infinity in the direction of the x-axis, and the point which corresponds to the real ordinary focus within the loop of the Folium of Descartes is lost.

* Since space is limited the figures for these are omitted.

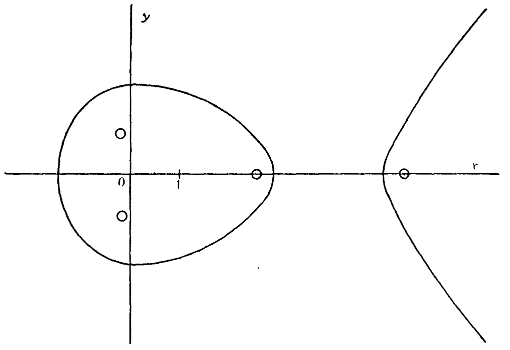


FIG. 1

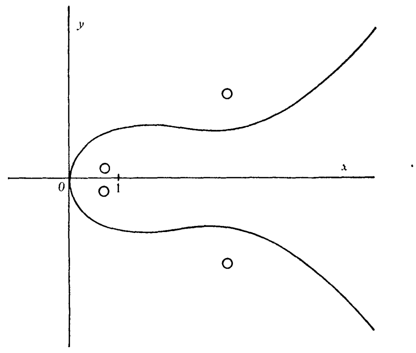


FIG. 2

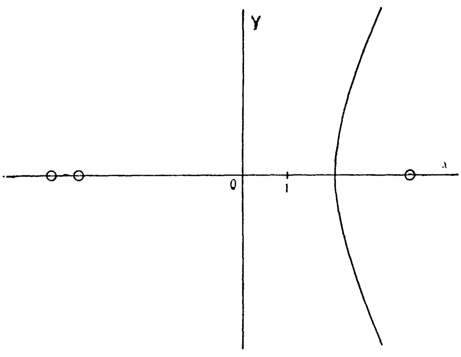


FIG. 3

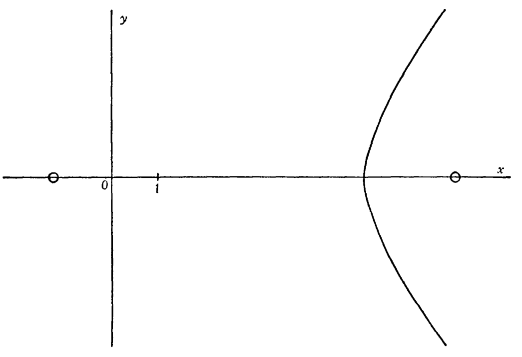


FIG. 4

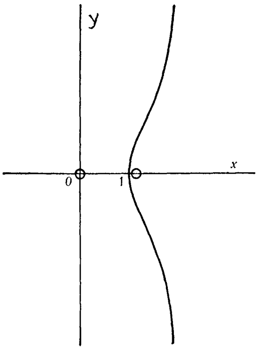


FIG. 5

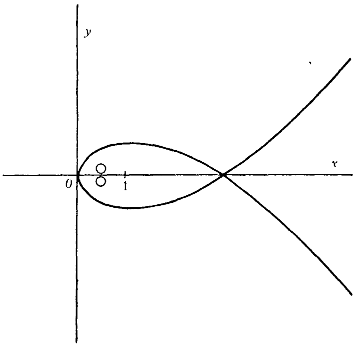


FIG. 6

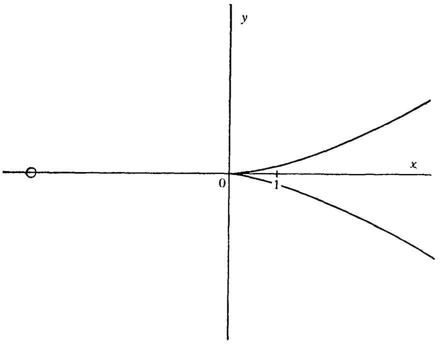


FIG. 7

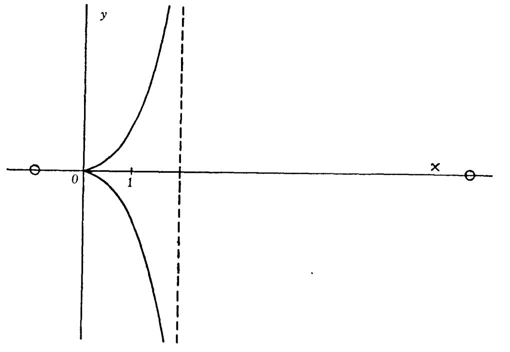


FIG. 8

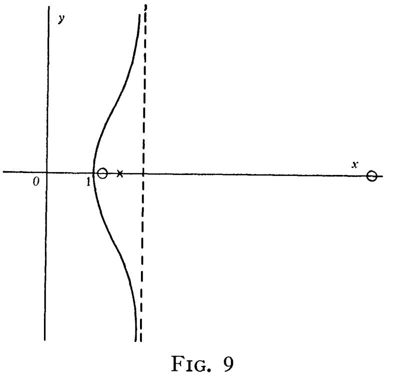


FIG. 9

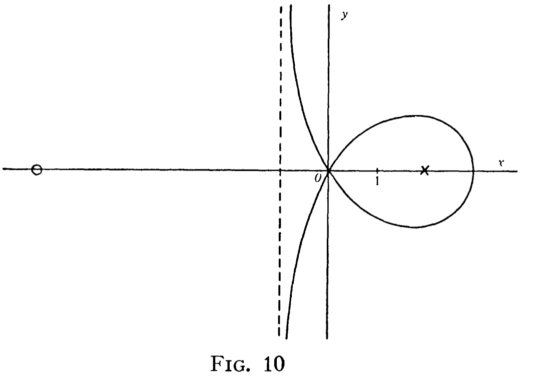


FIG. 10

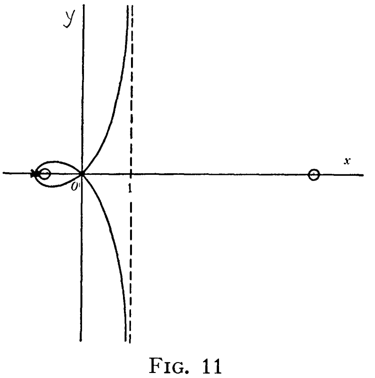


FIG. 11

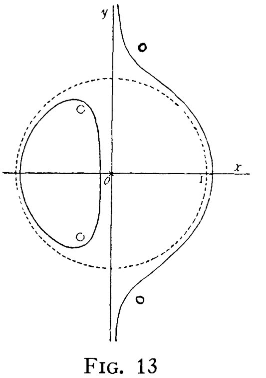


FIG. 13

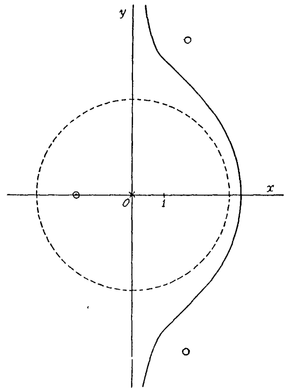


FIG. 14

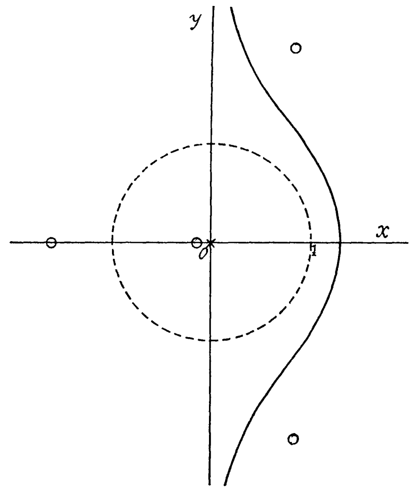


FIG. 15

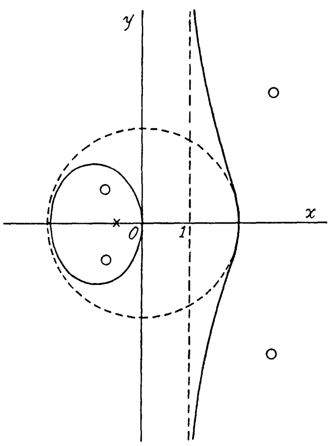


FIG. 16

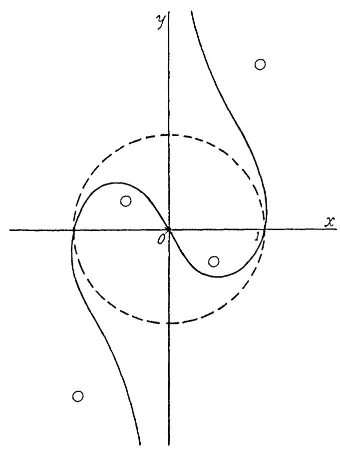


FIG. 17

EXAMPLE 20. *Cubical Parabola.* The real ordinary foci of the curve

$$y = ax^3$$

are $[\sqrt{6a}/(9a),\ \sqrt{6a}/(9a)]$ and $[-\sqrt{6a}/(9a),\ -\sqrt{6a}/(9a)]$.

The cubical parabola is of class three due to the fact that it has a cusp with the line at infinity as the cuspidal tangent. Hence the curve has two real ordinary foci. There is no singular focus since the curve is not circular.

EXAMPLE 21. *Serpentine.* The real ordinary foci of the curve

$$xy^2 + x - 2y = 0$$

are (2.51, 1.76), (−2.51, −1.76), (0.69, 0.84), and (−0.69, −0.84).

The curve is of class four because there is a double point at infinity in the

direction of the x-axis. There are four real ordinary foci but no singular focus because the curve is not circular. The foci are grouped in pairs which are symmetric with respect to the origin.

EXAMPLE 22. The real ordinary foci of the curve*

$$xy^2 = a^3$$

are $(-3a/\sqrt[3]{4}, 0)$, $(3a/\sqrt[3]{32}, 3a\sqrt{3}\sqrt[3]{32})$, and $(3a/\sqrt[3]{32}, -3a\sqrt{3}/\sqrt[3]{32})$.

This curve has a cusp at infinity with the x-axis as cuspidal tangent, and it has an inflexion at infinity with the y-axis as the inflexion tangent. The foci of such a curve have a particular relation. "When the curve has both its cusp and point of inflexion at infinity the foci form an equilateral triangle."†

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

HOMOTHETIC CENTERS AND CENTERS OF SIMILITUDE

By D. L. MACKAY, Evander Childs High School

I submit the following question for consideration by readers of the MONTHLY:

Is there anything in the history of the terms "center of similitude" and "homothetic center" which justifies their use as equivalent terms?

I give excerpts from various geometries.

A. *Exercices de géométrie*, F.G.M., 5th edition.

1. Page 483, No. 1150l.—Two circles in the same plane can be considered as two similar figures, either directly or inversely, in an infinite number of ways.
2. Page 484, No. 1150l.—If we designate the interior homothetic center by I and the exterior center by E, the circle described on IE as diameter is the locus of points whose distances to the centers (of the circles) O and O' are in the same ratio as the ratio of similitude.
3. Page 556, No. 1260.—Three circles taken in pairs have six centers of similitude; the three exterior centers are in a straight line and it is the same for two exterior and one interior center.
4. Page 99, No. 212.—Three circles taken in pairs have six homothetic centers.

B. *Traité de géométrie*. Rouché et de Comberousse, volume 1, 1900.

1. Page 252.—Given a system of points, A, B, C, $\cdots$ situated in a plane,

* A graph of this curve is in H. M. Jeffrey's article *On Plane Cubic of the Third Class with Three Single Foci*, Quarterly Journal of Mathematics, vol. 16 (1879), p. 348.

† R. A. Roberts, Loc. cit., p. 156.

if upon the rays SA, SB, SC, $\cdots$ issuing from any point S in the plane, we take from this point S the segments SA', SB', SC'. $\cdots$, so that

$$\frac{SA'}{SA} = \frac{SB'}{SB} = \frac{SC'}{SC} = \cdots = K = \cdots.$$

The point S is called the homothetic center $\cdots$

2. Page 255, line 12.—Then the two homothetic centers divide the line of centers OO' of the two circles harmonically.
3. Page 260.—Two circles are two similar figures; but their double point is not determined except if we indicate the point A' of the second circle that we wish to consider as homologous to a point A chosen arbitrarily on the first circle. If point A' homologous to A is not given, we know only one pair of homologous points, those formed by the centers C and C' of the two circles; and the double point is only subjected then to the condition of being situated upon the circle, locus of points the ratio of whose distances to the centers C and C' equals the ratio of the radii, that is to say the circle having for diameter the line which joins the *centers* of *similitude* of the given circles. (Reference is then made to page 254 section 367 which states that any two circles are directly or inversely homothetic and contains B2.)

C. *College Geometry*. Altshiller-Court, 1925.

1. Page 40, section 41.—Let S be the given homothetic center, K the ratio and O the center of the given circle.
2. Page 157, Section 301.—The point S is called the "external" and the point S' the internal center of similitude of the two circles.
3. Page 116, Section 214.—The orthocenter H is a homothetic center of the nine-point circle (N) and the circumcircle (O) of ABC.

D. *A Sequel to Elementary Geometry*, John W. Russell, 1924.

1. Page 3, Section 4.—Take any figure and any fixed point S. Let P be any point of the figure. Take P' upon SP so that $SP' = k \cdot SP \cdots$. Then P' is said to generate a homothetic or similar and similarly situated figure, S being the homothetic center $\cdots$.
2. Page 6, Section 9.—If we rotate one of two homothetic figures about the homothetic center, S, in the plane of the figures, we obtain figures which are similar and of the same kind but no longer homothetic. S is now called the center of similarity or center of similitude.
3. Page 99, Section 1.—If we divide the line joining the centers of two circles internally and externally in the ratio of the radii, the points so obtained are called the internal and external centers of similitude of the circles.
4. Page 100, Section 3.—The line joining any two of the six centers of similitude of three circles taken in pairs passes through a third center of similitude.

5. Page 103.—Two given circles are similar with respect to any point on their circle of similitude.

E. *A Treatise on the Circle and the Sphere*, Julian L. Coolidge, 1916.

1. Page 28, line 1.—If we divide the line joining the centers of two circles internally and externally in the ratio of the radii, the points so obtained are called the internal and external centers of similitude of the circles.
2. Page 28, Theorem 25.—If two circles of unequal radius lie outside of one another, their common tangents intersect at their centers of similitude $\cdots$.

F. *Modern Geometry*, Richard Townsend, volume 1, 1863, pages 37, 39, 273.—Uses only the term center of similitude.

G. *The Harpur Euclid*, Langley and Phillips.

1. Page 448, 449.—Use the term center of similitude.
2. Page 450.—When two similar figures have a center of similitude, they are called homothetic; $\cdots$.

H. *Modern Geometry*, Godfrey and Siddons, 1928.

1. Page 72.—When two similar figures are so placed that the join of each pair of points in the one figure is parallel to the join of the corresponding pair of points in the other figure the two figures are said to be similarly situated and the point of concurrence of the lines joining corresponding points is called the center of similitude.

The following conclusions would appear to be warranted:

1. There is confusion in the use of these terms by different writers and in some cases by the same writer.
2. The two terms are sometimes used as equivalent. See A2, 3, 4; D2, 3, 4.
3. In the case of A1 and A3 it is not clear which six of an infinite number are to be considered.
4. Of the two designations for the points I and E (using the lettering of A2) I believe that the term "homothetic centers" is preferable, reserving the term "center of similitude" for any point on the circle of similitude.
5. I consider the use of these terms in D1 and D2 to be the clearest to the reader. It seems to me that one should substitute homothetic center for center of similitude in D3 and D4.

On the Definition of $e^{i\theta}$

By G. B. Price, Brown University

The usual motivation of the definition of $e^{i\theta}$ requires a knowledge of infinite series; the following procedure which does not depend upon series may therefore have a certain interest and usefulness.

Let us begin by making the following assumptions:

(a) $e^{i\theta}$ is a complex number, i.e.,

$$e^{i\theta} = c(\theta) + is(\theta), \tag{1}$$

where $c(\theta)$ and $s(\theta)$ are functions which are to be determined;

(b) $e^{i\theta}$ obeys the usual laws of exponents;

(c) $e^{i\theta}$ has $e^{-i\theta}$ for its conjugate complex number, i.e.,

$$e^{-i\theta} = c(\theta) - is(\theta); \tag{2}$$

(d) $e^{i\theta}$ obeys the usual law for differentiation, i.e.,

$$\frac{de^{i\theta}}{d\theta} = ie^{i\theta}; \tag{3}$$

(e) $e^{i\theta}$ is a continuous function of θ for at least one value of θ, e.g.,

$$\lim_{\theta\to 0} e^{i\theta} = e^0 = 1. \tag{4}$$

From these assumptions it will be shown that the usual definition of $e^{i\theta}$ follows.

From (1), (2), and (b) we have first that

$$c^2(\theta) + s^2(\theta) = 1. \tag{5}$$

Next by (b) we have $e^{i\theta}\cdot e^{i\phi} = e^{i(\theta+\phi)}$, from which it follows that

$$[c(\theta)c(\phi) - s(\theta)s(\phi)] + i[s(\theta)c(\phi) + c(\theta)s(\phi)] = c(\theta + \phi) + is(\theta + \phi).$$

Then

$$\begin{aligned} c(\theta + \phi) &= c(\theta)c(\phi) - s(\theta)s(\phi), \\ s(\theta + \phi) &= s(\theta)c(\phi) + c(\theta)s(\phi). \end{aligned} \tag{6}$$

An obvious solution of the functional equations (6) subject to the restriction (5) is $c(\theta) = \cos k\theta$, $s(\theta) = \sin k\theta$. From (d) it follows that $k = 1$. Finally, by means of (e) we can show that this solution is unique.*

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent to the Review Editor, American Mathematical Monthly, Low Memorial Library, Columbia University, New York, N. Y.

REVIEWS

Challenging Problems in American Schools of Education. By David Eugene Smith, The 1934–35 Sachs Lectures. Bureau of Publications, Teachers College, Columbia University, 1935. 48 pages. 55 cents.

In these two lectures, delivered at Teachers College, Columbia, in February

* Osgood, *Lehrbuch der Funktionentheorie*, 4th edition (1923), Teubner, Berlin, pp. 582–586.

1935 under the provisions of the Julius and Rosa Sachs Endowment Fund, Dr. Smith arraigns the teachers colleges in no uncertain terms for foisting upon the schools of this country a vast mass of beings labeled "teacher" whose training in pedagogy has been uncoordinated and often ridiculous, whose knowledge of the subjects they expect to teach is quite inadequate, and whose general cultural background is wofully meagre. By way of contrast he depicts the scholarly interests and rich cultural background of that renowned teacher, Julius Sachs, and presents a plan to the teachers colleges whereby, if they will, they can shape their future product after the pattern afforded by Dr. Sachs himself.

Dr. Smith recalls recent efforts in certain quarters to make the training of educational administrators and of college teachers of education more substantial. He would make at least equal provision for prospective teachers in secondary schools. Though he holds up to ridicule a certain teachers college which offers twenty-two courses in the teaching of mathematics alone, he does not decry all training in Education. Though he praises the scholarly and professional spirit of the Ecole Normale Supérieure at Paris, with its great emphasis on command of subject and its slight attention to Education, he would not have our teachers colleges neglect either subject matter or Education, but emphasize both, offering substantial and dignified instruction in each, and adding a third feature, the broad cultural formation of the prospective teacher. Dr. Smith suggests that this last feature can be provided, in part at least, by master teachers giving general courses of broad cultural import, one of which, for example, might be entitled "The Nature of Man and the Universe" and might include such topics as "sociology, economics, conversation, the best music and other fine arts, deportment, and the cultivation of good taste." In this program he would secure proper balance between subject matter and Education by entrusting the administration of the program neither to the faculty of Arts and Sciences nor to the faculty of Education, but to both jointly.

Dr. Smith's insistence on the importance of a sure grasp of subject matter, of substantial and compact instruction in Education, and of broad cultural foundation serves to reinforce the conclusions reached by the recent committee of this Association appointed under the chairmanship of Professor Moulton to consider the proper relation between knowledge of Mathematics and knowledge of Education in the formation of the teacher of secondary mathematics. In more concrete form still, the general spirit of Dr. Smith's program and most of the details had already taken shape in the program initiated by President Conant of Harvard University at the very time these lectures were being delivered, whereby a new degree, the Master of Arts in Teaching, has been established, to be awarded jointly by the Faculties of Arts and Sciences and of Education on the basis of proved competence tested by a general examination in subject matter, by a general examination in Education, and by satisfaction of certain requirements of apprenticeship. Though Dr. Smith must have been unaware of these developments as he was speaking, he clearly implies that he would be pleased if a similar, or even somewhat more severe, program should

be adopted by Teachers College, Columbia, where both he and Dr. Sachs taught with distinction for so many years. We agree that this would be of tremendous effect in strengthening the tone of mathematical instruction throughout the country; we fervently hope that it may come to pass.

RALPH BEATLEY

Graphical Solutions. By C. O. Mackey, New York, John Wiley and Sons, Inc., 1936. viii+130 pages, with chart. $2.50.

This is an engineer's textbook, designed by its author as a short course in graphical methods for third or fourth year students. The treatment, however, is elementary, and assumes little mathematical knowledge on the part of the student, so that much of the material might well be offered in a freshman course. While emphasis is placed throughout on the value of graphical methods to engineers, the reviewer would like to suggest that these processes should also be considered in general mathematical courses.

The chapter headings indicate the contents and scope of the book: Stationary Adjacent Scales, Sliding Scales, Network or Intersection Charts, Alignment Charts, Empirical Equations, Non-Periodic Curves. The treatment is less detailed than that of Lipka's *Graphical and Mechanical Computation*, but the various graphical methods studied are copiously illustrated by examples drawn from engineering practice, while numerous exercises are also included.

It would have been helpful to the student if the author had included a discussion of the relative merits of the different methods possible in graphical solutions. When the same problem is solved by more than one method some guidance is needed to enable the student to decide which should be preferred. At the same time, any one who becomes familiar with the methods here outlined need never be at a loss in applying graphical analysis to given data.

The numerous graphs are very well reproduced and the typography is excellent. The only possible criticism of the format of the book is that cross-references would be more easily located if the page headings had included section numbers.

M. C. GRAY

Plane and Spherical Trigonometry. By A. L. Nelson and K. W. Folley. New York, Harper and Brothers, 1936. xii+186+92 pages. $2.20 (with tables).

A feature of this text which distinguishes it from many of its very numerous predecessors is a rather detailed article (three and one-half pages) treating of the number of digits to be retained in various computations. Of lesser importance but noteworthy perhaps is the collection of figures—107 in all. These diagrams, judiciously placed, are unusually impressive because of the extreme thickness of certain of the lines.

In addition to these points of merit, we noted one or two others, which although shared with several texts are worthy of commendation here. Superior to that given in the older books is the treatment of infinity and the behaviour

of the functions about the quadrantal angles. Furthermore, in our opinion, the sequence of the topics is good. The reviewer believes that, regardless of tradition, important mathematical ideas should be presented to the student as soon as feasible, preferably in their final form. Thus we feel that the authors are correct in defining the trigonometric functions by means of a coordinate system at the outset and in developing the analytical part of the subject before treating in detail the solution of triangles.

As a matter of act, our few disappointments with this work derive, for the most part, from the fact that this principle is not carried further. For example, in the section devoted to complex numbers there occurs the definition "an imaginary number is the indicated square root of a negative number." Perhaps a more "advanced" introduction to the new number system would be more comprehensible. Further, the reviewer would have welcomed the addition to the material on vectors of a section treating of the "dot" and "cross" products. The heavy hand of tradition is against it, but nevertheless the subject matter involved can be taught to freshmen; and such an innovation would open up new vistas to the student, contribute to his mathematical foundation for physics, and make possible simpler proofs of the fundamental identities. Finally, because of the regrettable reduction in the amount of mathematics given in the high schools it is quite likely that the chapter on spherical trigonometry will be found to be inadequate.

H. V. Craig

MATHEMATICS CLUBS

Edited by F. W. Owens and Helen B. Owens, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 463 East Foster Ave., State College, Pa.

Club Reports
1935–36

Junior Mathematics Club, University of Michigan

President, P. Rims; Vice-President, H. Bendler; Secretary, Claude Shannon; Faculty Advisers, Professor N. H. Anning and Dr. Elder. A feature of all meetings was a list of mathematical problems placed on the board and discussed informally after the regular program. A demonstration of mathematical instruments in the department's collection made an interesting program. Other meetings were given over to discussion of the following subjects: "Heisel's heretic hunt"; "The equations of regular polygons"; "Figures which are congruent to a part of themselves"; "The coloring of maps"; "The nature of mathematics."

Pi Mu Epsilon of the University of Kentucky

Director, Professor D. E. Small; Vice Director, L. P. Hutchison; Secretary, K. L. Palmquist; Treasurer, W. H. Pell. Two banquets which featured the initiation of new members and eight regular meetings were held with discussion of the following topics: "Implicit function theorem"; "Geometry and mechanics of a conical helix"; "Infinitesimal transformations"; "Convex regions"; "A group"; "Descriptive geometry"; "An infinite series"; "Topology."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 245. *Proposed by J. Rosenbaum, Hartford Federal College.*

Prove that the sum of the squares of the edges of an isosceles tetrahedron is equal to four times the square of the diameter of the circumscribed sphere. (A tetrahedron is isosceles when its faces are congruent.)

E 246. *Proposed by V. Thébault, Le Mans, France.*

In what system of enumeration can a four-place number of the form *abab* be the square of a two-place number of the form *ba*? Ascertain if the solution is unique.

E 247. *Proposed by N. A. Court, University of Oklahoma.*

Show that the six planes perpendicular to the six edges of a tetrahedron, and passing through the mid-points of the projections of the respective edges upon a given plane, have a point in common.

E 248. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Show that for any plane triangle,

$$\sum \frac{a^2}{(r_a - r)r_b r_c} = \frac{2}{r},$$

where r is the radius of the inscribed circle, r_a the radius of the escribed circle tangent to the side a between B and C, etc.

E 249. *Proposed by E. R. Ott, University of Buffalo.*

A circular track has a circumference of sixty miles. Three travelers, A, B and C, start in the same direction from the same point, and travel around the track continuously at the rates of 26, 10 and 2 miles per day, respectively. When are they next all three abreast?

E 250. *Proposed by A. Gloden, l'Athênée de Luxembourg.*

Find a seven-place number whose exact cube root equals the millions digit plus the number formed by the last three digits, minus the number formed by the remaining three digits.

E 251. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

After an exact long division had been made (in the decimal system), some of the digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, were replaced by the letter s wherever they occurred in the work, while all the other digits in the work were replaced by the letter t. This was the result:

$$
\begin{array}{r|cccccc|l}
s\ t\ s\,) & t & t & t & s & t & t & (\,s\ t\ s\ t \\
 & t & t & t & & & & \\
\cline{2-4}
 & s & s & t & s & & & \\
 & t & s & t & t & & & \\
\cline{2-5}
 & & t & s & t & t & & \\
 & & & t & t & s & & \\
\cline{3-6}
 & & & t & t & s & t & \\
 & & & t & t & s & t & \\
\cline{4-7}
\end{array}
$$

Reconstruct the original long division and show that the solution is unique.

SOLUTIONS

E 205 [1936, 241]. *Proposed by Emma Lehmer, Lehigh University.*

Prove that

$$\frac{1}{k+1}\sum_{t=1}^{n}(-1)^{t+1}\frac{{}_nC_t}{{}_{t+k}C_{k+1}} = H_{n+k} - H_k$$

where H_n is the sum of the reciprocals of the first n positive integers.

Solution by Frank Ayres, Jr., Dickinson College.

This problem requires that k be a positive integer. However, a problem having the above as a special case, can be stated as follows:

If n is a positive integer and $k \geqq 0$, prove that

$$\frac{1}{k+1}\sum_{t=1}^{n}(-1)^{t+1}\frac{{}_nB_t}{{}_{t+k}B_{t-1}} = G_{n+k} - G_k$$

where

$${}_mB_s = \frac{m(m-1)\cdots(m-s+1)}{\underline{|s}} \equiv \frac{{}_mR_s}{\underline{|s}}, \qquad G_m = \sum_{i=0}^{M}\frac{1}{m-i},$$

and M is an integer such that $M < m \leqq M+1$.

Having verified the formula for $n=1$, the proof by induction is completed by showing that

$$\text{(A)}\qquad \frac{1}{k+1}\sum_{t=1}^{n+1}(-1)^{t+1}\frac{{}_{n+1}B_t}{{}_{t+k}B_{t-1}} - \frac{1}{k+1}\sum_{t=1}^{n}(-1)^{t+1}\frac{{}_nB_t}{{}_{t+k}B_{t-1}} = \frac{1}{n+k+1}.$$

The left-hand member of (A) may be written

$$\frac{1}{k+1}\left\{\sum_{t=1}^{n}(-1)^{t+1}\frac{{}_{n+1}B_t-{}_nB_t}{{}_{t+k}B_{t-1}}+(-1)^n\frac{1}{{}_{k+n+1}B_n}\right\}$$

$$=\frac{1}{k+1}\sum_{t=1}^{n+1}(-1)^{t+1}\frac{{}_nB_{t-1}}{{}_{t+k}B_{t-1}}=\frac{1}{k+1}\sum_{s=0}^{n}(-1)^s\frac{{}_nR_s}{{}_{s+k+1}R_s}$$

$$=\frac{1}{k+1}\left\{\sum_{s=0}^{n-2}(-1)^s\frac{{}_nR_s}{{}_{s+k+1}R_s}+(-1)^{n-1}\frac{{}_nR_{n-1}}{{}_{n+k}R_{n-1}}+(-1)^n\frac{{}_nR_n}{{}_{n+k+1}R_n}\right\}$$

$$=\frac{1}{k+1}\left\{\sum_{s=0}^{n-2}(-1)^s\frac{{}_nR_s}{{}_{s+k+1}R_s}+(-1)^{n-1}\frac{{}_nR_{n-1}}{{}_{n+k-1}R_{n-2}}\cdot\frac{1}{n+k+1}\right\}$$

$$=\cdots=\frac{1}{n+k+1},$$

since, for $0<j<n$

$$\frac{{}_nR_{n-j}}{{}_{n+k+1-j}R_{n-j}}-\frac{{}_nR_{n-j+1}}{{}_{n+k+1-j}R_{n-j}}\cdot\frac{1}{n+k+1}=\frac{{}_nR_{n-j}}{{}_{n+k-j}R_{n-j-1}}\cdot\frac{1}{n+k+1},$$

and for $j=n$, the left member of the last equation above reduces to $1/(n+k+1)$.

Also solved by the proposer.

E 206 [1936, 241]. *Proposed by L. S. Johnston, University of Detroit.*

Show that for any plane quadrilateral (convex or otherwise) the intersection of the diagonals, the intersection of the medians, and the centroid are collinear, and that the distance from the intersection of the diagonals to the intersection of the medians is three times the distance from the intersection of the medians to the centroid. (A median of a quadrilateral is here understood to be a line joining the midpoints of opposite sides.)

Solution by W. B. Clarke, San Jose, California.

Let the corners of the quadrilateral be lettered in order, A, B, C and D. Let W, X, Y and Z be the mid-points of AB, BC, CD and DA respectively. Let P, Q, R and S be the respective centroids of triangles BCD, CDA, DAB and ABC. The centroid, G, of $ABCD$, must be the intersection of PR with QS. Let AC meet BD at K, and WY meet XZ at M.

Since P and Q are the respective trisection points of YB and YA, it follows that PQ is parallel to AB, and equals one-third of it. Similarly QR, RS and SP are respectively parallel to, and equal to one-third of, BC, CD and DA. By like reasoning, $PR\|WX\|AC$, and $QS\|XY\|BD$. Thus the two quadrilaterals, $PQRS$ and $ABCD$ are similar, and their corresponding sides and diagonals are parallel, and in the ratio of one to three.

WY bisects PQ and RS, since it is a median in triangles ABY and CDW, and hence in the similar triangles, PQY and RSW. Therefore M is the intersection of the medians in both quadrilaterals. Then MG, running from the median

points to the intersection of the diagonals in quadrilateral $PQRS$, must be parallel to, and equal to one-third of, MK, which runs from the median point to the intersection of the diagonals in quadrilateral $ABCD$. Consequently, K, M and G are collinear, with $KM = 3MG$, as was to be proved.

Also solved by K. W. Crain, L. M. Kelly, Samuel Kramer, C. E. Springer, C. W. Trigg, Simon Vatriquant and the proposer.

E 207 [1936, 242]. *Proposed by V. Thébault, Le Mans, France.*

Find a number of six digits which is a perfect square both in the decimal system and in the system of enumeration based on five. Show that the solution is unique.

Solution by Wm. Douglas, Courtenay, B. C.

The number we seek must be in the scale of five. Hence no digit can be greater than 4.

Inspection of a table of squares in the decimal system reveals the fact that there are but nineteen such six-digit squares.

The sum of the digits must reduce to zero or one by casting out fours. This narrows down the field of possible numbers to eleven, ten of which are readily eliminated by similar simple tests (such as quadratic residues modulo 24 and modulo 26), leaving 232324 as the only six-digit number which is a perfect square both in the system of enumeration based on five and in the decimal system.

Also solved by W. E. Buker, Mary L. Constable, J. E. Trevor, C. W. Trigg, Simon Vatriquant and the proposer.

E 208 [1936, 242]. *Proposed by C. A. Whittemore, New York City.*

How may eight married couples play a seven-round bridge tournament, if each man plays one round as partner with each lady except his wife, and everybody plays against everyone else except one's own spouse? How many solutions exist? Can this be generalized for $4n$ couples playing $4n-1$ rounds? Does a solution exist for $4n+2$ couples playing $4n+1$ rounds?

Editorial Note. Since no solutions have been received for this problem, it is suggested that it be brought to the attention of those who generally work only on the problems in the advanced section.

E 209 [1936, 242]. *Proposed by J. M. Feld, New York City.*

Show that the equation $a \cos \theta + b \sin \theta = c$, $0 < c$, can be solved graphically by the following method: With center having the rectanglar Cartesian coordinates $(a/2, b/2)$, describe the circle K passing through the origin O. With O as center and radius c, strike arcs cutting K at A and B. Then the angles XOA and XOB are the desired values of θ.

Solution by L. M. Kelly, Lawrence, Massachusetts.

Let D be the foot of the perpendicular from A on OX. From $P(a/2, b/2)$

drop perpendiculars to OA and OX at R and S respectively. Let the perpendicular from R to SP meet it at Q.

Then $DA/OD=QR/QP=(OD/2-a/2)/(b/2-DA/2)=(OD-a)/(b-DA)$. Consequently, $bDA-DA^2=OD^2-aOD$, or $aOD+bDA=OD^2+DA^2=c^2$. Thus $a(OD/c)+b(DA/c)=c$, or $a\cos XOA+b\sin XOA=c$, and $\angle XOA=\theta$. Similarly it is readily shown that XOB also satisfies the given equation.

Also solved by W. E. Buker, A. R. Jerbert, Samuel Kramer, Elmer Latshaw, M. J. Turner and Simon Vatriquant.

E 210 [1936, 242]. *Proposed by S. A. Anderson, New York City.*

In the well-known problem of the two ladders leaning across the street from opposite sides, it is customary to give the lengths of the ladders, a and b, and the height c of their intersection above the street. Assuming the touching sides of the two ladders to be in a vertical plane straight across the street, with each end of each ladder touching one of the two vertical walls bounding the street, it is usually required to find the width d of the street. In the present version of the problem however, it is required to find the set of positive integer values which may be consistently assigned to a, b, c and d, such that $b<a$, and such that no such set exists with a smaller a.

Solution by Simon Vatriquant, Brussels, Belgium.

If we denote by x and y respectively the heights of the tops of the ladders a and b above the level of the street, the consideration of two pairs of similar right triangles gives the equation

$$cx+cy=xy, \tag{1}$$

with the conditions that

$$a^2-d^2=x^2 \tag{2}$$

$$b^2-d^2=y^2. \tag{3}$$

Taking y in terms of x from (1) and squaring, we obtain

$$b^2-d^2=y^2=c^2x^2(x^2+2cx+c^2)/(x^2-c^2)^2. \tag{4}$$

This shows that if b and d are integers, y^2 is an integer, and x is rational. By reason of symmetry, x^2 is also an integer, and y is rational. Consequently, x and y are both integers.

Since every set of four integer values for a, b, c and d must satisfy equations (1), (2) and (3), with x and y integers, we may start with two Pythagorean triangles, multiply each by a suitable integer so that they may have a common base and satisfy equations (2) and (3), and then, since c is a rational function of the integers x and y, we can find another integer multiplier whose use will result in a complete solution. Since from (1), $c=xy/(x+y)$, our x and y must have their product divisible by their sum.

For example, the two simplest Pythagorean triangles are the 3, 4, 5 and the 5, 12, 13 ones. If we try $d=12$, we get either 15, 13, 9, 5 or 20, 13, 16, 5 for a, b, x,

y respectively. In neither of these cases is xy divisible by $(x+y)$, but in the first we may multiply through by this sum, 14, getting 210, 182, 45, 168, 126 and 70 for a, b, c, d, x and y respectively. This is an integral solution for our problem, but not necessarily the one with the smallest a.

As there are a finite number of Pythagorean triangles with hypotenuses less than 210, we may consider all possible such pairs and proceed as in the preceding paragraph. This task is not very difficult because most cases indicate that they give larger a's. The solution with the smallest a comes from the 3, 4, 5 triangle properly combined with the 20, 21, 29 triangle, first using 21 as common base, and thus arriving at the set 105, 87, 35, 63, 84 and 60 for a, b, c, d, x and y.

Also solved by S. A. Anderson, Michael Goldberg and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTIONS

3807. *Proposed by J. R. Musselman, Western Reserve University.*

Any four straight lines are touched by a parabola. It is well known that the three mid-points of the diagonals of the four lines are collinear, on l. Show that, if the points of tangency of the parabola with the four lines are concyclic, the line l is the axis of the parabola.

3808. *Proposed by N. A. Court, University of Oklahoma.*

A, B, C are three given points on three given spheres (P), (Q), (R). Find a point D on the radical axis u of the three given spheres such that, if A', B', C' are the points of intersection of the lines DA, DB, DC with the spheres (P), (Q), (R), respectively, the plane $A'B'C'$ shall be parallel to the plane determined by the centers P, Q, R of the three spheres.

Note. A similar problem relative to two circles was discussed in the *Nouvelles Annales de Mathêmatiques*, 1846, p. 260, and 1848, p. 231.

3809. *Proposed by Don Wallace, Charlottesville, Va.*

Let P be that point for which the product of its distances to the vertices of a triangle is an extreme. Then P is the symmedian point of its pedal triangle with respect to the first triangle.

3810. *Proposed by Oystein Ore, Yale University.*

One may define a new "multiplication" in the system of all positive real numbers by putting

$$[a, b] = a^b.$$

Determine all positive *rational* numbers for which this multiplication is

(I) *Commutative* $[a, b]=[b, a]$
(II) *Associative* $[a, [b, c]]=[[a, b], c]$
(III) *Right-hand* or *left-hand distributive*

$$[(a+b), c] = [a, c] + [b, c], \; [c, (a+b)] = [c, a] + [c, b].$$

SOLUTIONS

3723 [1935, 115]. *Proposed by William Hoover, Columbus, Ohio.*

The envelope of all paraboloids with respect to which a given tetrahedron is self-conjugate is the set of seven planes each of which bisects three edges of the tetrahedron.

Solution by the Proposer.

The equation of a quadric surface S in homogeneous coordinates α, β, γ, δ, for which the tetrahedron of reference is self-conjugate, has the form

$$q\alpha^2 + r\beta^2 + s\gamma^2 + t\delta^2 = 0. \tag{1}$$

If S is a paraboloid then

$$q^{-1} + r^{-1} + s^{-1} + t^{-1} = 0; \tag{2}$$

see Smith's *Solid Geometry*, Ex. 1, p. 175. Then S is tangent to the eight planes

$$\alpha \pm \beta \pm \gamma \pm \delta = 0; \tag{3}$$

and these planes, excluding the one for which all the signs are positive, bisect the edges of the tetrahedron in sets of three: see in the same text Ex. 7, p. 168. Hence (3) gives the required envelope.

Editorial Note. That the equation of S must have the form (1) if the tetrahedron of reference is self-conjugate is obvious. The conditions for the metric properties which follow in the proposer's solution could not be expressed in general "homogeneous coordinates." The proofs of the conditions as given, however, are simple if the proper kind of metric homogeneous coordinates are specified. The equation of the tangent plane at $P_0(\alpha_0, \beta_0, \gamma_0, \delta_0)$ is

$$q\alpha_0\alpha + r\beta_0\beta + s\gamma_0\gamma + t\delta_0\delta = 0. \tag{4}$$

Since S is a paraboloid it is tangent to the plane at infinity. If P_0 is the point of tangency at infinity, (4) must be the equation of the plane at infinity. The equation of the plane at infinity depends upon the nature of the coordinates used. If $ABCD$ is the given tetrahedron, a convenient system of coordinates may be obtained by taking α, β, γ, δ as proportional to the volumes of the tetrahedrons $P-BCD$, $P-ACD$, $P-ABD$, $P-ABC$. It is then easily shown that in this system P is the centroid of the masses α, β, γ, δ placed at A, B, C, D, respec-

tively. If P is inside the tetrahedron of reference the coordinates are considered as having the same sign. Such coordinates are sometimes called barycentric coordinates. Thus the middle point of AB has the coordinates (1, 1, 0, 0), while the point at infinity on AB is $(1, -1, 0, 0)$. There are six middle points of the edges, whose coordinates are obtained by taking in all possible orders two positive units and two zeros. The corresponding points at infinity are obtained by making one unit negative in each set of coordinates. It is now obvious that the six points at infinity satisfy the equation

$$\alpha + \beta + \gamma + \delta = 0; \tag{5}$$

and this is the equation of the plane at infinity. We also obtain this result by considering α', β', γ', δ' as equal to the volumes mentioned above with the proper signs. Then for any point P

$$\alpha' + \beta' + \gamma' + \delta' = V = \text{volume } ABCD. \tag{6}$$

Hence, if P is at infinity, (5) is true for its coordinates, which are taken as finite and proportional to α', β', γ', δ'. In this case at least one of the latter is infinite. The planes whose equations are

$$\begin{aligned} -\alpha + \beta + \gamma + \delta &= 0,\\ \alpha - \beta + \gamma + \delta &= 0,\\ \alpha + \beta - \gamma + \delta &= 0,\\ \alpha + \beta + \gamma - \delta &= 0, \end{aligned} \tag{7}$$

obviously, pass through the mid-points of the three edges meeting, respectively, in A, B, C, D. Now consider the equations with two negative signs

$$\begin{aligned} \alpha + \beta - \gamma - \delta &= 0,\\ \alpha - \beta + \gamma - \delta &= 0,\\ \alpha - \beta - \gamma + \delta &= 0. \end{aligned} \tag{8}$$

The first gives a plane through (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), and these are mid-points of AC, AD, BC, BD. The other two are interpreted similarly. In order for (4) to be the plane at infinity we must have $q\alpha_0 = r\beta_0 = s\gamma_0 = t\delta_0$; and in order for $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ to satisfy (1) the equation (2) must be true. The point of tangency of S with the plane at infinity is then $(q^{-1}, r^{-1}, s^{-1}, t^{-1})$. Moreover, $(q^{-1}, \epsilon_1 s^{-1}, \epsilon_2 s^{-1}, \epsilon_3 t^{-1})$, $\epsilon_i = \pm 1$, , are eight points on S taking account of (2). Inserting these sets of coordinates in (4) we obtain the equations of eight tangent planes which separate into the sets (5), (7), (8). The same results are easily obtained also by the usual calculus methods for finding the envelope.

The greater part of the above is almost obvious without the use of coordinates. For, let EF be the direction of the point of contact of S with the plane at infinity. Let the straight line through A parallel to EF cut S in P and BCD in M.

Then $AP=PM$, since BCD is in the polar plane of A. Also the tangent plane at P is parallel to BCD. Hence the tangent plane at P bisects the edges of $ABCD$ which meet in A. The same kind of result follows for the vertices B, C, D.

The rest is not quite so obvious. Denote the mid-point of AB by M_{ab}, and the other mid-points in a similar manner. The plane $M_{ab}M_{ac}M_{ad}$ is tangent to S at P by the above proof. Let BP cut S again at Q and ACD in R. Since ACD is in the polar plane of B, the four points B, P, R, Q form an harmonic set. Hence the four planes through $M_{ac}M_{ad}$ as an axis and these four points cut the plane of BCD in four straight lines of an harmonic pencil one ray of which is along CD and the other three are parallel to CD. Since plane $M_{ac}M_{ad}P$ is parallel to BCD, the plane $M_{ac}M_{ad}Q$ bisects the edges BC and BD. The two straight lines $M_{ac}M_{ad}$ and BP are conjugate with respect to S. For the polar plane of any point on $M_{ac}M_{ad}$ must pass through B since ACD is the polar plane of B, and it must pass through P since $M_{ac}M_{ad}P$ is the polar plane of P. Since Q is on the surface and BQ is conjugate to $M_{ac}M_{ad}$, the plane $M_{ac}M_{ad}Q$ is the tangent plane at Q. Thus the plane $M_{ac}M_{ad}M_{bc}M_{bd}$ is tangent to S at Q. This gives three more tangent planes and the proof is complete. Thus, if the direction EF is given, the above gives the construction for the seven finite tangent planes and their points of contact with S.

3725 [1935, 115]. *Proposed by G. W. Petrie III, Pittsburgh, Pa.*

A storekeeper is interested in purchasing a balance and set of weights which will enable him to measure any weight in ounces up to and including 205 lbs., using weights in both pans when necessary. One company agrees to sell him such an outfit for \$100. Another company proposes to charge \$10 for the balance, scale pans, etc., and \$.90 for each weight plus one cent an ounce for the total amount of metal involved in making the weights. Which company should receive the storekeeper's order?

Solution by M. T. Bird, Southwestern College, Winfield, Kans.

The essential features of this problem may be stated in the more general terms: A number n of weights of total combined weight w are to be selected in such a way that all quantities weighing from 1 to m can be weighed in a balance. Furthermore, this selection is to be such that w and n are minima.

The solution is found as follows: Let n be the integer such that

$$(3^{n-1}-1)/2 < m \leqq (3^n - 1)/2.$$

Consider the n weights $1, 3, 3^2, \cdots, 3^{n-2}, p$, where p is the integer

$$p = m - (3^{n-1} - 1)/2.$$

The combined total of this set of n weights is seen to be equal to m. Clearly this set has a minimum combined total weight.

Furthermore, it weighs all quantities weighing from 1 to m. For it is readily established by induction that the $n-1$ weights $1, 3, 3^2, \cdots, 3^{n-2}$ weigh all

quantities from 1 to $(3^{n-1}-1)/2$. Then the n weights $1, 3, 3^2, \cdots, 3^{n-2}, p$ weigh all quantities from

$$p - (3^{n-1} - 1)/2 \quad \text{to} \quad p + (3^{n-1} - 1)/2$$

as well as the quantities from 1 to $(3^{n-1}-1)/2$. As a consequence of the inequality

$$(3^{n-1} + 1)/2 \geqq p - (3^{n-1} - 1)/2$$

we see that the n weights selected weigh all quantities from 1 to m.

It remains to show that fewer than n weights cannot weigh all the quantities in the desired range. Each weight may be put in any one of 3 places, i.e., in either of the balance pans or on the "shelf." Then k weights may be arranged in the 3 places in 3^k different ways. Now one of these arrangements does not weigh a quantity because all of the weights are on the "shelf." For each of the other arrangements there is another which weighs the same quantity. Consequently, k weights can weigh at most $(3^k-1)/2$ different quantities. But the first $n-1$ weights which we have selected weigh $(3^{n-1}-1)/2$ different quantities. Hence at least one more weight is necessary to weigh m quantities and we have selected the fewest possible weights.

For the particular problem assigned this selection of weights costs the merchant \$50.00 according to the one contract. The problem is not worded in such a way that the best choice of weights is necessary in order to see which firm should have the contract because if weights are placed only in one pan the cost need not exceed \$100.00. If 1, 2, 4, and 8 ounce weights together with 1, 2, 2, 5, 10, 20, 20, 50, 100, and 200 lb. weights are used in a single pan the outfit would cost only \$88.35 according to the contract.

Solved also by E. L. Harp, Jr., A. J. Lewis, F. A. Lewis, J. K. Peterson, W. T. Short, E. P. Starke and the proposer.

Editorial Note. Starke's solution concluded with a remark similar to the one at the end of the above solution. Starke and A. J. Lewis gave a reference regarding weights to Carmichael's *Theory of Numbers*, pp. 22–24. F. A. Lewis referred to Sanford's *History of Mathematics*, p. 225. There is also a brief account of this matter in *Mathematical Recreations and Problems*, W. W. Rouse Ball, 3rd. ed. pp. 28–30.

3727 [1935, 177]. *Proposed by J. R. Musselman, Western Reserve University.*

It is well known that, if P is a point on the circle O circumscribing a quadrilateral $ABCD$, where AB is parallel to CD, the feet of the perpendiculars from P to AC, AD, BC and BD are concyclic. Show that the locus of the center Q of this circle is that diameter of O which is perpendicular to AB and CD; in fact, for a given point P to construct the point Q associated with it, one merely drops the perpendicular from P upon the aformentioned diameter of O.

Solution by J. Rosenbaum, Hartford Federal College.

Let the feet of the perpendiculars from P to AC, AD, BC, and BD be E, F, G, H respectively. Also, let PQ cut the circle O again at P', and let the feet of the prependiculars from P' to the four lines be E', F', G', H', where each primed point is symmetrical to its corresponding blank point, with respect to the diameter OQ as an axis. Now from the isosceles trapezoid $EH'P'P$, since Q is the midpoint of the leg PP', Q lies on the perpendicular bisector of EH', and hence, $QE=QH'$. From symmetry, $QH=QH'$. Hence $QE=QH$, and hence Q lies on the perpendicular bisector of EH.

Similarly, Q is proved to lie on the perpendicular bisector of FG, and since it is known that E, F, G, H lie on a circle, Q is the center of that circle.

Note. The reasoning above breaks down when the two perpendicular bisectors form the same line, but this will take place only when P is an extremity of the diameter mentioned above, for which case the statement of the problem is easily proved.

Solved also by M. G. Boyce, J. W. Clawson, R. A. Johnson, Leon Recht, F. Underwood, S. Vatriquant, and the proposer.

3728 [1935, 177]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the principal axes of an ellipse and the angle between two conjugate diameters, by a ruler-compass construction locate the diameters.

I. *Solution by G. E. Raynor, Lehigh University.*

Draw the given principal axes in their proper relative positions as major and minor axes of the ellipse. Let them be $A'A$ and $B'B$ respectively and O the mid-point of each. With $B'B$ as chord construct a circle K such that $B'B$ subtends the given angle, or its supplement, at each point of K. In addition to B and B', K will meet the ellipse in two points E and E' which we now proceed to find. Let the lines AB and $A'B'$ meet K at C and C' respectively. The line CC' will meet $A'A$ at a point D which lies on the common chord EE'. To see this, consider the two ordered hexagons $BAA'B'EE'$ and $BCC'B'EE'$, the first inscribed in the ellipse and the second in the circle K. The Pascal line of the first hexagon is determined by the intersections of the pairs of opposite sides $(BA)(B'E)$ and $(A'B')(E'B)$ while that of the second is determined by the intersections of the pairs $(BC)(B'E)$ and $(C'B')(E'B)$. But since line BA is the same as BC and $A'B'$ the same as $C'B'$ the two Pascal lines coincide and from the first hexagon are seen to meet EE' at its intersection with AA' and from the second to meet EE' at its intersection with CC'. Thus D, the intersection of AA' and CC', lies on EE'. Hence from the symmetry of the ellipse and K with respect to $A'A$ the points E and E' are determined as the intersections of the perpendicular to $A'A$ at D with K.

If D falls outside the segment $A'A$ points E' and E will be imaginary and our problem has no real solution. In this case the given angle, or its supplement,

is less than the minimum angle between the pairs of conjugate diameters of the given ellipse. If E and E' are real, join each of them to B and B'. Let H and H' be the mid-points of the segments EB and EB' respectively. Then the lines l and l' joining O to H and H' will determine the directions of a pair of conjugate diameters having the given angle between them. That they contain the required angle is obvious and that they are conjugate follows because each of them bisects a chord of the ellipse parallel to the other. Similarly E' will determine another pair of conjugate diameters satisfying the given conditions. If E and E' coincide these two pairs coalesce and the given angle will be the minimum angle mentioned above. Also E and E' may coincide with B and B', which will happen if the given angle is a right angle, and our construction will give the principal axes, as it should.

To obtain the *magnitudes* of the conjugate diameters draw through B the line t parallel to $A'A$; t is, of course, a tangent to the ellipse. Let l and l' meet t at G and G' respectively. Since two conjugate diameters and the line at infinity form a self-conjugate triangle with respect to the ellipse, it follows that the polar of G will be parallel to l'. Also, since G lies on the tangent t this polar will pass through B and is therefore the line BE which is met by l in H. G and H are thus conjugate points in the involution determined on l by the ellipse, O is the center of the involution and the points M and M' in which l meets the ellipse are the double points. Hence $OM=OM'$ may be determined as the mean proportional between OH and OG. Similarly the points of intersection of l' with the ellipse will be determined by finding a mean proportional between OH' and OG'.

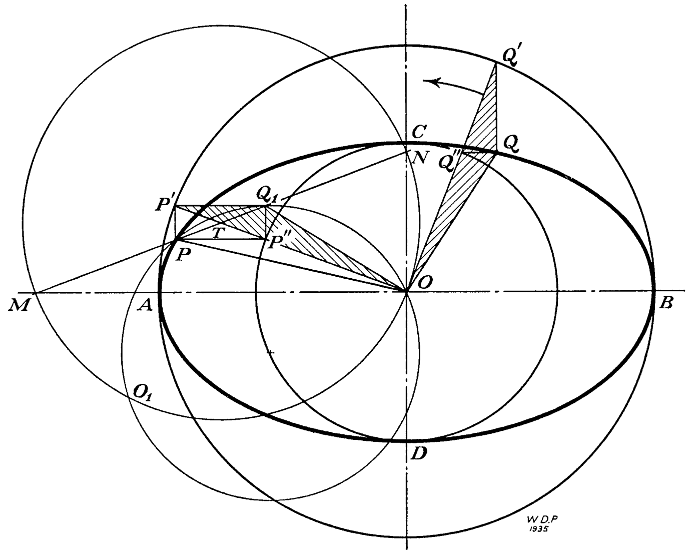
Since the foci of an ellipse are very simply constructed when the principal axes are known, and then as many points of the ellipse as desired may be found, our problem may be looked upon as a simplification of the following

Problem. Given five points on a conic, to construct a pair of conjugate diameters which shall make with each other a given angle.

This problem is quite old and is ascribed by Cremona to De la Hire who published it in his *Sectiones conicae* in 1685. See Cremona, *Elements of Projective Geometry*, second edition, p. 292, art. 402.

II. *Solution by the Proposer.*

In the accompanying figure let AB and CD be the given axes, OP and OQ the required conjugate semi-diameters, and angle POQ the given angle. On AB and CD as diameters describe the auxiliary circles, and through P and Q draw perpendiculars to AB. Join the points, P' and Q', in which these perpendiculars cut the major auxiliary circle, to center O. These radii, which are, of course, perpendicular to each other, cut the minor auxiliary circle in points P'' and Q''. Hold the figure $OQ''Q'Q$ rigid and revolve it counter-clockwise about point O through an angle of 90°. Thus Q' and Q'' will be brought into coincidence with P' and P'' respectively, and Q will take a position Q_1 such that angle Q_1OQ is a right angle, and quadrangle $PP'Q_1P''$ a rectangle. Join points P and Q_1 and



produce PQ_1 to cut the major and minor axes of the ellipse, both produced if necessary, at points M and N, respectively. Then, obviously, $OT=MT=TN$, $PQ_1=P'P''=OA-OC$, $PM=P''O=OC$, $MQ_1=P'O=OA$, and angle POQ_1 is the difference between the given angle POQ and a right angle.

Thus to solve the problem we have the construction: Lay down a linear segment $MN(=OA+OC)$, and on it as a diameter describe a circle. Lay off $MP(=OC)$ along MN and $PQ_1=OA-OC$. On PQ_1 describe the segment of a circle in which the given angle POQ_1 $(=POQ-90°)$ can be inscribed. Either of the two points, O or O_1 in which this circle cuts the circle whose diameter is MN will be the center of the given ellipse, OP and OQ_1 will be the lengths of the required conjugate semi-diameters, and along OM and ON will lie the given principal axes. Moreover, OP will be in position and the position of OQ is determined by revolving OQ_1 clockwise about O through an angle of 90°.

Incidentally, the solution of the problem gives a ruler-compass construction for finding the principal axes of an ellipse when a pair of conjugate diameters is given.

Solved also by W. B. Campbell, J. A. Greenwood, L. M. Kelly, H. A. Pool, E. P. Starke, and S. Vatriquant.

Editorial Note. The solutions by Kelly and Pool used the geometrical theorems that, if a' and b' are the lengths of a pair of conjugate semi-diameters and θ is the acute angle between them,

$$a'^2 + b'^2 = a^2 + b^2 = AB^2, \qquad a'b' \sin\theta = ab,$$

where $a=OA$, $b=OB$ are the principal semi-axes. This method was also indicated by Vatriquant. Campbell, Greenwood and Starke used the slopes of the conjugate axes. Solution I uses the theorem that, if DOD' is any diameter of the ellipse and M is any point on the ellipse, the lines joining the center O with

the mid-points of the chords DM and $D'M$ are a pair of conjugate diameters. Chords MD and MD' have been designated as supplemental chords. Vatriquant's solution also made use of supplemental chords, taking for the diameter the principal diameter AOA'. The center of the circle circumscribing AMA', where the angle at M is known, is then easily found. The intersections of the circle and the ellipse were then determined algebraically. The geometric construction made use of the properties of the foci.

A construction similar to II may be obtained by using the theorem stated at the end of the note on the solutions of 3718 [1936, 440]. Draw the segment of a straight line $OE=a-b$, and with the mid-point K of this segment as center describe a circle of radius $(a+b)/2$. On OE as a chord describe an arc of a circle such that the ang e subtended by the chord at any point of this arc is $90°-\theta$, where θ is the given acute angle between the conjugate diameters. Let P be one of the points of intersection of the two circles. Draw the straight line segment EP and produce it to D so that $PD=EP$. Then $PD=a+b$. The point P is on the ellipse with semi-axes a and b, and the major semi-axis a lies along the bisector OX of angle EOD. Also, as stated in the reference, DPE is normal to the ellipse at P. Hence the diameter OQ conjugate to OP lies along the perpendicular from O to DE. The angle between these conjugate diameters is obviously θ. The end Q of the second diameter is easily constructed. It is clear that there may be one, two, or no constructions according to the magnitude of θ.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The editors of the Annals of Mathematics desire to secure the following numbers of that journal: Vol. 1, No. 1; Vol. 2, Nos. 1, 3; Vol. 3, Nos. 2, 3, 4; Vol. 4, No. 1. Readers having any of these copies for disposal are asked to write directly to Professor S. Lefschetz, Princeton, N. J.

Dr. A. H. Bailey of Ohio State University has been appointed to a professorship at Central Normal College, Danville, Indiana.

Assistant Professor W. D. Baten of the University of Michigan has been appointed associate professor of mathematical statistics and research associate in statistics at Michigan State College. His work will include that of statistical adviser in the State College Experiment Station.

Dr. A. H. Black of Cornell University has been appointed to a professorship at Lebanon Valley College, Annville, Pa.

Assistant Professor I. L. Hebel of the Colorado School of Mines has been promoted to an associate professorship.

Dr. C. F. Luther of Stanford University has been appointed an associate professor at Willamette University.

Assistant Professor A. F. Moursund of the University of Oregon has been promoted to an associate professorship.

Dr. Rufus Oldenburger has been promoted to an assistant professorship at the Armour Institute of Technology.

E. W. Ploenges of Kansas Wesleyan University has been appointed associate professor at James Millikin University.

Dr. Helen G. Russell of Wellesley College has been promoted to an assistant professorship.

S. J. Smith of the State Teachers College, Lockhaven, Pennsylvania, has been appointed director of teacher training there. Margaret E. Waldron of Providence (R.I.) Teachers College has been appointed to succeed him as instructor in mathematics.

Dr. Ruth W. Stokes has been appointed professor of mathematics and astronomy at Winthrop College, Rock Hill, South Carolina.

Assistant Professor J. C. Trainor has been promoted to an associate professorship at Washington State Normal School at Ellensburg.

Dr. Henry VanEngen of Western Reserve University has been appointed an assistant professor at Kansas State College.

At Northwestern University, Dr. H. S. Wall has been promoted to an associate professorship and Dr. H. L. Garabedian to an assistant professorship.

Assistant Professor Louis Weisner of Hunter College has been promoted to an associate professorship.

The following appointments to instructorships have been announced:

Hunter College: Meta A. Wood
Michigan State College: Dr. C. C. Hurd
Southern Methodist University: L. D. Rodabaugh

Dr. C. L. Herron, dean of Hillsdale College, a charter member of the Mathematical Association, died November 13, 1936, at the age of seventy-five. He had taught mathematics there for thirty-four years.

Dr. G. A. Plimpton, senior member of the firm of Ginn and Company, Publishers, died July 1, 1936, at the age of eighty years. He had for some years been a member of the Association.

J. A. Reising, teacher at Central High School and in the Indiana University Extension at Fort Wayne, Indiana, died September 9, 1936, at the age of sixty-six. He had been a member of the Association for a number of years.

Dr. H. I. Thomsen of Baltimore, Maryland, a member of the Mathematical Association since 1917, died on October 20, 1936.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, Low Memorial Library, Columbia University, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of addresss to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-first Annual Meeting, Duke University, Durham, N. C., Dec. 31, 1936–Jan. 1, 1937.
Twenty-first Summer Meeting, Pennsylvania State College, Sept., 1937.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1936 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, May 2.
ILLINOIS, Normal, May 8–9.
INDIANA, North Manchester, May 1–2.
IOWA, Iowa City, April 3–4.
KANSAS, Topeka, March 14.
KENTUCKY, Richmond, May; Nashville, Tenn., Nov. 20–21.
LOUISIANA-MISSISSIPPI, Hattiesburg, Miss., Mar. 13–14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, May 9; Washington, Dec. 5.
MICHIGAN, Ann Arbor, March 21; Albion, Nov. 28.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
OKLAHOMA, Oklahoma City, Feb.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Denver, April.
SOUTHEASTERN, Columbia, S.C., April 17–18.
SOUTHERN CALIFORNIA, Fullerton, Mar. 7.
TEXAS, College Station, April 11.
WISCONSIN, Madison, May 9.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN